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On Positiveness of the Cauchy Function of a Singular Linear Functional Differential Equation

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Abstract—We study first-order functional differential equations which are singular in the independent variable, establish criteria for the constant sign property of the Cauchy function, and prove an assertion analogous to the Vallée-Poussin theorem.

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In the present paper we consider a linear singular in the independent variable functional differential equation of the first order. The singularity is concentrated at the left endpoint of the segment, where the equation is defined. We obtain criteria for the constant sign property of the Cauchy function for this equation. These criteria are formulated as a theorem of the Vallée-Poussin type ([1], P. 359).

The term "Vallée-Poussin theorem " was proposed by N. V. Azbelev in 70th of the last century for theorems on the equivalence of certain properties of differential equations. For functional differential equations, in this context we consider the following properties: The existence of a constant-sign solution, the validity of the differential inequality theorem, and the assertion on the spectral radius of a certain integral operator.

The monograph [1] (pp. 356–363) contains a general assertion enabling us to prove theorems of the Vallée-Poussin type for an extensive class of functional differential equations.

Vallée-Poussin type theorems for concrete functional differential equations were proved by A. I. Domoshnitskii [2], G. G. Islamov [3], [3, 4], S. M. Labovskii [5], E. S. Chichkin [6], M. Zh. Alves [7], and others. See the monograph [1] (pp. 83, 278) for the detailed bibliography and publications review. For studying a singular equation one uses an approach based on the theory of abstract functional differential equations ([1], P. 18).

The Vallée-Poussin type theorem is related to differential inequalities. The latter are actively used, for instance, in the monograph [8].

Consider the equation

$$(\mathcal{L}x)(t) \equiv \dot{x}(t) + a(t)x(t) + (Tx)(t) = f(t), \quad t \in [0, b]. \tag{1}$$

Here the right-hand side f belongs to the space L^p of summable with power $p \in (1, \infty)$ scalar real functions $z : [0, b] \to \mathbb{R}$ with the standard norm

$$\|z\|_{L^p} = \left(\int_0^b |z(t)|^p \, dt \right)^{1/p}.$$

A solution x belongs to the space D_0^p of absolutely continuous functions $x:[0, b]\to\mathbb{R}$, whose derivatives belong to the space L^p, satisfying the additional condition $x(0) = 0$. The norm of this space is $\|x\|_{D_0^p} = \|\dot{x}\|_{L^p}$.

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