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PETROV CATEGORY OF NONCOMMUTATIVE EINSTEIN SPACES

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Abstract

We give a definition of Petrov category **Petrov-NC-Einst** of noncommutative Einstein spaces *NC-Einst*, which is used for construction of noncommutative topological quantum field theory (NC TQFT). We suggest extensions of these ideas, which may be useful for further development of NC TQFT, and apply them to higher dimensions.

Key words: noncommutative Einstein spaces, Petrov category, noncommutative topological quantum field theory.

Introduction

The subjects of the noncommutative Einstein spaces, double category and TQFT have been studied in [1–10]. In [3, 11] some noncommutative geometric aspects of twisted deformations were described, and it was shown that the universal enveloping algebra of vector fields can be deformed in two different ways:

- $U\Xi_\star$

This is a Hopf algebra [11] defined by deforming the structure functions of $U\Xi$:

$$u \star v = \overline{f}^\alpha(u)\overline{f}_\alpha(v),$$

$$\Delta_\star(u) = u \otimes \mathbf{1} + \overline{R}^\alpha \otimes \overline{R}_\alpha(u),$$

$$\epsilon_\star(u) = \epsilon(u) = 0,$$

$$S_\star(u) = -\overline{R}^\alpha(u)\overline{R}_\alpha,$$

where $\overline{R}^\alpha(u)$ is the usual Lie derivative of u along the vector field $\overline{R}^\alpha$.

There is a natural action of $\Xi_\star$ on the algebra of functions $\mathcal{A}_\star$ given in terms of the usual undeformed Lie derivative,

$$\mathcal{L}_u^\star(h) := \overline{f}^\alpha(u)(\overline{f}_\alpha(h)),$$

which can be extended to $U\Xi_\star$.

The $\star$-Lie algebra of vector fields $\Xi_\star$ is generating the Hopf algebra $U\Xi_\star$.

- $U\Xi^{\mathcal{F}}$

We have the following structure maps:

$$u \cdot^{\mathcal{F}} v = u \cdot v,$$

$$S^{\mathcal{F}}(u) = S(u),$$

$$\epsilon^{\mathcal{F}}(u) = \epsilon(u),$$

$$\Delta^{\mathcal{F}}(u) = \mathcal{F}\Delta(u)\mathcal{F}^{-1}.$$

However, $U\Xi_\star$ and $U\Xi^{\mathcal{F}}$ turn out to be isomorphic Hopf algebras.

The star-connection $\nabla^\star$ is defined to satisfy the following axioms:

$$\begin{aligned} &\nabla^*_{u+v} z = \nabla^*_u z + \nabla^*_v z, \\ &\nabla_{h\star u} v = h \star \nabla^*_u v, \\ &\nabla^*_u (h \star v) = \mathcal{L}^*_u(h) \star v + \overline{R}^\alpha(h) \star \nabla^*_{\overline{R}_\alpha(u)} v, \end{aligned} \tag{1}$$

where u, v and z are vector fields. Next, we define connection coefficients by

$$\nabla^\star_\mu \widehat{\partial}_\nu := \Gamma^\sigma_{\mu\nu} \star \widehat{\partial}_\sigma,$$

using the basis $\{\widehat{\partial}_\mu\}$. The action of the covariant derivative on a one-form can be obtained employing the star-dual pairing of a vector field v with a one-form ω,

$$\nabla^*_u < v, w >_\star = \mathcal{L}^*_u \langle v, w\rangle_\star = \langle \nabla^*_u v, w\rangle_\star + \langle \overline{R}^\alpha(v), \nabla^*_{\overline{R}_\alpha(u)} w\rangle_\star,$$

which equivalently can be written as

$$\langle v, \nabla^*_u w\rangle_\star = \mathcal{L}_{\overline{R}^\alpha(u)} \langle \overline{R}_\alpha(v), w\rangle_\star - \langle \nabla^*_{\overline{R}^\alpha(u)} (\overline{R}_\alpha(v)), w\rangle_\star . \tag{2}$$

For a given metric

$$g = g_{\mu\nu} \star d\widehat{x}^\mu \otimes_\star d\widehat{x}^\nu,$$

the connection that leaves it invariant is called a Levi – Civita connection:

$$\nabla^\star_\mu g = 0.$$

For general twist $\mathcal{F}^{-1} = \overline{f}^\alpha \otimes \overline{f}_\alpha$, torsion and curvature tensors are given by [3]

$$\begin{aligned} &T(u, v) = \nabla^*_u v - \nabla^*_{\overline{R}^\alpha(v)} \overline{R}_\alpha(u) - [u, v]_*, \\ &R(u, v, z) \equiv R(u, v) z = \nabla^*_u \nabla^*_v z - \nabla^*_{\overline{R}^\alpha(v)} \nabla^*_{\overline{R}_\alpha(u)} z - \nabla^*_{[u,v]_*} z\, . \end{aligned}$$

It is enough to calculate the tensor on a basis $\widehat{\partial}_\mu$, because of the tensorial property, i.e.

$$T(u, v) = u^\nu \star T(\widehat{\partial}_\nu, \widehat{\partial}_\mu) \star v^\mu.$$

In this frame, the star-connection is given by

$$\nabla^*_z u = \mathcal{L}^*_z(u^\nu) * \widehat{\partial}_\nu + \overline{R}^\alpha(u^\nu) * \overline{R}_\alpha(z)^\mu * \Gamma^\sigma_{\mu\nu} * \widehat{\partial}_\sigma . \tag{3}$$

We will need to compute the components of the curvature tensor in this base. They can be expressed in the following way:

$$R_{ijk}{}^l = \langle R(\widehat{\partial}_i, \widehat{\partial}_j, \widehat{\partial}_k),\, d\widehat{x}^k\rangle_* .$$

Consequently, we have for the deformed Ricci tensor

$$R_{ij} = R_{ijk}{}^i .$$

Classical Einstein spaces have a Ricci tensor proportional to the metric. In the noncommutative case, we are looking for spaces satisfying the same property:

$$R_{ij} = c g_{ij},$$

where c is some constant.

1. Noncommutative Einstein spaces

1.1. Weyl – Moyal plane $\mathbb{R}^4_\theta$. The metric is the usual Minkowski or Euclidean one; the twist is Abelian [11]:

$$\mathcal{F} = \exp\left(-\frac{i}{2}\,\theta^{\mu\nu}\,\partial_\mu \otimes \partial_\nu\right),$$

where $\theta^{\mu\nu} = -\theta^{\nu\mu} \in \mathbb{R}$. The covariant derivative is given by

$$\nabla^*_z u = z^\mu \star \partial_\mu(u^\nu) \star \partial_\nu + z^\mu \star u^\nu \star \Gamma^\sigma_{\mu\nu} \star \partial_\sigma\,. \tag{4}$$

In a first step, let us show that the choice $\Gamma^\sigma_{\mu\nu} = 0$ is a good choice and renders the affine connection to be a Levi – Civita connection. Thus, the expression for the covariant derivative (4) becomes

$$\nabla^*_z u = z^\mu \star \partial_\mu(u^\nu) \star \partial_\nu.$$

Let us show that the axioms (1) are satisfied:

$$\nabla^*_{u+v} z = (u+v)^\mu \star \partial_\mu(z^\nu) \star \partial_\nu = \nabla^*_u z + \nabla^*_v z, \tag{5}$$

$$\nabla_{h\star u} v = (h \star u^\mu) \star \partial_\mu(v^\nu) \star \partial_\nu = h \star (u^\mu \star \partial_\mu v^\nu \star \partial_\nu) = h \star \nabla^*_u v, \tag{6}$$

$$\begin{aligned}\nabla^*_u(h \star v) = u^\mu \star \partial_\mu(h \star v^\nu) \star \partial_\nu = \mathcal{L}^*_u(h) \star v + u^\mu \star h \star (\partial_\mu v^\nu) \star \partial_\nu \;=\\ =\; \mathcal{L}^*_u(h) \star v + \overline{R}^\alpha(h) \star \overline{R}_\alpha(u^\mu) \star (\partial_\mu v^\nu) \star \partial_\nu \;=\\ =\; \mathcal{L}^*_u(h) \star v + \overline{R}^\alpha(h) \star \nabla^*_{\overline{R}_\alpha(u)} v.\end{aligned} \tag{7}$$

In a next step, we show that the curvature and torsion vanish. The torsion is given by

$$T(\partial_\mu, \partial_\nu) = \nabla^*_\mu \partial_\nu - \nabla^*_\nu \partial_\mu - [\partial_\mu, \partial_\nu]_* = 0,$$

since the Christoffel symbols are all zero and the derivatives commute. Similarly, we see that the curvature tensor also vanishes:

$$R(\partial_\nu, \partial_\beta, \partial_\mu) = \nabla^*_\nu \nabla^*_\beta \partial_\mu - \nabla^*_{\overline{R}^\alpha(\partial_\beta)} \nabla^*_{\overline{R}_\alpha(\partial_\nu)} \partial_\mu - \nabla^*_{[\partial_\nu, \partial_\beta]_*} z = 0.$$

At last, we consider the covariant derivative of the metric:

$$\begin{aligned}\nabla^*_\mu g = \nabla^*_\mu(g_{\alpha\beta}\, dx^\alpha \otimes_* dx^\beta) \;=\\ =\; \partial_\mu(g_{\alpha\beta})\, dx^\alpha \otimes_* dx^\beta - g_{\alpha\beta} \Gamma^\alpha_{\mu\sigma}\, dx^\sigma \otimes_* dx^\beta - g_{\alpha\beta}\, dx^\alpha \otimes_* \Gamma^\beta_{\mu\sigma} dx^\sigma = 0\end{aligned}$$

since we get from the star-dual pairing (2)

$$\nabla^*_\mu dx^\alpha = -\Gamma^\alpha_{\mu\sigma} \star dx^\sigma = 0.$$

Among these metrics those that are classically Einstein metrics are also shown to be noncommutative Einstein metrics.

1.2. $\mathbb{R}_q^5$. The algebra is generated by the coordinates $\widehat{x}^1, \ldots, \widehat{x}^5$ satisfying the relations [11]

$$\begin{aligned} &\widehat{x}^1\,\widehat{x}^2 = q\,\widehat{x}^2\,\widehat{x}^1, \quad \widehat{x}^1\,\widehat{x}^4 = q^{-1}\widehat{x}^4\,\widehat{x}^1,\\ &\widehat{x}^1\,\widehat{x}^5 = \widehat{x}^5\,\widehat{x}^1, \quad \widehat{x}^2\,\widehat{x}^4 = \widehat{x}^4\,\widehat{x}^2,\\ &\widehat{x}^2\,\widehat{x}^5 = q\,\widehat{x}^5\,\widehat{x}^2, \quad \widehat{x}^4\,\widehat{x}^5 = q^{-1}\widehat{x}^5\,\widehat{x}^4. \end{aligned} \tag{8}$$

The coordinate $\widehat{x}^3$ is central. Conjugation is given by

$$\widehat{x}^{1*} = \widehat{x}^5, \quad \widehat{x}^{2*} = \widehat{x}^4, \quad \widehat{x}^{3*} = \widehat{x}^3.$$

Hence, the twist (for symmetrical ordering) reads

$$\mathcal{F} = \exp\left(\frac{ih}{2}(\chi_1 \otimes \chi_2 - \chi_2 \otimes \chi_1)\right), \tag{9}$$

where χ_1 and χ_2 are the following commuting vector fields:

$$\chi_1 = x^2\partial_2 - x^4\partial_4, \quad \chi_2 = x^1\partial_1 - x^5\partial_5.$$

Thus, we have for the inverse $\mathcal{R}$ matrix

$$\begin{aligned} \mathcal{R}^{-1} &= \overline{R}^{\alpha} \otimes \overline{R}_{\alpha} = f^{\alpha}\overline{f}_{\beta} \otimes f_{\alpha}\overline{f}^{\beta} = \\ &= \sum (-1)^{m+k-l}\left(\frac{h}{2}\right)^{n+k} \frac{\binom{n}{m}\binom{k}{l}}{n!k!}\,\chi_1^{n-m+l}\chi_2^{m+k-l} \otimes \chi_1^{m+k-l}\chi_2^{n-m+l}. \end{aligned}$$

1.3. Twisted sphere. The twisted sphere is defined by the relations (8) and the additional condition [12]

$$r^2 = 2(\widehat{x}^1\,\widehat{x}^5 + \widehat{x}^2\,\widehat{x}^4) + (\widehat{x}^3)^2.$$

Using stereographic coordinates y^i, $i = 1, 2, 4, 5$, the metric is given by

$$g^* = \frac{4r^2}{(r^2+\kappa^2)^2} \star C_{ij}dy^i \otimes_* dy^j,$$

where

$$(C_{ij}) = \begin{pmatrix} & & & 1\\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}.$$

In order to simplify the notation, we introduce the following definitions. For the vector fields, let us define

$$t_i := y^i\frac{\partial}{\partial y^i} = y^i\partial_i;$$

note that here no summation over the index i is implied. Hence, we write for the twist

$$\mathcal{F} = \exp\left(-\frac{ih}{2}\varphi_{ij}t_i \otimes t_j\right),$$

with

$$\begin{aligned} &\varphi_{ij} = -\varphi_{ji} = -\varphi_{ij'},\\ &\varphi_{12} = 1, \quad \varphi_{ii} = \varphi_{ii'} = 0, \end{aligned}$$

and $i' = 6 - i$. Furthermore, let us introduce P_{ij} and its square,

$$P_{ij} = \exp\left(\frac{ih}{2}\varphi_{ij}\right), \quad q_{ij} = P_{ij}^2.$$

Using these definitions, we can write for the metric

$$g^* = \sum_{ij} g_{ij} dy^i \otimes_* dy^j = \frac{4r^2}{(r^2+\kappa^2)^2} \sum_{i,j} C_{ij} P_{ij}\, dy^i \otimes dy^j. \tag{10}$$

The Levi – Civita connection can be obtained by demanding vanishing torsion and vanishing covariant derivative of the metric. The former condition reads

$$\Gamma^{*\ k}_{ij} = q_{ij}\Gamma^{*\ k}_{ji}.$$

The latter condition then leads to

$$\Gamma^{*\ k}_{ij} = \frac{1}{2} g^{lk} \left(q_{ij}\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}\right). \tag{11}$$

As a result, the universal connection is the same as in the undeformed case:

$$\nabla^* = \nabla. \tag{12}$$

The converse is also true: assuming (12), we obtain (11) for the connection coefficients.

Similarily, we obtain for the Riemann curvature:

$$R^* = R,$$

and in terms of components

$$R^* = R^{*\ \ m}_{ikl} dy^i \otimes_* dy^j \otimes_* dy^k \otimes_* \partial_m,$$
$$R^*_{ijkl} = \frac{1}{r^2}\left(g_{li} g_{jk} - q_{ik}\, g_{lj} g_{ik}\right).$$

Now let us consider a possible transformation between 5d theta-deformed plane (see Subsection 1.1.) and a 5d q-deformed one (see Subsection 1.2.). The theta-deformed space is chosen in the following way: $[x_i, x_j] = i\theta_{ij}$ with the coordinate x_3 commuting with all other coordinates and

$$\theta_{ij} = \begin{pmatrix} 0 & h & -h & 0 \\ -h & 0 & 0 & h \\ h & 0 & 0 & -h \\ 0 & -h & h & 0 \end{pmatrix}.$$

Then with the map $y_i = \exp(x_i)$ we obtain the correct commutation relations (8). But unfortunately this map does not respect the complex structure, and the induced metric seems not to be the proper metric for the q-deformed plane. But another possible map is from the q-deformed sphere to a plane, via stereographic projection. Starting with the q-deformed sphere: commutation relations (8) and the constraint $r^2 = 2(x^1x^5 + x^2x^4) + (x^3)^2$, one defines a map to the plane in the usual way by $y^3 = x^3,\ y^i = (x^i r)/(r - x^3),\ i = 1, 2, 4, 5$. The induced metric is then given by (10).

2. Petrov category

Definition 1. A category is a quadruple $(\mathbf{Obj}, \mathbf{Mor}, \mathrm{id}, \circ)$ consisting of:
(C1) a class $\mathbf{Obj}$ of objects;
(C2) a set $\mathbf{Mor}(A, B)$ of morphisms for each ordered pair (A, B) of objects;
(C3) a morphism $\mathrm{id}_A \in \mathbf{Mor}(A, A)$ for each object A: the identity of A;
(C4) a composition law associating to each pair of morphisms $f \in \mathbf{Mor}(A, B)$ and $g \in \mathbf{Mor}(B, C)$: a morphism $g \circ f \in \mathbf{Mor}(A, C)$;
which is such that:
(M1) $h \circ (g \circ f) = (h \circ g) \circ f$ for all $f \in \mathbf{Mor}(A, B)$, $g \in \mathbf{Mor}(B, C)$ and $h \in \mathbf{Mor}(C, D)$;
(M2) $\mathrm{id}_B \circ f = f \circ \mathrm{id}_A = f$ for all $f \in \mathbf{Mor}(A, B)$;
(M3) the sets $\mathbf{Mor}(A, B)$ are pairwise disjoint.

Definition 2. The category **Petrov-NC-Einst**. Objects of the category **Petrov-NC-Einst** are noncommutative Einstein spaces *NC Einst* defined in Subsection 1.1.–1.3. by the induced metric (10). For morphisms s, t ($\mathit{NC\ Einst} \to \mathit{NC\ Einst}'$) we define a map to the plane in the usual way by $y^3 = x^3$, $y^i = (x^i r)/(r - x^3)$, $i = 1, 2, 4, 5$. A product of such morphisms of the category **Petrov-NC-Einst** is again a morphism of the category **Petrov-NC-Einst**. So, the category **Petrov-NC-Einst** is well defined.

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Резюме

С.С. Москалюк. Категория А.З. Петрова некоммутативных пространств Эйнштейна.

В статье построены некоммутативные пространства Эйнштейна, которые являются объектами в определении категории А.З. Петрова. Концепция категории А.З. Петрова позволяет получить некоммутативные топологические теории поля и их расширения в многомерных пространствах.

Ключевые слова: некоммутативные пространства Эйнштейна, категория А.З. Петрова, некоммутативная топологическая квантовая теория поля.

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