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Generalization of the Plana Formula

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Abstract—We obtain an analog of the Plana formula, which is essential in obtaining the functional equation for the classical Riemann zeta-function.

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INTRODUCTION

The aim of the paper is obtaining an analog of the Plana formula (see, e.g., [1], Chap. 7, exercise 7) which is essential for finding the functional equation (see, e.g., [2], Chap. 2, section 9) for the classical Riemann zeta-function. Kuzovatov and A. M. Kytmanov [3] obtained an analog of the Plana formula under more strong restrictions on rational function. In the paper we remove these restrictions.

Let φ be a holomorphic function, bounded for all z such that $x_1 \leq \operatorname{Re} z \leq x_2$; x_1, x_2 are integers. The classical Plana formula connects the sum of values of $\varphi(z)$ at integer points with some integrals:

$$\frac{1}{2}\varphi(x_1) + \varphi(x_1+1) + \varphi(x_1+2) + \cdots + \varphi(x_2-1) + \frac{1}{2}\varphi(x_2)$$
$$= \int_{x_1}^{x_2} \varphi(z)\, dz + \frac{1}{i}\int_0^{\infty} \frac{\varphi(x_2+iy) - \varphi(x_2-iy) + \varphi(x_1-iy) - \varphi(x_1+iy)}{e^{2\pi y}-1}\, dy. \quad (1)$$

Concerning generalizations of the zeta-function, we note that in 1950s Gelfand, Levitan, and Dikii (see, e.g., [4–6]) studied the zeta-function associated to eigenvalues of the Sturm–Lioville operator. As it turned out, its value is connected with the trace of the operator. Further their approach was developed by Lidskii and Sadovnichii [7] who considered a class of entire functions of one variable, defined the zeta-function of their zeroes and investigated its domain of analytic continuation. Smagin and Shubin [8] constructed the zeta-functions for elliptic operators, as long for operators of more general type, proved a possibility of meromorphic continuation of the zeta-function and gave some information on its poles.

Multidimensional results were obtained by A. M. Kytmanov and Myslivetz [9]. They introduced the concept of zeta-function associated with a system of meromorphic functions $f = (f_1, \ldots, f_n)$ in $\mathbb{C}^n$. With the help of the residue theory, an integral representation for the zeta-function was given but under fulfillment of some strong conditions on the system $f_1, \ldots, f_n$.

In [10], with the help of the residue theory, Kuzovatov and A. A. Kytmanov obtained two integral representation for zeta-function constructed by zeroes of an entire function of finite order on the complex plane. With the help of these representations, they described a domain which the zeta-functions can be extended to.

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