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Integral Bounds for Simple Partial Fractions

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Abstract—For $p \geq 2$ we obtain bounds for L_p-norms of the Fourier transform of real parts of simple partial fractions. For even p our estimate is sharp. We also prove a new inequality for L_p-norms of simple partial fractions which in some cases is stronger than the corresponding inequality obtained by V. Yu. Protasov.

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1. INTRODUCTION

A simple partial fraction is defined as

$$g_n(t) = \sum_{k=1}^{n} \frac{1}{t - z_k}.$$

Simple partial fractions pose an advantage over polynomials. For instance, they can be used for approximating holomorphic functions in unbounded domains. Many useful properties of simple partial fractions were obtained by V. Yu. Protasov [1], V. I. Danchenko [2], and P. A. Borodin [3]. The mentioned papers also contain references to other works in this direction.

In this paper we assume that $z_k = x_k + iy_k \in \mathbb{C} \setminus \mathbb{R}$, $1 \leq k \leq n$. We set

$$f(t) = \operatorname{Re} g_n(t), \quad \|f\|_p = \left(\int_{\mathbb{R}} |f(t)|^p dt \right)^{1/p},$$

$$\gamma_p = \left(\int_{\mathbb{R}} \frac{1}{(t^2+1)^p} dt \right)^{1/p} = \left(\sqrt{\pi} \frac{\Gamma(p-1/2)}{\Gamma(p)} \right)^{1/p},$$

where $\Gamma(p)$ is the Euler gamma function.

V. Yu. Protasov [1] has obtained the following interesting and important property: In the case $p > 1$ the L_p-norm of the real part of a simple partial fraction is bounded below by some strictly positive constant independent of $x_1, x_2, \ldots, x_n$. Moreover, he has proved the next theorem.

Theorem A. *The following inequality holds*:

$$\|f\|_p \geq \gamma_p \|F\|_p^{-1} \left(\sum_{k=1}^{n} \frac{1}{|y_k|^{p-1}} \right)^{1/p}, \tag{1}$$

it is sharp (for fixed p) in the sense that for each $\varepsilon > 0$ and any $y_1, \ldots, y_n$ there exist x_k such that

$$\|f\|_p \leq (1+\varepsilon) \gamma_p \|F\|_p \left(\sum_{k=1}^{n} \frac{1}{|y_k|^{p-1}} \right)^{1/p} \tag{2}$$

where $\|F\|_p$ is the norm of the Hilbert transform in $L_p(\mathbb{R})$; it equals $\tan(\pi/2p)$ for $p \in (1, 2]$ and $\cot(\pi/2p)$ for $p > 2$.

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