Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 44, No. 6, pp.1–7, 2000 *UDC 517.518.822*

APPROXIMATION OF FUNCTIONS OF SEVERAL VARIABLES WITH THE CHEBYSHEV–HERMITE WEIGHT

D.V. Alekseyev

In this article we prove direct theorems for approximation of functions of several variables by means of "angle" and "rectangle" of algebraic polynomials. We also prove the inverse theorem for approximation "by angle". Earlier, the corresponding results for a function of one variable were obtained in [1] and [2], while for the approximation "by angle" in the metric of L_2 in [3].

Let us introduce the notation. Let $\rho(x) = \exp\big(-\frac{1}{2}\sum\limits_{i=1}^{n} x_i^2\big)$ be a weight function, $L_{p,\rho}(\mathbb{R}^N)$ a set of Lebesgue-measurable functions of N variables $f(x_1,\dots,x_N)$ with a finite norm

$$\|f\|_{p,\rho} = \begin{cases} \Big(\displaystyle\int_{\mathbb{R}^N} |f(x)\rho(x)|^p dx\Big)^{1/p} & \text{for } 1 \le p < \infty; \\ \operatorname*{ess\,sup}\limits_{x\in\mathbb{R}^N} |f(x)\rho(x)| & \text{for } p = \infty. \end{cases}$$

We define the generalized translation operator with respect to i-th variable as follows:

$$T_h^i f(x_1,\dots,x_N) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x_1,\dots,x_{i-1},u,x_{i+1},\dots,x_N)\exp(-y^2)dy,$$

where $u = e^{-h}(x_i + y\sqrt{e^{2h}-1})$. We introduce a differential operator $D_i = \frac{1}{2}\frac{\partial^2}{\partial x_i^2} - x_i\frac{\partial}{\partial x_i}$. This operator is given on the class $\Lambda_p(D_i)$ of functions $f \in L_{p,\rho}(\mathbb{R}^n)$, for which a function g exists equivalent to f and such that, for almost all $(x_1,\dots,x_{i-1},x_{i+1},\dots,x_N)$, $\frac{\partial g}{\partial x_i}$ exists being absolutely continuous with respect to x_i on any finite segment and, in addition, $\frac{\partial^2 g}{\partial x_i^2}, x_i\frac{\partial g}{\partial x_i} \in L_{p,\rho}$. Write

$$\Lambda_p(D_i^{r_i}) = \{f \in \Lambda_p(D_i) \mid D_i f \in \Lambda_p(D_i^{r_i-1})\}, \quad \Lambda_p(D_{i_1\dots i_k}^{r_1\dots r_k}) = \{f \in \Lambda_p(D_{i_k}^{r_k}) \mid D_{i_k}^{r_k} f \in \Lambda_p(D_{i_1\dots i_{k-1}}^{r_1\dots r_{k-1}})\}.$$

Let us define the generalized smoothness module

$$\Omega_{i_1\dots i_k}^{r_1\dots r_k}(f,\delta_1,\dots,\delta_k)_{p,\rho} = \sup_{|h_j|\le\delta_j,\ j=1\div k} \|(I-T_{h_1}^{i_1})^{r_1}\cdot\ldots\cdot(I-T_{h_k}^{i_k})^{r_k} f\|_{p,\rho},$$

where I is the identical operator. Let the K-functional be defined as follows

$$K_{i_1\dots i_k}^{r_1\dots r_k}(f,\delta_1,\dots,\delta_k)_{p,\rho} = \inf_{g_{l_1\dots l_j}\in\Lambda_p(D_{i_{l_1}\dots i_{l_j}}^{r_1\dots r_k})} \{\|f - \widehat{\Sigma} g_{l_1\dots l_j}\|_{p,\rho} + \widehat{\Sigma}\delta_{l_1}^{r_{l_1}}\cdots\delta_{l_j}^{r_{l_j}} \|D_{i_{l_1}}^{r_{l_1}}\dots d_{i_{l_j}}^{r_{l_j}} g_{l_1\dots l_j}\|_{p,\rho}\},$$

where $\widehat{\Sigma}$ stands for $\sum\limits_{j=1}^{k}\sum\limits_{1\le l_1<l_2<\dots<l_j\le k}$. Let $A \le cB$, where c is a constant depending probably on p, N, k, and r_i if these parameters enter into A or B, and independent of other parameters entering into expressions A and B. In this case we shall use the notation $A \preceq B$ and say that A does not exceed B with a constant.

Supported by Russian Foundation for Basic Research (grant no. 97-01-00010).