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A Relative Variant of the Morse Theory

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Abstract—We develop a relative variant of the Morse theory for Lipschitz functionals defined on closed subsets of a Banach manifold. We prove the invariance of topological characteristics of functionals under uniform deformations.

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The well-known infinite-dimensional versions of the Morse theory have respect to functionals from the class $C^{1,1}$ defined on a Hilbert manifold. In this paper the smoothness of the functionals is reduced by one. It is important that functionals are considered not on the whole Banach manifold, but only on its closed subset whose boundary, possibly, is not a smooth manifold. The main results are new even for functions of a finite set of arguments.

1. Recall some definitions of the nonsmooth analysis. Let X and Y be real Banach spaces and let C be a subset of X. The set of all maps from C to Y satisfying the local Lipschitz condition is denoted as $\Lambda(C, Y)$, i.e., $F \in \Lambda(C, Y)$ if for all x in C there exist a neighborhood V and a constant k such that

$$\|F(x_1) - F(x_2)\| \leq k\, \|x_1 - x_2\| \quad (x_i \in C \cap V,\ \ i = 1, 2).$$

Below $B = \{x \in X : \|x\| \leq 1\}$ is the unit ball in the space X, $B(x, r) = x + rB$ $(x \in X,\ r \geq 0)$; A^0, ∂A, $\overline{A}$ are, respectively, the interior, the boundary, and the closure of the set $A \subset X$, $\Lambda(C) = \Lambda(C, \mathbb{R})$ is the class of real functions defined on C satisfying the local Lipschitz condition.

Let $x \in C \subset X$. A vector v in X is said to be ([1], P. 59; [2]) *hypertangent* to the set C at the point x, if with some $\varepsilon > 0$ it holds

$$y + tw \in C \ \ \forall y \in C \cap B(x, \varepsilon),\ \ w \in B(v, \varepsilon),\ \ t \in (0, \varepsilon).$$

If there exists at least one vector hypertangent to C at x, then the set $K_C(x)$ of all such hypertangent vectors is a convex open cone in the space X [2]. The set $N_C(x) = \{x^* \in X^* : \langle x^*, z\rangle \leq 0 \ \ \forall z \in K_C(x)\}$ is said to be a *normal cone* to the set C at the point x.

If $f \in \Lambda(C)$, $x \in C^0$, then the equality

$$f^0(x; v) = \varlimsup_{y \to x,\ t \to +0} \frac{f(y + tv) - f(y)}{t}$$

defines a sublinear functional $f^0(x; \cdot) : X \to \mathbb{R}$ ([1], P. 32). The following equality [3] is valid:

$$f^0(x; v) = \varlimsup_{z \to x,\ z \in C^0} f^0(z; v). \tag{1}$$

The limit in right-hand side of (1) also exists for x in $C \cap \overline{C^0}$. In this case we use (1) as the definition of the sublinear functional $f^0(x; \cdot) : X \to \mathbb{R}$.

The set $\partial f(x) = \{x^* \in X^* : \langle x^*, v\rangle \leq f^0(x; v) \quad \forall v \in X\}$ is called a *generalized gradient* of the functional f at the point x. By virtue of the known properties of sublinear functionals (for

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