ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2012, Vol. 56, No. 6, pp. 20–29. © Allerton Press, Inc., 2012.
Original Russian Text © V.V. Karachik, 2012, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2012, No. 6, pp. 24–35.

Application of the Almansi Formula for Constructing Polynomial Solutions to the Dirichlet Problem for a Second-Order Equation

V. V. Karachik[1*]

[1] *Southern Ural State University, pr. Lenina 76, Chelyabinsk, 454080 Russia*

Received June 14, 2011

Abstract—We obtain the Almansi decomposition for the second-order partial differential operators with constant coefficients. This decomposition is used for constructing a polynomial solution to the Dirichlet problem.

DOI: 10.3103/S1066369X12060035

Keywords and phrases: *Almansi decomposition, polynomial solutions, Dirichlet problem.*

1. INTRODUCTION

The following remarkable assertion stated by E. Almansi [1] is well known: If $f(x)$ is a semiharmonic function of order m in a star domain Ω centered at the origin of coordinates, then there exist unique functions $f_1(x), \dots, f_m(x)$ harmonic in Ω such that $f(x) = f_0(x) + |x|^2 f_1(x) + \cdots + |x|^{2(m-1)} f_{m-1}(x)$. In the paper [2] one proves the existence of infinite Almansi decompositions. M. Nicolescu [3] obtained the Almansi decomposition for a class of operators of two variables including the heat-conduction operator. In the paper [4] we obtained Almansi decompositions for differential operators in the form $\Delta_\lambda = \lambda_1 \frac{\partial^2}{\partial x_1^2} + \cdots + \lambda_n \frac{\partial^2}{\partial x_n^2}$, $\lambda_k \in \mathbb{R} \setminus \{0\}$, in a bounded star domain $\Omega \subset \mathbb{R}^n$. In this paper we prove the existence of Almansi-type decompositions $f(x) = \sum\limits_{k=1}^{\infty} |x|_A^{2k} u_k(x)$, $x \in \Omega \subset \mathbb{R}^n$, in a star domain Ω for any solution of the equation $L^m(D)f(x) = 0$ with finite m, where $L(D) = (A\nabla, \nabla)$ is a second-order operator, A is an $n \times n$ nondegenerate symmetric matrix, $|x|_A^2 = (A^{-1}x, x)$, and $u_k(x)$ are some classical solutions to the equation $L(D)u(x) = 0$ in the domain Ω uniquely defined with the help of a given function $f(x)$. The obtained decomposition is applied for constructing polynomial solutions to the Dirichlet problem for the equation $L(D)u(x) = P(x)$.

2. CONSTRUCTION OF A NORMED SYSTEM OF FUNCTIONS

Let us use the method developed in [4] for studying Almansi-type decompositions. This method is based on the notion of an f-normed system of functions $\{f_k(x) : k \in \mathbb{N}\}$ with respect to a linear differential operator L in a domain Ω. This means that the system of functions $\{f_k(x) : k \in \mathbb{N}\}$ in the domain Ω satisfies the conditions

$$L f_1(x) = f(x); \quad L f_k(x) = f_{k-1}(x), \quad k \ge 2. \qquad (*)$$

In what follows we use systems of functions with property $(*)$. One has constructed normed systems of functions for the Laplace operator in star domains in [5], and for the wave operator and the Dunkl one in [6].

Let $u \in C^2(\Omega)$ be a classical solution to the second-order differential equation with constant coefficients

$$L(D)u(x) \equiv \sum_{i,j=1}^{n} a_{j\,i} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0, \quad x \in \Omega \subset \mathbb{R}^n, \qquad (1)$$

[*] E-mail: karachik@susu.ru.