УЧЕНЫЕ ЗАПИСКИ КАЗАНСКОГО УНИВЕРСИТЕТА.
СЕРИЯ ФИЗИКО-МАТЕМАТИЧЕСКИЕ НАУКИ

2018, Т. 160, кн. 2 ISSN 2541-7746 (Print)
С. 243–249 ISSN 2500-2198 (Online)

UDK 517.983:517.986

ON AN ANALOG OF THE M.G. KREIN THEOREM FOR MEASURABLE OPERATORS

A.M. Bikchentaev

Kazan Federal University, Kazan, 420008 Russia

Abstract

Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$ and τ be a faithful normal semifinite trace on $\mathcal{M}$. Let $\mu_t(T)$, $t > 0$, be a rearrangement of a τ-measurable operator T. Let us consider a τ-measurable operator A, such that $\mu_t(A) > 0$ for all $t > 0$ and assume that $\mu_{2t}(A)/\mu_t(A) \to 1$ as $t \to \infty$. Let a τ-compact operator S be so that the operator $I + S$ is right invertible, where I is the unit of $\mathcal{M}$. Then, for a τ-measurable operator B, such that $A = B(I + S)$, we have $\mu_t(A)/\mu_t(B) \to 1$ as $t \to \infty$. It is an analog of the M.G. Krein theorem (for $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\tau = \mathrm{tr}$, theorem 11.4, ch. V [Gohberg I.C., Krein M.G. Introduction to the theory of linear nonselfadjoint operators. In: *Translations of Mathematical Monographs.* Vol. 18. Providence, R.I., Amer. Math. Soc., 1969. 378 p.] for τ-measurable operators.

Keywords: Hilbert space, von Neumann algebra, normal trace, τ-measurable operator, distribution function, rearrangement, τ-compact operator

---

Introduction

Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$ and τ be a faithful normal semifinite trace on $\mathcal{M}$. In theorem 3.5, we prove an analog of the M.G. Krein theorem (for $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\tau = \mathrm{tr}$, theorem 11.4, ch. V, [1]) for τ-measurable operators. We also describe asymptotics of the generalized singular numbers for a product of almost commuting τ-measurable operators.

1. Notation, definitions, and preliminaries

Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$. Let $\mathcal{M}^{\mathrm{pr}}$ be the lattice of projections in $\mathcal{M}$. Let I be the unit of $\mathcal{M}$. Let $P^{\perp} = I - P$ for $P \in \mathcal{M}^{\mathrm{pr}}$. Let $\mathcal{M}^+$ be the cone of positive elements in $\mathcal{M}$.

A mapping $\varphi : \mathcal{M}^+ \to [0, +\infty]$ is called *a trace*, if $\varphi(X + Y) = \varphi(X) + \varphi(Y)$, $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{M}^+$, $\lambda \geq 0$ (moreover, $0 \cdot (+\infty) \equiv 0$) and $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{M}$. A trace φ is called as follows: *faithful* if $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$, $X \neq 0$; *finite* if $\varphi(X) < +\infty$ for all $X \in \mathcal{M}^+$; *semifinite* if $\varphi(X) = \sup\{\varphi(Y) :\ Y \in \mathcal{M}^+,\ Y \leq X,\ \varphi(Y) < +\infty\}$ for every $X \in \mathcal{M}^+$; *normal* if $X_i \nearrow X$ $(X_i, X \in \mathcal{M}^+) \Rightarrow \varphi(X) = \sup \varphi(X_i)$.

An operator on $\mathcal{H}$ (not necessarily bounded or densely defined) is said to be *affiliated with a von Neumann algebra* $\mathcal{M}$ if it commutes with any unitary operator from the commutant $\mathcal{M}'$ of the algebra $\mathcal{M}$. A self-adjoint operator is affiliated with $\mathcal{M}$ if and only if all the projections from its spectral decomposition of unity belong to $\mathcal{M}$.

Let τ be a faithful normal semifinite trace on $\mathcal{M}$. A closed operator X of everywhere dense in $\mathcal{H}$ domain $\mathcal{D}(X)$ and affiliated with $\mathcal{M}$ is said to be τ-*measurable* if there exists such a projection $P \in \mathcal{M}^{\mathrm{pr}}$ for any $\varepsilon > 0$ that $P\mathcal{H} \subset \mathcal{D}(X)$ and $\tau(P^{\perp}) < \varepsilon$. The set $\widetilde{\mathcal{M}}$ of all τ-measurable operators is a *-algebra under transition to the adjoint operator, multiplication by a scalar, and strong addition and multiplication operations defined as closure of the usual operations [2, 3].

If X is a closed densely defined linear operator affiliated with $\mathcal{M}$ and $|X| = \sqrt{X^*X}$, then the spectral decomposition $P^{|X|}(\cdot)$ is contained in $\mathcal{M}$ and X belongs to $\widetilde{\mathcal{M}}$ if and only if there exists a number $\lambda \in \mathbb{R}$, such that $\tau(P^{|X|}((\lambda, +\infty))) < +\infty$. Let $\mu_t(X)$ denote the *rearrangement* of the operator $X \in \widetilde{\mathcal{M}}$, i.e., the nonincreasing right continuous function $\mu(X)\colon (0,\infty) \to [0,\infty)$ given by the formula

$$\mu_t(X) = \inf\{\|XP\| : \ P \in \mathcal{M}^{\mathrm{pr}}, \quad \tau(P^{\perp}) \le t\}, \quad t > 0.$$

Then, $\mu_t(X) = \inf\{s \ge 0 : \ \lambda_s(X) \le t\}$, where $\lambda_s(X) = \tau(P^{|X|}((s, \infty)))$ is the distribution function of X. The set of τ-compact operators $\widetilde{\mathcal{M}}_0 = \{X \in \widetilde{\mathcal{M}} : \lim\limits_{t \to +\infty} \mu_t(X) = 0\}$ is an ideal in $\widetilde{\mathcal{M}}$ [4].

Lemma 1 (see [4–6]). *Let $X, Y \in \widetilde{\mathcal{M}}$. Then*
1) $\mu_t(X) = \mu_t(|X|) = \mu_t(X^*)$ *for all* $t > 0$;
2) $\mu_{s+t}(X + Y) \le \mu_s(X) + \mu_t(Y)$ *for all* $s, t > 0$;
3) $\mu_{s+t}(XY) \le \mu_s(X)\mu_t(Y)$ *for all* $s, t > 0$;
4) $\mu_t(|X|^p) = \mu_t(X)^p$ *for all* $p, t > 0$.

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, i.e., the *-algebra of all linear bounded operators on $\mathcal{H}$, and $\tau = \mathrm{tr}$ is the canonical trace, then $\widetilde{\mathcal{M}}$ coincides with $\mathcal{B}(\mathcal{H})$. In this case, $\widetilde{\mathcal{M}}_0$ is the compact operators ideal on $\mathcal{H}$ and

$$\mu_t(X) = \sum_{n=1}^{\infty} s_n(X)\chi_{[n-1,n)}(t), \quad t > 0,$$

where $\{s_n(X)\}_{n=1}^{+\infty}$ is a sequence of an operator X s-numbers [1]; here, χ_A is the indicator function of a set $A \subset \mathbb{R}$.

2. A generalization of the M.G. Krein theorem for τ-measurable operators

Lemma 2. *The following conditions are equivalent for a nonincreasing function* $f\colon (0,\infty) \to (0,\infty)$:

(i) *there exists* $\lim\limits_{t\to\infty} \dfrac{f(at)}{f(t)} = 1$ *for some number* $0 < a \ne 1$;

(ii) *there exists* $\lim\limits_{t\to\infty} \dfrac{f(bt)}{f(t)} = 1$ *for every number* $b > 0$.

Proof. (i)$\Rightarrow$(ii). We have

$$1 = \lim_{t\to\infty} \frac{f(at)}{f(t)} = \left[\lim_{t\to\infty} \frac{f(at)}{f(t)}\right]^{-1} = \lim_{t\to\infty} \left[\frac{f(at)}{f(t)}\right]^{-1} = \\ = \lim_{t\to\infty} \frac{f(t)}{f(at)} = \lim_{u\to\infty} \frac{f(a^{-1}u)}{f(u)}, \quad (1)$$

where $u = at$ for all $t > 0$. Hence, we assume that $a, b > 1$.

Case 1: $1 < b < a$. Then, we have

$$\frac{f(a^{-1}t)}{f(t)} \ge \frac{f(bt)}{f(t)} \ge \frac{f(at)}{f(t)} \quad \text{for all } \ t > 0$$

and the lemma follows from (1) and the squeeze theorem.

Case 2: $1 < a < b$. Then, for $k \equiv \min\left\{n \in \mathbb{N} : \dfrac{b}{a^{n+1}} < a\right\}$ and for all $t > 0$, we have

$$\frac{f(a^{-1}t)}{f(t)} \geq \frac{f(bt)}{f(t)} = \frac{f(bt)}{f\left(\frac{b}{a}t\right)} \frac{f\left(\frac{b}{a}t\right)}{f\left(\frac{b}{a^2}t\right)} \cdots \frac{f\left(\frac{b}{a^k}t\right)}{f\left(\frac{b}{a^{k+1}}t\right)} \frac{f\left(\frac{b}{a^{k+1}}t\right)}{f(t)} \geq$$

$$\geq \frac{f(bt)}{f\left(\frac{b}{a}t\right)} \frac{f\left(\frac{b}{a}t\right)}{f\left(\frac{b}{a^2}t\right)} \cdots \frac{f\left(\frac{b}{a^k}t\right)}{f\left(\frac{b}{a^{k+1}}t\right)} \frac{f(at)}{f(t)}$$

and the lemma follows from relations (1) and

$$\lim_{t\to\infty} \frac{f(bt)}{f\left(\frac{b}{a}t\right)} = \lim_{t\to\infty} \frac{f\left(\frac{b}{a}t\right)}{f\left(\frac{b}{a^2}t\right)} = \cdots = \lim_{t\to\infty} \frac{f\left(\frac{b}{a^k}t\right)}{f\left(\frac{b}{a^{k+1}}t\right)} = 1,$$

combined with theorem on the limit of product of functions and the squeeze theorem. The lemma is proved.

□

Example 1. 1) The conditions of lemma 2 hold if there exists $\lim\limits_{t\to\infty} f(t) = x > 0$.

2) Let us consider $f(t) = \dfrac{1}{\log(1+t)}$ for all $t > 0$. Then, there exists $\lim\limits_{t\to\infty} f(t) = x = 0$ and the conditions of lemma 2 also hold by the L'Hospital theorem for $\dfrac{f(2t)}{f(t)} = \dfrac{\log(1+t)}{\log(1+2t)} = \left\{\dfrac{\infty}{\infty}\right\}$ as $t \to \infty$. Induction helps us to prove the same result for n-iterated function $f_n(t) = \dfrac{1}{\log\log\cdots\log(e^{n-1}+t)}$ for all $n \in \mathbb{N}$ and $t > 0$.

3) If functions f, g satisfy the conditions of lemma 2, then, for the functions $f_{p*}(t) = f(pt)$, $f_{p+}(t) = f(t+p)$, $\psi_{f,p}(t) = \displaystyle\int_t^{t+p} f(u)du$, $f(t^p)$, f^p $(0 < p < \infty)$, $\log(1+f)$, $f+g$, $\dfrac{f}{g}$ (if $\dfrac{f}{g}$ is nonincreasing), and fg, the conditions of lemma 2 also hold.

We prove it for f_{p+}, $\psi_{f,p}$, $\log(1+f)$ and $f+g$. The case of $x = \lim\limits_{t\to\infty} f(t) > 0$ is trivial. Let us put $x = 0$. Since

$$\frac{f(t+p)}{f(2t+2p)} \leq \frac{f(t+p)}{f(2t+p)} = \frac{f_p(t)}{f_p(2t)} \leq \frac{f(t+p/2)}{f(2t+p)} \quad \text{for all} \quad t > 0,$$

we can apply the squeeze theorem.

Since $pf(t+p) \leq \psi_{f,p}(t) \leq pf(t)$, we have for all $t > p$ the estimates

$$\frac{f(3t)}{f(t)} \leq \frac{f(2t+p)}{f(t)} = \frac{pf(2t+p)}{p\ f(t)} \leq \frac{\psi_{f,p}(2t)}{\psi_{f,p}(t)} \leq \frac{pf(2t)}{pf(t+p)} = \frac{f(2t)}{f(t+p)} \leq 1$$

and are able to apply the squeeze theorem.

We have $\log(1+u) = u + o(u)$ as $u \to 0$ and $f(2t) = f(t) + o(f(t))$ as $t \to \infty$. Therefore

$$\frac{\log(1+f(2t))}{\log(1+f(t))} = \frac{f(2t)+o(f(2t))}{f(t)+o(f(t))} = \frac{f(2t)+o(f(t))}{f(t)+o(f(t))} = 1 + o(f(t))$$

as $t \to \infty$. For $h = f + g$ we have $o(f(t)) + o(g(t)) = o(h(t))$ and

$$\frac{h(2t)}{h(t)} - 1 = \frac{f(2t) - f(t) + g(2t) - g(t)}{f(t)+g(t)} = \\ = \frac{o(f(t)) + o(g(t))}{f(t)+g(t)} = \frac{o(h(t))}{h(t)} = o(1) \quad \text{as} \quad t \to \infty.$$

4) Let us consider f, as in lemma 2, numbers $\alpha, \beta > 0$ and a nonincreasing function $g\colon (0,\infty) \to (0,\infty)$, so that $f(\alpha t) \le g(t) \le f(\beta t)$ for all $t > 0$. Then, for the function g, the conditions of lemma 1 also hold.

Lemma 3. *Let $\mathcal{J}$ be a left ideal in a unital algebra $\mathcal{A}$ and $S \in \mathcal{J}$ be so that the element $I + S$ is right invertible (i.e., there exists $T \in \mathcal{A}$ with $(I+S)T = I$). Then, $T = I + X$ for some $X \in \mathcal{J}$.*

Proof. Since $(I+S)T = I$, we have $T = I - ST \equiv I + X$ with $X \equiv -ST \in \mathcal{J}$. The lemma is proved.

□

Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$ and $\tau(I) = +\infty$.

Proposition 1 (cf. lemma 3). *Let an isometry operator $U \in \mathcal{M}$ and a selfadjoint operator $A \in \widetilde{\mathcal{M}}$ be so that $I + A$ is invertible in $\widetilde{\mathcal{M}}$. Then, the following conditions are equivalent:*
(i) $U - A \in \widetilde{\mathcal{M}}_0$;
(ii) $I - A,\ I - U \in \widetilde{\mathcal{M}}_0$.

Proof. (i)⇒(ii). We have $U^* - A = (U - A)^* \in \widetilde{\mathcal{M}}_0$ and

$$-U^*A + AU = U^*(U-A) - (U^* - A)U \in \widetilde{\mathcal{M}}_0.$$

Therefore, $I - A^2 = (U^* - A)(U + A) - U^*A + AU \in \widetilde{\mathcal{M}}_0$ and $I - A = (I - A^2)(I + A)^{-1} \in \widetilde{\mathcal{M}}_0$. Thus, $I - U = I - A - (U - A) \in \widetilde{\mathcal{M}}_0$.

(ii)⇒(i). We have $U - A = (I - A) - (I - U) \in \widetilde{\mathcal{M}}_0$. The proposition is proved.

□

Theorem 1. *Let an operator $A \in \widetilde{\mathcal{M}}$ be such that $\mu_t(A) > 0$ for all $t > 0$ and assume that there exists $\lim\limits_{t\to\infty} \dfrac{\mu_{2t}(A)}{\mu_t(A)} = 1$. Let an operator $S \in \widetilde{\mathcal{M}}_0$ be so that the operator $I+S$ is right invertible in $\widetilde{\mathcal{M}}$. Then, for an operator $B \in \widetilde{\mathcal{M}}$, such that $A = B(I+S)$, there exists $\lim\limits_{t\to\infty} \dfrac{\mu_t(A)}{\mu_t(B)} = 1$.*

Proof. Let a number $\varepsilon > 0$ be arbitrary and let a number $t_1 > 0$ be such that $\mu_{t/3}(S) < \varepsilon$ for $t \ge t_1$. Then, by items 2) and 3) of lemma 1, we have the following estimates for all $t \ge t_1$:

$$\mu_t(A) = \mu_t(B + BS) \le \mu_{t/3}(B) + \mu_{2t/3}(BS) \le \\ \le \mu_{t/3}(B) + \mu_{t/3}(B)\mu_{t/3}(S) < (1+\varepsilon)\mu_{t/3}(B). \quad (2)$$

Let an operator $T \in \widetilde{\mathcal{M}}$ be such that $(I+S)T = I$. Then, $T = I + X$ with some $X \in \widetilde{\mathcal{M}}_0$, see lemma 3. Since

$$AT = B(I+S)T = B = A(I+X),$$

for number $t_2 > 0$ with $\mu_{t/3}(X) < \varepsilon$ for $t \geq t_2$, we obtain, analogously to estimates (2), the relation

$$\mu_t(B) < (1+\varepsilon)\mu_{t/3}(A) \quad \text{for all} \quad t \geq t_2. \tag{3}$$

Let a number $t_3 > 0$ be such that

$$1 \leq \frac{\mu_{t/9}(A)}{\mu_{t/3}(A)} < 1 + \varepsilon \quad \text{for all} \quad t \geq t_3,$$

see lemma 2. Let us put $t_0 = \max\{t_1, t_2, t_3\}$. From (2) and (3) we obtain for all $t > t_0$

$$\mu_t(A) < (1+\varepsilon)\mu_{t/3}(B) < (1+\varepsilon)^2\mu_{t/9}(A),$$

hence,

$$1 \leq \frac{\mu_t(A)}{\mu_{t/3}(A)} < (1+\varepsilon)\frac{\mu_{t/3}(B)}{\mu_{t/3}(A)} < (1+\varepsilon)^2\frac{\mu_{t/9}(A)}{\mu_{t/3}(A)} < (1+\varepsilon)^3.$$

Therefore,

$$1 < (1+\varepsilon)\frac{\mu_{t/3}(B)}{\mu_{t/3}(A)} < (1+\varepsilon)^3 \quad \text{for all} \quad t > t_0.$$

The theorem is proved. □

Corollary 1. *Let an operator $A \in \widetilde{\mathcal{M}}$ be such that $\mu_t(A) > 0$ for all $t > 0$ and assume that there exists $\lim\limits_{t\to\infty} \dfrac{\mu_{2t}(A)}{\mu_t(A)} = 1$. Let an operator $S \in \widetilde{\mathcal{M}}_0$ be so that the operator $I + S$ is left invertible in $\widetilde{\mathcal{M}}$. Then, for an operator $B \in \widetilde{\mathcal{M}}$, such that $A = (I+S)B$, there exists $\lim\limits_{t\to\infty} \dfrac{\mu_t(A)}{\mu_t(B)} = 1$.*

Proof. We have $S^* \in \widetilde{\mathcal{M}}_0$ and since $(XY)^* = Y^*X^*$ for all $X, Y \in \widetilde{\mathcal{M}}$, the operator $I + S^*$ is right invertible in $\widetilde{\mathcal{M}}$. Therefore, $A^* = B^*(I + S^*)$. Then, we apply theorem 1 for the operators A^*, B^*, S^* and recall item 1) of lemma 1. The corollary is proved.

□

Example 2. Let operators $X, Y \in \widetilde{\mathcal{M}}$ be almost commuting, i.e., the commutator $[X,Y] = XY - YX \in \widetilde{\mathcal{M}}_0$. Let us put $K = [X,Y]$ and let the operator YX possess a right inverse $T \in \widetilde{\mathcal{M}}$. Hence, $XY = YX(I+TK)$. Since the operator YX is right invertible by item 3) of lemma 1, we have $1 = \mu_t(I) = \mu_t(YXT) \leq \mu_{t/2}(YX)\mu_{t/2}(T)$ for all $t > 0$. Hence, $\mu_t(YX) > 0$ for all $t > 0$. Now, if the operator $I + TK$ possess a right inverse $R \in \widetilde{\mathcal{M}}$ (then $XYR = YX(I+TK)R = YX$ and by item 3) of lemma 1, we have $0 < \mu_t(YX) \leq \mu_{t/2}(XY)\mu_{t/2}(R)$ for all $t > 0$; hence, $\mu_t(XY) > 0$ for all $t > 0$) and there exists $\lim\limits_{t\to\infty} \dfrac{\mu_{2t}(XY)}{\mu_t(XY)} = 1$, then there exists $\lim\limits_{t\to\infty} \dfrac{\mu_t(XY)}{\mu_t(YX)} = 1$ by theorem 1. For any normal operators $X, Y \in \widetilde{\mathcal{M}}$, we have $\mu_t(XY) = \mu_t(YX)$ for all $t > 0$ [7, corollary 3.6].

Remark 1. In theorem 1 and corollary 1 by item 4) of lemma 1, there exists $\lim\limits_{t\to\infty}\dfrac{\mu_t(|A|^p)}{\mu_t(|B|^p)}=1$ for every $p>0$. For $\mathcal{M}=\mathcal{B}(\mathcal{H})$ and $\tau=\mathrm{tr}$, the condition "there exists $\lim\limits_{t\to\infty}\dfrac{\mu_{2t}(A)}{\mu_t(A)}=1$" also appeared in [8].

Example 3. Let (Ω,ν) be a measure space and $\mathcal{M}$ be the von Neumann algebra of multiplicator operators M_f by functions f from $L_\infty(\Omega,\nu)$ on a space $L_2(\Omega,\nu)$. The algebra $\mathcal{M}$ containes no compact operators $\Leftrightarrow$ the measure ν has no atoms [9, theorem 8.4]. Let $\mathcal{M}=L_\infty(0,\infty)$ and $\mathcal{H}=L_2(0,\infty)$. Then, for any right continuous nonincreasing function $f\colon(0,\infty)\to(0,\infty)$, we have $\mu_t(M_f)=f(t)$ for all $t>0$, see definition 2.2, ch. II, [10]. Example 1 shows that the set of multiplicator operators M_f, such that there exists $\lim\limits_{t\to\infty}\dfrac{\mu_{2t}(M_f)}{\mu_t(M_f)}=1$, is relatively rich.

Acknowledgements. This work was supported by subsidies allocated to Kazan Federal University for the state assignment in the sphere of scientific activities (projects nos. 1.1515.2017/4.6 and 1.9773.2017/8.9).

References

1. Gohberg I.C., Krein M.G. Introduction to the theory of linear nonselfadjoint operators. In: *Translations of Mathematical Monographs.* Vol. 18. Providence, R.I., Amer. Math. Soc., 1969. 378 p.
2. Segal I.E. A non-commutative extension of abstract integration. *Ann. Math.*, 1953, vol. 57, no. 3, pp. 401–457. doi: 10.2307/1969729.
3. Nelson E. Notes on non-commutative integration. *J. Funct. Anal.*, 1974, vol. 15, no. 2, pp. 103–116. doi: 10.1016/0022-1236(74)90014-7.
4. Yeadon F.J. Non-commutative L^p-spaces. *Math. Proc. Cambridge Philos. Soc.*, 1975, vol. 77, no. 1, pp. 91–102. doi: 10.1017/S0305004100049434.
5. Ovchinnikov V.I. Symmetric spaces of measurable operators, *Dokl. Akad. Nauk SSSR*, 1970, vol. 191, no. 4, pp. 769–771. (In Russian)
6. Fack T., Kosaki H. Generalized s-numbers of τ-measurable operators. *Pac. J. Math.*, 1986, vol. 123, no. 2, pp. 269–300.
7. Bikchentaev A.M. On normal τ-measurable operators affiliated with semifinite von Neumann algebras. *Math. Notes*, 2014, vol. 96, nos. 3–4, pp. 332–341. doi: 10.1134/S0001434614090053.
8. Matsaev V.I., Mogul'ski E.Z. On the possibility of weak perturbation of a complete operator up to a Volterra operator. *Dokl. Akad. Nauk SSSR*, 1972, vol. 207, no. 3, pp. 534–537. (In Russian)
9. Antonevich A.B. *Linear functional equations. Operator Approach.* Basel, Birkhäuser, 1996. viii, 183 p.
10. Krein S.G., Petunin Ju.I., Semenov E.M. Interpolation of linear operators. In: *Translations of Mathematical Monographs.* Vol. 54. Providence, R.I., Amer. Math. Soc., 1982. 375 p.

Received
October 12, 2017

Bikchentaev Airat Midkhatovich, Doctor of Physical and Mathematical Sciences, Leading Research Fellow of Department of Theory of Functions and Approximations

Kazan Federal University
ul. Kremlevskaya, 18, Kazan, 420008 Russia
E-mail: *Airat.Bikchentaev@kpfu.ru*

УДК 517.983:517.986

Об аналоге теоремы М.Г. Крейна для измеримых операторов

А.М. Бикчентаев

Казанский (Приволжский) федеральный университет, г. Казань, 420008, Россия

Аннотация

Пусть алгебра фон Неймана операторов $\mathcal{M}$ действует в гильбертовом пространстве $\mathcal{H}$ и τ – точный нормальный полуконечный след на $\mathcal{M}$. Пусть $\mu_t(T)$, $t > 0$, – перестановка τ-измеримого оператора T. Пусть τ-измеримый оператор A такой, что $\mu_t(A) > 0$ для всех $t > 0$ и пусть $\mu_{2t}(A)/\mu_t(A) \to 1$ при $t \to \infty$. Пусть τ-компактный оператор S такой, что оператор $I + S$ является обратимым справа, где I – единица алгебры $\mathcal{M}$. Тогда для τ-измеримого оператора B такого, что $A = B(I + S)$, имеем $\mu_t(A)/\mu_t(B) \to 1$ при $t \to \infty$. Это является аналогом теоремы М.Г. Крейна (для $\mathcal{M} = \mathcal{B}(\mathcal{H})$ и $\tau = \mathrm{tr}$ (теорема 11.4, гл. V, [Гохберг И.Ц., Крейн М.Г. Введение в теорию линейных несамосопряженных операторов. – М.: Наука, 1965. – 448 с.]), для τ-измеримых операторов.

Ключевые слова: гильбертово пространство, алгебра фон Неймана, нормальный след, τ-измеримый оператор, функция распределения, перестановка, τ-компактный оператор

Поступила в редакцию
12.10.17

Бикчентаев Айрат Мидхатович, доктор физико-математических наук, ведущий научный сотрудник кафедры теории функций и приближений

Казанский (Приволжский) федеральный университет
ул. Кремлевская, д. 18, г. Казань, 420008, Россия
E-mail: *Airat.Bikchentaev@kpfu.ru*

***For citation*:** Bikchentaev A.M. On an analog of the M.G. Krein theorem for measurable operators. *Uchenye Zapiski Kazanskogo Universiteta. Seriya Fiziko-Matematicheskie Nauki*, 2018, vol. 160, no. 2, pp. 243–249.

***Для цитирования*:** *Bikchentaev A.M.* On an analog of the M.G. Krein theorem for measurable operators // Учен. зап. Казан. ун-та. Сер. Физ.-матем. науки. – 2018. – Т. 160, кн. 2. – С. 243–249.