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NECESSARY CONDITIONS FOR QUALIFIED CONVERGENCE OF METHODS FOR SOLVING LINEAR ILL-POSED PROBLEMS

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1. Introduction

In a Hilbert space H we consider the operator equation

$$Au = f, \tag{1}$$

where $A : H \to H$ is a linear continuous selfadjoint nonnegative operator, i.e., $A^* = A$, $(Au, u) \geq 0$ $\forall u \in H$. We assume that in (1) the right side f belongs to $R(A)$, where $R(A) = \{v \in H : v = Au, u \in H\}$ is the image of the operator A. In our conditions the set U_* of solutions of equation (1) is nonempty. In what follows the closedness of the image $R(A)$ is not assumed, so problem (1) is ill-posed (see [1], p. 9). As is well-known, for finding a solution of such problems it is necessary to use the regularization methods (see, e.g., [1]–[3]), which specially take into account this peculiarity of the problems.

In the article we investigate a class of methods for stable solving equation (1) (see [1], Chap. 2; [2], Chap. 2), which in absence of errors in the input data A, f can be written as follows

$$u_r = (I - Ag_r(A))u_0 + g_r(A)f, \quad r \geq r_0 \ \ (r_0 > 0). \tag{2}$$

Here and in what follows u_0 is a fixed initial approximation, r the parameter of regularization, $\{g_r(\lambda)\}_{r\in[r_0,+\infty)}$ a family of Borel-measurable functions on the segment $[0, M]$ which contains the spectrum $\sigma(A)$ of the operator A. Following [2] (p. 28), we assume that the functions $g_r(\lambda)$ satisfy the condition

$$\sup_{\lambda\in[0,M]} \lambda^p |1 - \lambda g_r(\lambda)| \leq Cr^{-p} \quad \forall r \geq r_0, \ \ C = C(p), \ \ p \in [0, p_0], \ \ p_0 > 0. \tag{3}$$

In considering problems with errors in their data, along with (3) we shall use the condition

$$\sup_{\lambda\in[0,M]} |\lambda^q g_r(\lambda)| \leq Cr^d \quad \forall r \geq r_0; \ \ d \in [0, 1], \ \ d = d(q), \ \ q \geq 0. \tag{4}$$

The greatest value of the parameter p_0, for which inequality (3) is valid, is called the qualification of method (2); besides, the case $p_0 = \infty$ is not excluded. In these and further relations we denote by C positive absolute constants, which, in general, may be different.

To the class of methods (2), in particular, the group of iteration procedures

$$u_{n+1} = u_n - g(A)(Au_n - f), \quad n = 0, 1, \dots, \tag{5}$$

which correspond to a special choice of generating functions $g_r(\lambda)$ (see (17)), belongs; in addition, the parameter of regularization runs over the values $r = n \in \mathbb{N}$, where $\mathbb{N} = \{1, 2, \dots\}$. The