Theories of Massive Gravity

Mikhail S. Volkov

LMPT, University of Tours, FRANCE
Kazan Federal University, RUSSIA

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Theories with massive gravitons

Motivations for massive gravity

- Modification of gravity:

$$\text{Newton} \quad \frac{1}{r} \quad \rightarrow \quad \text{Yukawa} \quad \frac{1}{r}\, e^{-mr}$$

  gravity is weaker at large distances $\Rightarrow$ the cosmic acceleration, $m \sim 1/(\text{cosm. horizon size})$.

- Purely theoretical, interesting history.

Massive gravity theory

$$S = \frac{1}{\kappa^2} \int \sqrt{-g} \left(\frac{1}{2} R - m^2 \mathcal{U} \right) d^4x + S_{\mathrm{matter}}.$$

The potential $\mathcal{U}$ is a scalar function of $H^{\mu}_{\ \nu} = \delta^{\mu}_{\nu} - g^{\mu\alpha} f_{\alpha\nu}$ where $f_{\mu\nu} = \eta_{\mu\nu}$ is a flat reference metric, and

$$\mathcal{U} = \frac{1}{8} \left(H^{\mu}_{\ \nu} H^{\nu}_{\ \mu} - \alpha (H^{\mu}_{\ \mu})^2 \right) + \dots$$

higher order term can be arbitrary. In order to have the correct weak field limit one should set $\alpha = 1$. Equations

$$G_{\mu\nu} = m^2 T_{\mu\nu} + 8\pi G T^{\mathrm{m}}_{\mu\nu}$$

with

$$T_{\mu\nu} = 2 \frac{\partial \mathcal{U}}{\partial g^{\mu\nu}} - g_{\mu\nu} \mathcal{U}$$

Linear massive gravity of Fierz and Pauli /1939/

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

Linearizing the equations with respect to $h_{\mu\nu}$ gives

$$\Box h_{\mu\nu} + \ldots = m^2(h_{\mu\nu} - \alpha\, h\, \eta_{\mu\nu}) + 16\pi G T_{\mu\nu}$$

One should have $2s + 1 = 5$ Dof. Taking the divergence gives 4 constraints

$$m^2(\partial^\mu h_{\mu\nu} - \alpha \partial_\nu h) = 0.$$

Taking the trace,

$$(1 - \alpha)\Box h + m^2(4\alpha - 1)h = 16\pi T.$$

If $\alpha = 1$ one gets the fifth constraint

$$h = \frac{16\pi T}{3m^2}$$

$\Rightarrow$ 5 Dof= polarizations of massive graviton. For $\alpha \neq 1$ there is a sixth Dof with a negative kinetic energy.

VdVZ discontinuity /1970/

$$
\begin{aligned}
(\Box - m^2) h_{\mu\nu} &= 16\pi G T_{\mu\nu} \\
\partial^\mu h_{\mu\nu} &= \partial_\nu h \\
h &= \frac{16\pi G T}{3m^2}
\end{aligned}
$$

The limit $m \to 0$ is not smooth. Let us make $5 = 2 + 2 + 1$ split

$$
h_{\mu\nu} = \gamma_{\mu\nu} + \frac{1}{m}(\partial_\mu A_\nu + \partial_\nu A_\mu) + \frac{1}{m^2}\partial_\mu \partial_\nu \phi
$$

where $\partial^\mu \gamma_{\mu\nu} = \gamma^\mu_\mu = \partial^\mu A_\mu = 0$. Then for $m \to 0$ one gets

$$
\begin{aligned}
\Box \gamma_{\mu\nu} &= 16\pi G T_{\mu\nu} && \text{tensor modes} \\
\Box A_\mu &= 0 && \text{vector modes} \\
\Box \phi &= \frac{16\pi G}{3} T && \text{scalar graviton}
\end{aligned}
$$

$\Rightarrow$ extra attraction, different Newton's laws for massive bodies.

Explicit VdVZ

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} R'^2(r) dr^2 - r^2 e^{\mu(r)} d\Omega^2$$

where $R = re^{\mu}$ and $\nu, \lambda, \mu \ll 1$. Linearizing,

$$\nu = -\frac{r_g}{r} e^{-mr}, \qquad \lambda = \frac{r_g}{2r} (1 + mr) e^{-mr}$$

$$\mu = r_g \frac{1 + mr + (mr)^2}{2m^2 r^3} e^{-mr}$$

For $mr \ll 1$ one has

$$\nu = -\frac{r_g}{r}, \qquad \lambda = \frac{r_g}{2r}, \qquad \mu \sim \frac{1}{r^3}$$

One has $\nu = -2\lambda$ instead of $\nu = -\lambda \Rightarrow$ either the Newton law is wrong or the bending of light is wrong, depending on choice of r_g.

Is massive gravity ruled out by the Solar System observations?

Vainshtein /1972/

Non-linear corrections to the VdVZ solution are proportional to

$$\frac{r_g}{m^4 r^5}$$

This should be small as compared to unity, but it is $\sim 10^{32}$ at the edge of solar system. Becomes small only for

$$r \gg r_V = \left(\frac{r_g}{m^4}\right)^{1/5} \sim 100\ Kps$$

For $r \ll r_V$ one cannot use the linear theory $\Rightarrow$ linearized equations for ν, λ but non-linear for μ

$$\nu = -\frac{r_g}{r} + \ldots, \quad \lambda = \frac{r_g}{r} + \ldots, \quad \mu = \sqrt{\frac{a r_g}{r}} + \ldots$$

The GR is recovered, the scalar graviton is bound.

The VdVZ discontinuity is visible only for $r \gg r_V$,
for $r \ll r_V$ it is cured by the non-linear effects.

Hamiltonian formulation, Boulware-Deser ghost

Non-linear Fierz-Pauli

$$S = \frac{1}{\kappa^2} \int \sqrt{-g} \left(\frac{1}{2} R - m^2 \mathcal{U} \right) d^4x \equiv \frac{1}{\kappa^2} \int \mathcal{L} \, d^4x \, .$$

The potential $\mathcal{U}$ is a scalar function of $H^{\mu}_{\nu} = \delta^{\mu}_{\nu} - g^{\mu\alpha} f_{\alpha\nu}$ where $f_{\mu\nu}$ is a flat reference metric. The potential should reproduce the Fierz-Pauli in the weak field, therefore

$$\mathcal{U} = \frac{1}{8} \left(H^{\mu}{}_{\nu} H^{\nu}{}_{\mu} - (H^{\mu}{}_{\mu})^2 \right) + \ldots$$

Higher order term can be arbitrary.

ADM decomposition

With

$$ds^2 = -N^2 dt^2 + \gamma_{ik}(dx^i + N^i dt)(dx^k + N^k dt)$$

the Lagrangian becomes

$$\mathcal{L} = \sqrt{\gamma} N \left(\frac{1}{2} \{ K_{ik} K^{ik} - K^2 + R^{(3)} \} - m^2 \mathcal{U}(N^\nu, \gamma_{ik}) \right) + \text{total derivative} \, ,$$

where the second fundamental form

$$K_{ik} = \frac{1}{2N} (\dot{\gamma}_{ik} - \nabla_i^{(3)} N_k - \nabla_k^{(3)} N_i).$$

Hamiltonian

Canonical momenta for γ_{ik} and $N^\mu = (N, N^k)$

$$\pi^{ik} = \frac{\partial \mathcal{L}}{\partial \dot{\gamma}_{ik}} = \frac{1}{2}\sqrt{\gamma}(K^{ik} - K\gamma^{ik}), \qquad p_{N_\mu} = \frac{\partial \mathcal{L}}{\partial \dot{N}^\mu} = 0.$$

Hamiltonian

$$\mathcal{H} = \pi^{ik}\dot{\gamma}_{ik} - \mathcal{L} = N^\mu \mathcal{H}_\mu + m^2 \mathcal{V}$$

with $\mathcal{V} = \sqrt{\gamma} N \mathcal{U}$ and

$$\mathcal{H}_0 = \frac{1}{\sqrt{\gamma}}(2\pi^{ik}\pi_{ik} - (\pi^k_k)^2) - \frac{1}{2}\sqrt{\gamma}R^{(3)}, \quad \mathcal{H}_k = -2\nabla^{(3)}_i \pi^i_k$$

N^μ are non-dynamical, phase space is spanned by 12 (π^{ik}, h_{ik}).

$$0 = -\dot{p}_{N_\mu} = \frac{\partial \mathcal{H}}{\partial N^\mu} = \mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) + m^2 \frac{\partial \mathcal{V}(N^\alpha, \gamma_{ik})}{\partial N^\mu},$$

This condition determines the number of Dof.

GR, $m = 0$

$$\frac{\partial \mathcal{H}}{\partial N^\mu} = \mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) = 0. \qquad \text{4 constraints}$$

Since

$$\{\mathcal{H}_\mu, \mathcal{H}_\nu\} \sim \mathcal{H}_\alpha$$

they are first class and generate gauge symmetries $\Rightarrow$ one can impose 4 gauge conditions. There remain

$$12 - 4 - 4 = 4 = 2 \times (2 \text{ DoF})$$

independent phase space variables describing 2 graviton polarizations.

Energy is zero on the constraint surface (up to surface terms)

$$H = N^\mu \mathcal{H}_\mu = 0$$

Boulware-Deser ghost /1972/

$$\frac{\partial \mathcal{H}}{\partial N^\mu} = \mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) + m^2 \frac{\partial \mathcal{V}(N^\alpha, \gamma_{ik})}{\partial N^\mu} = 0$$

These are not constraints but equations for N^μ whose solution is $N^\mu(\pi^{ik}, h_{ik})$. No constraints $\Rightarrow$ all 12 phase space variables are independent $\Rightarrow 6 = 5 + 1$ DoF $=$ 5 graviton polarizations $+$ ghost

Inserting $N^\mu(\pi^{ik}, h_{ik})$ to $\mathcal{H}$ gives a non-positive-definite in π^{ik} expression. No constraints $\Rightarrow$ energy is unbounded from below.

In the non-linear theory one can overcome the VdVZ problem, but one finds the BD ghost

Ghost-free massive gravity

dRGT theory /de Rham, Gabadadze, Tolley 2010/

$$S = \frac{1}{\kappa^2} \int \sqrt{-g} \left(\frac{1}{2} R - m^2 \mathcal{U} \right) d^4x \equiv \frac{1}{\kappa^2} \int \mathcal{L} \, d^4x \,,$$

with the potential made of $H^\mu_\nu = \delta^\mu_\nu - g^{\mu\alpha} f_{\alpha\nu}$

$$\mathcal{U} = \frac{1}{8} (H^\mu{}_\nu H^\nu{}_\mu - (H^\mu{}_\mu)^2) + \dots$$

For the dRGT theory the higher order terms are chosen such that

$$\mathcal{U} = b_0 + b_1 \sum_a \lambda_a + b_2 \sum_{a<b} \lambda_a \lambda_b + b_3 \sum_{a<b<c} \lambda_a \lambda_b \lambda_c + b_4 \, \lambda_0 \lambda_1 \lambda_2 \lambda_3$$

where λ_a are eigenvalues of

$$\gamma^\mu{}_\nu = \sqrt{g^{\mu\alpha} f_{\alpha\nu}}$$

and the parameters

$$\begin{aligned} b_0 &= 4c_3 + c_4 - 6, \quad b_1 = 3 - 3c_3 - c_4, \quad b_2 = 2c_3 + c_4 - 1, \\ b_3 &= -(c_3 + c_4), \quad b_4 = c_4 \end{aligned}$$

Degrees of freedom

The Hessian matrix

$$\frac{\partial^2 \mathcal{V}(N^\alpha, \gamma_{ik})}{\partial N^\mu \partial N^\nu}$$

has rank 3 $\Rightarrow$ equations

$$\frac{\partial \mathcal{H}}{\partial N^\mu} = \mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) + m^2 \frac{\partial \mathcal{V}(N^\alpha, \gamma_{ik})}{\partial N^\mu} = 0$$

determine the shifts $\boxed{N^k = N^k(N, \pi^{ik}, \gamma_{ik})}$ but the lapse N remains undetermined. Inserting N^k to $\mathcal{H}$ gives

$$\mathcal{H} = \mathcal{E}(\pi^{ik}, \gamma_{ik}) + N\mathcal{C}(\pi^{ik}, \gamma_{ik})$$

Varying with respect to N gives the primary constraint $\mathcal{C} = 0 \Rightarrow$ the secondary constraint $\mathcal{S} = \{\mathcal{C}, H\} = 0 \Rightarrow$ only 5 DoF. The energy density is $\mathcal{E}(\pi^{ik}, \gamma_{ik})$ restricted to the constraint surface. Difficult to calculate $\mathcal{E}, \mathcal{C}$.

Energy in the s-sector

Spherical symmetry

$$
\begin{aligned}
ds_g^2 &= -N^2 dt^2 + \frac{1}{\Delta^2}(dr + \beta\, dt)^2 + R^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) \\
ds_f^2 &= -dt^2 + dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)
\end{aligned}
$$

where N, β, Δ, R depend on t, r. With the canonical momenta $p_\Delta = \partial\mathcal{L}/\partial\dot{\Delta}$ and $p_R = \partial\mathcal{L}/\partial\dot{R}$ the Hamiltonian

$$
\mathcal{H} = \dot{\Delta}\pi_\Delta + \dot{R}\pi_R - \mathcal{L} = N\mathcal{H}_0 + \beta\mathcal{H}_r + m^2\mathcal{V}
$$

where

$$
\begin{aligned}
\mathcal{H}_0 &= \frac{\Delta^3}{4R^2}\, p_\Delta^2 + \frac{\Delta^2}{2R}\, p_\Delta p_R + \Delta R R'^2 + 2R(\Delta R')' - \frac{1}{\Delta}\,, \\
\mathcal{H}_r &= \Delta p_\Delta' + 2\Delta' p_\Delta + R' p_R
\end{aligned}
$$

Phase space is spanned by 4 variables $\Delta, R, p_\Delta, p_R \equiv (q^i, p_k)$.

Generic case

- $m = 0 \Rightarrow$ varying $\mathcal{H}$ with respect to N, β yields 2 first class constraints $\mathcal{H}_0 = 0$, $\mathcal{H}_r = 0 \Rightarrow$ there are $4 - 2 - 2 = 0$ independent variables $\Rightarrow$ no dynamics $=$ Birkhoff theorem.
- $m \neq 0$ and generic $\mathcal{V} \Rightarrow$

$$\mathcal{H}_0 + m^2 \frac{\partial \mathcal{V}}{\partial N} = 0, \qquad \mathcal{H}_r + m^2 \frac{\partial \mathcal{V}}{\partial \beta} = 0,$$

$\Rightarrow N = N(q^i, p_k)$, $\beta = \beta(q^i, p_k)$, no constraints $\Rightarrow$ all 4 phase space variables are independent $\Rightarrow$

2 DoF=scalar graviton+ghost

Inserting $N(q^i, p_k)$, $\beta(q^i, p_k)$ to $\mathcal{H}$, the result is unbounded from below

dRGT theory

$$\mathcal{V} = \frac{NR^2 P_0}{\Delta} + \frac{R^2 P_1}{\Delta} \sqrt{(\Delta N + 1)^2 - \beta^2} + R^2 P_2 \,,$$

with

$$P_n = b_n + 2b_{n+1} \frac{r}{R} + b_{n+2} \frac{r^2}{R^2}.$$

in which case

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial N} &= \mathcal{H}_0 + m^2 \frac{R^2 P_0}{\Delta} + m^2 R^2 P_1 \frac{N\Delta + 1}{\sqrt{(N\Delta + 1)^2 - \beta^2}} = 0 \\
\frac{\partial \mathcal{H}}{\partial \beta} &= \mathcal{H}_r - m^2 \frac{R^2 P_1}{\Delta} \frac{\beta}{\sqrt{(N\Delta + 1)^2 - \beta^2}} = 0.
\end{aligned}$$

The second of these conditions determines β,

$$\beta = (N\Delta + 1) \frac{\Delta \mathcal{H}_r}{Y}$$

while the first condition gives

Constraints

$$\mathcal{C} \equiv \mathcal{H}_0 + Y + m^2 \frac{R^2 P_0}{\Delta} = 0$$

The Hamiltonian becomes $\mathcal{H} = \mathcal{E} + N\mathcal{C}$ where

$$\mathcal{E} = \frac{Y}{\Delta} + m^2 R^2 P_2 \quad \text{with} \quad Y \equiv \sqrt{(\Delta \mathcal{H}_r)^2 + (m^2 R^2 P_1)^2}$$

Since $\{\mathcal{C}(r_1), \mathcal{C}(r_2)\} = 0 \Rightarrow$ the secondary constraint

$$\begin{aligned} \mathcal{S} &= \{\mathcal{C}, H\} = \frac{m^4 R^2 P_1^2}{2Y} (\Delta p_\Delta + R p_R) - Y \left(\frac{\Delta \mathcal{H}_r}{Y} \right)' \\ &- \frac{\Delta^2 p_\Delta}{2R} \left\{ \frac{m^4}{2\Delta Y} \partial_R (R^4 P_1^2) + m^2 \partial_R (R^2 P_2) \right\} \\ &- \frac{m^2 \mathcal{H}_r}{Y} \left\{ \Delta \left(R^2 P_2 \right)' + R^2 \partial_r (P_0 - \Delta^2 P_2) \right\} = 0 \end{aligned}$$

No ghost. $E = \int_0^\infty \mathcal{E} dr$ *assuming that* $\mathcal{C} = \mathcal{S} = 0$.

Potential energy sector

$$\text{Let}\quad p_\Delta = p_R = 0, \qquad \Delta = \frac{g}{h}, \quad R = rh, \qquad \Rightarrow \qquad \mathcal{S} = 0,$$

$$\begin{aligned} \mathcal{C} &= -h'' - \frac{2}{x}h' + \frac{h'^2}{2h} - \frac{(xh)'g'}{xg} + \frac{h(1-g^2)}{2x^2g^2} \\ &+ \frac{h(2-3h)}{2g} + \frac{h(1-6h+6h^2)}{2g^2} = 0, \end{aligned}$$

with $h_0 \leftarrow h \rightarrow 1$, $1 \leftarrow g \rightarrow 1$ for $0 \leftarrow x \rightarrow \infty$; the energy

$$\mathcal{E} = \frac{x^2h^2(3h-g-2)}{g}.$$

Special solutions, also fulfill the Hamilton equations

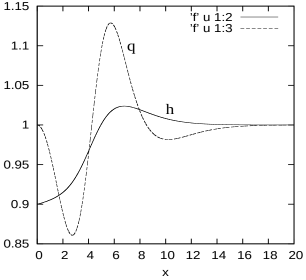
- $h = 1,\ g = 1,\ ds_g^2 = ds_f^2, \quad \mathcal{E} = 0$ flat space
- $h = \frac{1}{2},\ g = 1,\ ds_g^2 = \frac{1}{4}\,ds_f^2, \quad \mathcal{E} = -3x^2/8$ tachyon universe

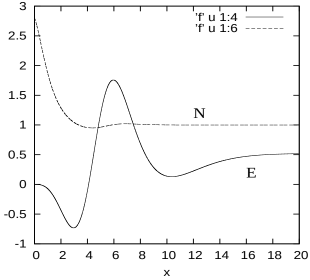
Solving the constraint

Setting $g = qh/(xh)'$, the constraint is solved with

$$
\begin{aligned}
Q &= xh(1 - q^2) + x^3 h(2h - 1)(h - 1), \\
Q' &= r^2 h(3h - 2)(q - 1)
\end{aligned}
$$

for any $Q(x)$. Let $Q = A\Theta(x - x_0)(x - x_0)^p e^{-x}$





Energy is positive for smooth, asymptotically flat fields.

Tachyon branch

Solutions of the constraint with

$$h_0 \leftarrow h \rightarrow \frac{1}{2}$$

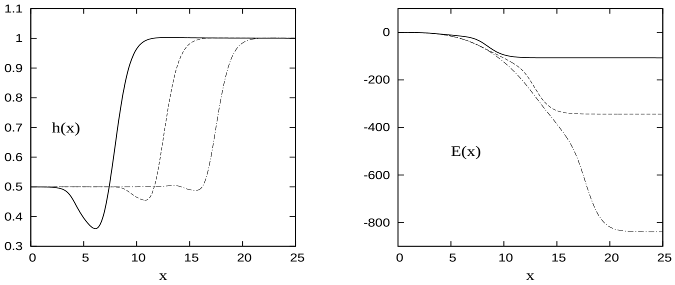
The energy is negative and infinite. However, this does not affect stability of flat space, because the asymptotic condition at infinity is different:

$h_0 \leftarrow h \rightarrow 1$ flat space branch
$h_0 \leftarrow h \rightarrow \frac{1}{2}$ tachyon branch

Negative energies comprise a disjoint branch and so they are harmless.

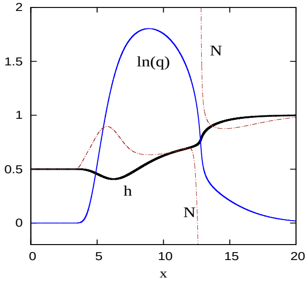
Tachyon branch vs. normal branch

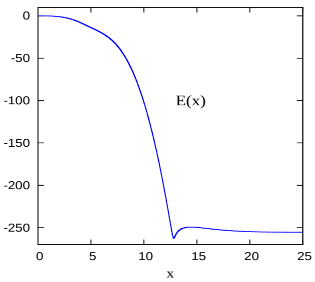
There are solutions which start from the tachyon branch at the origin and approach the flat space at infinity.
The energy is finite and negative – tachyon bubbles.



Does this affect the stability of flat space ?

Tachyon bubbles





The lapse N is singular. Can be proven for $c_3 = c_4 = 0$. For other values of the parameters – numerical evidence. One does not find negative energy solutions which would describe initial data for a decay of flat space $\Rightarrow$ negative energy decouple and are harmless.

Summary

- Two constraints of the dRGT massive gravity remove one of the 2 DoF in s-sector $\Rightarrow$ only 1 DoF propagates.
- It is natural to think that the removed mode is the ghost. Then the energy should be positive, but in fact it is unbounded from below.
- However, for smooth deformations of flat space the energy is positive – the physical sector. It seems that negative energy states belong to disjoint sectors, so they are harmless.

$\Rightarrow$ The evidence that the theory is healthy in its physical sector, where the energy is positive and the ghost is suppressed.

Cosmologies

Massive gravity FLRW

De Sitter:

$$ds_g^2 = \alpha^2 \left(-dt^2 + dr^2 + dX_1^2 + dX_2^2 + dX_3^2\right)$$

with $\alpha = \sqrt{3/\Lambda}$, $\Lambda = m^2 f(b_k)$,

$$-t^2 + r^2 + X_1^2 + X_2^2 + X_3^2 = 1$$

while

$$ds_f^2 = \alpha^2 u^2 \left(-dT^2 + dX_1^2 + dX_2^2 + dX_3^2\right)$$

where u is a constant, $T(t, r)$ fulfills

$$\dot{T}^2 - T'^2 = 1$$

$\Rightarrow$ infinitely many solutions. Only the $T = t$ solution has been studied. The g-metric can be generalized to the FLRW for a matter+Λ

The ghost-free bigravity

$$
\begin{aligned}
S &= \frac{1}{2(\kappa\cos\eta)^2}\int R\sqrt{-g}\,d^4x + \frac{1}{2(\kappa\sin\eta)^2}\int \mathcal{R}\sqrt{-f}\,d^4x \\
&- \frac{m^2}{\kappa^2}\int \mathcal{U}\sqrt{-g}\,d^4x + S_{\rm m}[g,\text{g-matter}] + \mathcal{S}_{\rm m}[f,\text{f-matter}]\,;
\end{aligned}
$$

two gravitons, one massless and one massive. Equations

$$
\begin{aligned}
G^{\rho}_{\lambda} &= m^2\cos^2\eta\, T^{\rho}_{\lambda} + \kappa^2\, T^{[\rm m]\,\rho}_{\ \ \ \ \lambda}\,, \\
\mathcal{G}^{\rho}_{\lambda} &= m^2\sin^2\eta\, \mathcal{T}^{\rho}_{\lambda} + \kappa^2\mathcal{T}^{[\rm m]\,\rho}_{\ \ \ \ \lambda}\,,
\end{aligned}
$$

Massive gravity for $\eta \to 0$ if $f_{\mu\nu}$ becomes flat.

Bigravity FLRW

$$ds_g^2 = -dt^2 + \mathbf{a}^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right),$$
$$\dot{\mathbf{a}}^2 - \frac{\mathbf{a}^2}{3}(\Lambda + \rho) = -k,$$

where $\Lambda = m^2 f(b_k)$, and

$$ds_f^2 = -\Delta(U)\, dT^2 + \frac{dU^2}{\Delta(U)} + U^2 d\Omega^2\,. \qquad (AdS)$$

where $U(t,r) = u\,\mathbf{a}(t)\,r$ and $\Delta(U) = 1 - \frac{\Lambda_f}{3}\,U^2$ and while $T(t,r)$ fulfills a PDE

$$\mathbf{a}\sqrt{1-kr^2}\,(\dot{U}T' - \dot{T}U') - u^2\mathbf{a}^2 + u\mathbf{a}\sqrt{\frac{A_+A_-}{\Delta}} = 0 \quad (\dagger)$$

with $A_\pm = \mathbf{a}\,(\Delta\,\dot{T} \pm \dot{U}) + \sqrt{1-kr^2}(U' \pm \Delta\,T')$

More complex FLRW

$$ds_g^2 = -dt^2 + e^{2\Omega}\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right), \qquad k = 0, \pm 1$$
$$ds_f^2 = -\mathcal{A}^2 dt^2 + e^{2\mathcal{W}}\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right).$$

With

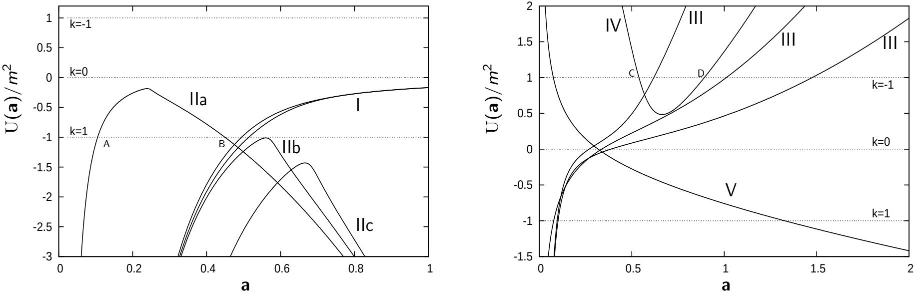
$$\left[\left(e^{\mathcal{W}}\right)^{\cdot} - \mathcal{A}\left(e^{\Omega}\right)^{\cdot}\right] = 0$$

the equations reduce to a Friedman equation

$$\dot{\mathbf{a}}^2 + \mathrm{U}(\mathbf{a}) = -k$$

where $\mathbf{a} = 2e^{\Omega}$ and $\mathrm{U}(\mathbf{a})$ is determined by roots of an algebraic equations. There are several roots $\Rightarrow$ several types of $\mathrm{U}(\mathbf{a})$, and several different type of solutions.

Effective potential – physical and exotic cosmologies



- physical: $\rho \gg m^2 T_0^0$ for small **a**, $\rho \ll m^2 T_0^0$ for large **a**
- exotic: $\rho \ll m^2 T_0^0$ for any **a**.

Bianchi class A types

$$
\begin{aligned}
ds_g^2 &= -\alpha(t)^2 dt^2 + h_{ab}(t)\, \omega^a \otimes \omega^b, \\
ds_f^2 &= -\mathcal{A}^2(t) dt^2 + \mathcal{H}_{ab}(t)\, \omega^a \otimes \omega^b \,.
\end{aligned}
$$

$$
[e_a, e_b] = C^c{}_{ab} e_c, \quad C^c{}_{ab} = n^{cd} \epsilon_{dab}, \quad n^{ab} = \mathrm{diag}[n^{(1)}, n^{(2)}, n^{(3)}]
$$

	I	II	VI_0	VII_0	VIII	IX
$n^{(1)}$	0	1	1	1	1	1
$n^{(2)}$	0	0	-1	1	1	1
$n^{(3)}$	0	0	0	0	-1	1

If $h_{ab}, \mathcal{H}_{ab}$ are diagonal $\Rightarrow$ $G_r^0 = \mathcal{G}_r^0 = 0 \Rightarrow$ no radial fluxes.

$$
h_{ab} = \mathrm{diag}[\alpha_1^2, \alpha_2^2, \alpha_3^2], \quad \mathcal{H}_{ab} = \mathrm{diag}[\mathcal{A}_1^2, \mathcal{A}_2^2, \mathcal{A}_3^2].
$$

Kei-ichi Maeda, M.S.V. 2013

Parametrization

$$ds_g^2 = -\alpha^2 dt^2 + dl_g^2, \qquad ds_f^2 = -\mathcal{A} dt^2 + dl_f^2$$

$$\begin{aligned} dl_g^2 &= e^{2\Omega}\left(e^{2\beta_+ + 2\sqrt{3}\beta_-}(\omega^1)^2 + e^{2\beta_+ - 2\sqrt{3}\beta_-}(\omega^2)^2 + e^{-4\beta_+}(\omega^3)^2\right) \\ dl_f^2 &= e^{2\mathcal{W}}\left(e^{2\mathcal{B}_+ + 2\sqrt{3}\mathcal{B}_-}(\omega^1)^2 + e^{2\mathcal{B}_+ - 2\sqrt{3}\mathcal{B}_-}(\omega^2)^2 + e^{-4\mathcal{B}_+}(\omega^3)^2\right) \end{aligned}$$

In the Bianchi I case $\omega^a = dx^a$.

Anisotropic attractor

At the initial moment $t = t_0$ the universe is chosen to be an anisotropic deformation of a finite size FLRW. The f-sector is empty, $\rho_f = 0$; the g-sector contains radiation $+$ dust,

$$\rho_g = 0.25 \times e^{-4\Omega} + 0.25 \times e^{-3\Omega}$$

For all Bianchi types, the solutions rapidly approach a state with a constant expansion rate and constant and non-zero anisotropies $=$ late time attractor.

Late time anisotropies

At late times anisotropy parameters oscillate around constant values

$$\begin{aligned} \beta_{\pm}(t) &\rightarrow \beta_{\pm}(\infty) + \text{const.} \times e^{-3Ht/2} \cos(H\omega t) \\ \mathcal{B}_{\pm}(t) &\rightarrow \beta_{\pm}(\infty) + \text{const.} \times e^{-3Ht/2} \cos(H\omega t) \end{aligned}$$

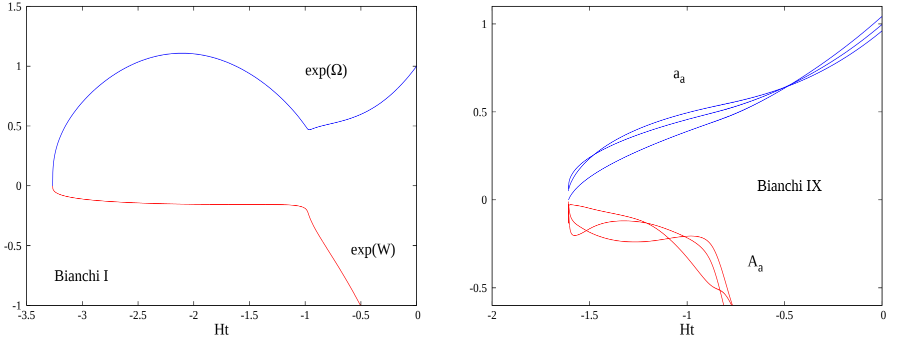
The shear energy in bigravity

$$\dot{\beta}_+^2 + \dot{\beta}_-^2 \sim e^{-3\Omega} \sim 1/\mathbf{a}^3$$

falls off as the energy of a non-relativistic (dark ?) matter.
In GR one has

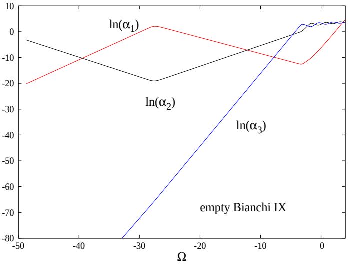
$$\dot{\beta}_+^2 + \dot{\beta}_-^2 \sim e^{-6\Omega} \sim 1/\mathbf{a}^6$$

Near singularity behaviour



When continued to the past, the solutions show a singularity where both e^{Ω} and $e^{\mathcal{W}}$ vanish. For Bianchi IX anisotropies start fluctuating near singularity.

Bainchi IX – chaos



Near singularity – a sequence of Kasner-like periods with

$$\alpha_a \propto t^{p_a} \quad \text{with} \quad p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1.$$

Matter cannot change this, as ρ grows slower than shears,

$$1/\mathbf{a}^6 \leftarrow \quad \text{shear energy} = \dot{\beta}_+^2 + \dot{\beta}_-^2 \quad \rightarrow 1/\mathbf{a}^3$$

Black holes

Schwarzschild-de Sitter

$$
\begin{aligned}
ds_g^2 &= -D(r)dt^2 + \frac{dr^2}{D(r)} + r^2 d\Omega^2, \\
ds_f^2 &= -\Delta(U)\, dT^2 + \frac{dU^2}{\Delta(U)} + U^2 d\Omega^2 .
\end{aligned}
$$

where $U = ur$ and $T = T(t, r)$,

$$
D(r) = 1 - \frac{2M}{r} - \frac{\Lambda_g}{3} U^2, \qquad \Delta(U) = 1 - \frac{\Lambda_f}{3} U^2
$$

$$
\boxed{\frac{\Delta}{D} \dot{T}^2 + \frac{D\Delta}{D - \Delta} T'^2 = 1}
$$

A solution

$$
T = t + \int \frac{dr}{D} - \int \frac{dr}{\Delta}
$$

If $\eta \to 0$ then $\Lambda_f \sim \sin\eta \to 0$, f-metric becomes flat, one obtains all known massive gravity black holes.

Hairy black holes

$$
\begin{aligned}
ds_g^2 &= Q^2 dt^2 - \frac{dr^2}{N^2} - r^2 d\Omega^2, \\
ds_f^2 &= A^2 dt^2 - \frac{U'^2}{Y^2} dr^2 - U^2 d\Omega^2
\end{aligned}
$$

Q, N, Y, U, A are 5 functions of r, they fulfill 5 equations

$$
\begin{aligned}
G_0^0 &= m^2 \cos^2\eta \, T_0^0, \\
G_r^r &= m^2 \cos^2\eta \, T_r^r, \\
\mathcal{G}_0^0 &= m^2 \sin^2\eta \, \mathcal{T}_0^0, \\
\mathcal{G}_r^r &= m^2 \sin^2\eta \, \mathcal{T}_r^r, \\
T_r^{r\prime} &+ \frac{Q'}{Q}(T_r^r - T_0^0) + \frac{2}{r}(T_\vartheta^\vartheta - T_r^r) = 0.
\end{aligned}
$$

/M.S.V. 2012/

Simplest solutions

$$f_{\mu\nu} = C^2 g_{\mu\nu}$$

$\Rightarrow$

$$C^4 + A_3 C^3 + A_2 C^2 + A_1 C + A_0 = 0$$

and

$$G^\mu_\nu + \Lambda_g(C) = 0$$

with

$$\Rightarrow \quad C = \{C_k\}, \Lambda_g = \Lambda_g(C)$$

$\Rightarrow$ Schwarzschild, Schwarzschild-dS, Schwarzschild-AdS

One wants to study deformations of these solutions by changing their boundary conditions at the horizon.

Event horizon at $r = r_h$

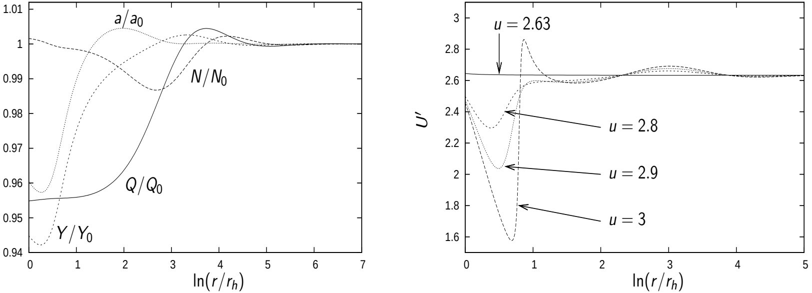
$$N^2 = \sum_{n\geq 1} a_n (r-r_h)^n, \quad Y^2 = \sum_{n\geq 1} b_n (r-r_h)^n, \quad U = u r_h + \sum_{n\geq 1} c_n (r-r_h)^n,$$

all coefficients depend on one free parameter u.

- Horizon is common for both metrics
- Set of all black holes is one-dimensional and labeled by $u = U(r_h)/r_h$ = ratio of the even horizon radius measured by $f_{\mu\nu}$ to that measured by $g_{\mu\nu}$.
- Horizon temperatures and surface gravities are the same. ($T = \kappa/2\pi$),

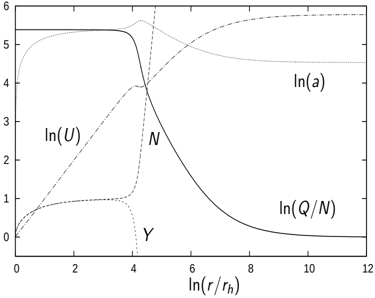
Hairy deformations of Schwarzschild-AdS

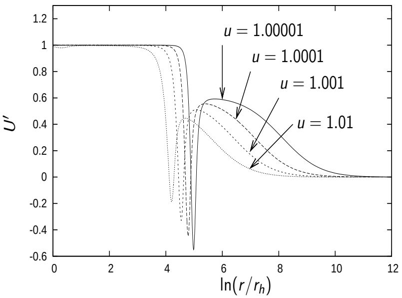
Deformations (hair) stay close to the horizon



N_0, Q_0, Y_0, a_0 correspond to the background AdS.

Deforming Schwarzschild





- Deformations are small close to the horizon but then grow and change the asymptotic behavior at $r \to \infty$.
- The only asymptotically flat is pure Schwarzschild

Stability of Schwarzschild

Let us perturb $g^{(0)}_{\mu\nu} = f^{(0)}_{\mu\nu} =$ Schwarzschild;

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}, \qquad f_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta f_{\mu\nu}$$

The linear combinations

$$h_{\mu\nu} = \cos\eta\,\delta g_{\mu\nu} + \sin\eta\,\delta f_{\mu\nu}, \quad h^{(0)}_{\mu\nu} = \cos\eta\,\delta f_{\mu\nu} - \sin\eta\,\delta g_{\mu\nu}$$

can be identified with the massive and massless gravitons. One has

$$\overset{(0)}{\Box}\, h_{\mu\nu} + 2\,\overset{(0)}{R}_{\mu\alpha\nu\beta}\, h^{\alpha\beta} = m^2 h_{\mu\nu}$$

which coincides with the equation for black string perturbation $\Rightarrow$ Gregory-Laflamme-type instability for

$$mr_h = \frac{\text{black hole radius}}{\text{graviton's Compton length}} < 0.86$$

$\Rightarrow$ Schwarzschild is unstable

Summary of black holes

- Solutions with non-bidiagonal metrics describe Schwarschild-de Sitter black holes. Admit the massive gravity limit with flat f-metric when $\eta \to 0$. The only black holes in massive gravity.

- Solutions with bidiagonal metrics describe hairy black holes in bigravity. None of them is asymptotically flat, apart from the pure Schwarzschild.

- Schwarzschild solution is (mildly) unstable