Modern problems of cosmology

Alexei A. Starobinsky

Landau Institute for Theoretical Physics RAS,
Moscow, Russia

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Basics of cosmology

Subject of cosmology:

general properties of the present Universe at large scales, its past and future

3 "whales" on which "old" classic cosmology is based:

1. The Einstein gravity (General Theory of Relativity)

Gravitational field is described by a space-time metric satisfying the Einstein equations.

Verified with $\sim 10^{-4}$ accuracy in Solar system experiments. Gravitational radiation from double radio-pulsars.

2. Approximate homogeneity and isotropy of the Universe

a) space-time metric:

$$\mathbf{v} = H_0 \mathbf{r}, \quad H_0 = (70 \pm 3)\frac{km}{s \cdot Mpc}$$

the Hubble law with H_0 - isotropic

b) matter:
isotropic spatial distribution of galaxies and clusters, isotropy of galaxy counts;
isotropy of the X-ray background

c) radiation (CMB):
black-body with the almost isotropic
$T_\gamma = (2.72548 \pm 0.00057)\mathrm{K}$

3. Hot past (Big Bang)

Change from the "old" to "new" standard cosmology

1. Understanding that all 3 basic foundations are approximate.
a) There exist natural generalizations of the Einstein equations, and we need them.
b) The Universe may be and generically is indeed very anisotropic and homogeneous at very large scales not observable now.
c) The very early Universe may still be "cold".

2. Discovery of two new kinds of dark "entities": dark matter and dark energy.

3. Discovery of two new periods in the evolution of the Universe in the very remote past and at the present time when its expansion is accelerated.

Present matter content of the Universe

In terms of the critical density

$\rho_{crit} = \frac{3H_0^2}{8\pi G} \approx 0.9 \times 10^{-29}\ \mathrm{g/cm^3}, \quad \Omega_i = \frac{\rho_i}{\rho_{crit}}, \quad \sum_i \Omega_i = 1$

(neglecting spatial curvature - less than 0.7%):

- Baryons (p,n) and leptons (e^-) 5%
  No primordial antimatter.
- Photons (γ) 4×10^{-5}
  $T_\gamma = (2.72548 \pm 0.00057)\mathrm{K}$
- 3 types of neutrinos ($\nu_e,\ \nu_\mu,\ \nu_\tau$) $< 0.5\%$

$$\sum_i m_{\nu i} < 0.23\ \mathrm{eV}, \quad \sum_i m_{\nu i} = 94\Omega_\nu h^2\ \mathrm{eV}.$$

- Non-relativistic non-baryonic dark matter $\approx 25\%$
- Dark energy $\approx 70\%$

Four epochs of the history of the Universe

$H \equiv \frac{\dot{a}}{a}$ where $a(t)$ is a scale factor of an isotropic homogeneous spatially flat universe (a Friedmann-Robertson-Walker background):

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)\,.$$

The history of the Universe in one line according to the present paradigm:

$$? \longrightarrow DS \Longrightarrow FRWRD \Longrightarrow FRWMD \Longrightarrow \overline{DS} \longrightarrow ?$$

$$|\dot{H}| << H^2 \Longrightarrow H = \frac{1}{2t} \Longrightarrow H = \frac{2}{3t} \Longrightarrow |\dot{H}| << H^2$$

$$p \approx -\rho \Longrightarrow p = \rho/3 \Longrightarrow p \ll \rho \Longrightarrow p \approx -\rho$$

Dark matter

Dark matter and dark energy are seen through gravitational interaction only – we know the structure of their effective energy-momentum tensor.

DM - non-relativistic, gravitationally clustered.
DE - relativistic, unclustered.
Definition of their effective EMT – through equations (conventional).
DM - through the generalized Poisson equation:

$$\frac{\triangle\Phi}{a^2} = 4\pi G(\rho - \rho_0(t)).$$

$\Phi(\mathbf{r}, t)$ is measured using the motion of 'test particles' in it.
a) Stars in galaxies → rotation curves.
b) Galaxies → peculiar velocities.
c) Hot gas in rich galaxy clusters → X-ray profiles.
d) Photons → gravitational lensing (strong and weak).

Observations: DM is non-relativistic, has a dust-like EMT – $p \ll \epsilon = \rho c^2, \quad p > 0$, collisionless in the first approximation – $\sigma/m < 1 \ \mathrm{cm}^2/\mathrm{g}$, and has the same spatial distribution as visible matter for scales exceeding a few Mpc.

Ground experiments: very weakly interacting with baryonic matter, $\sigma < 10^{-43} \ \mathrm{cm}^2$ for $m \sim (50 - 100)$ GeV.

Dark energy

Two cases where DE shows itself:
1) inflation in the early Universe – primordial DE,
2) present accelerated expansion of the Universe – present DE.

Quantitative and internally self-consistent definition of its effective EMT - through gravitational field equations conventionally written in the Einstein form:

$$\frac{1}{8\pi G}\left(R_\mu^\nu - \frac{1}{2}\,\delta_\mu^\nu R\right) = -\left(T_{\mu\,(vis)}^\nu + T_{\mu\,(DM)}^\nu + T_{\mu\,(DE)}^\nu\right) \;,$$

$G = G_0 = const$ - the Newton gravitational constant measured in laboratory.
In the absence of direct interaction between DM and DE:

$$T_{\mu\,(DE);\nu}^\nu = 0 \;.$$

Possible forms of DE

- Physical DE.
  New non-gravitational field of matter. DE proper place – in the rhs of gravity equations.
- Geometrical DE.
  Modified gravity. DE proper place – in the lhs of gravity equations.
- Λ - intermediate case.

Generically, DE can be both physical and geometrical, e.g. in the case of a non-minimally coupled scalar field or, more generically, in scalar-tensor gravity. So, there is no alternative "(either) dark energy or modified gravity".

Background evolution

Neglecting the spatial curvature (less than 0.7% of the critical density):

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)$$

The reconstruction programme: determination of the Universe evolution in the past from observational data.
The basic quantity to be found: the Hubble parameter $H \equiv \frac{\dot{a}}{a}$ as a function of redshift $z \equiv \frac{a(t_0)}{a} - 1$.

All components of the Riemann tensor can be expressed through $H(z)$ and $\frac{dH(z)}{dz}$.

EMT of present DE from the definition above:

$$\rho_{DE} = \frac{3H_0^2}{8\pi G}\left(h^2(z) - \Omega_{m0}(1+z)^3\right)$$

$$p_{DE} = \frac{3H_0^2}{8\pi G}\left(-h^2(z) + \frac{1}{3}\frac{dh^2(z)}{dz}\right)$$

where $h(z) = \frac{H(z)}{H_0}$, $H_0 = H(t_0)$ is the Hubble constant and Ω_{m0} is the present density of non-relativistic matter in terms of the critical one.

The DE effective equation of state $w_{DE} \equiv \frac{p_{DE}}{\rho_{DE}}$.

$w_{DE} > -1$ – normal case,
$w_{DE} < -1$ – phantom case,
$w_{DE} \equiv -1$ – the exact cosmological constant.

Luminosity distance from SNIa

The largest clean set at present: the Union 2.1 set (H. Suzuki *et al.*, Astroph. J. **746**, 85, (2012)): consists of 580 type Ia supernovae sampling the redshift range $0.015 \leq z \leq 1.414$. It provides us with the luminosity distance $D_L(z) = (1+z)\int_0^z \frac{dz}{H(z)}$.

$$H^{-1}(z) = \frac{d}{dz}\left(\frac{D_l(z)}{1+z}\right)$$

The main problem of the reconstruction programme: differentiation is not a proper operation in the presence of observational errors.

Ways to avoids it:

- Comparison of concrete theoretical models with data.
- Best fit to some arbitrary chosen analytical expressions for $H(z)$ or w_{DE}. The most widely known is the CPL (Chevallier-Polarski-Linder) fit

$$w_{DE} = w_0 + w_1 \frac{z}{1+z}$$

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- Smoothing.
  Many working proposals, e.g. A. Shafieloo et al., MNRAS **300**, 1081 (2006) and A. Shafieloo, arXiv:1204.1109.
- The principal components method and many others.

Acoustic oscillations in matter and CMB perturbation spectra

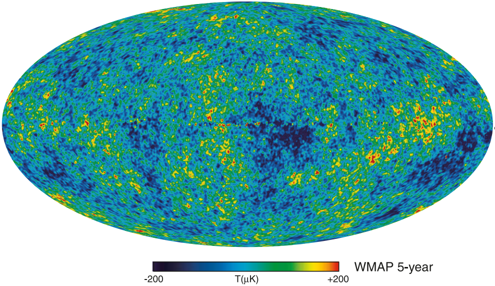
Origin of the effect: the Universe was isotropic at least from the BBN time $\longrightarrow$ half of large-scale scalar (density) perturbations – the so called decaying mode – are absent. Standing acoustic waves at the radiation-dominated stage.

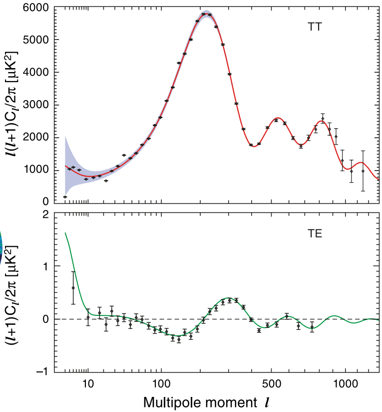
I. Acoustic oscillations in CMB angular temperature fluctuations (the effect is seen in CMB polarization, too).

Leads to a very accurate measured shift parameter

$$\mathcal{R} = \sqrt{\Omega_{m0}} \int_0^{z_{rec}} \frac{dz}{h(z)} = 1.725 \pm 0.018$$

(E. Komatsu et al., Astroph. J. Suppl. **192**, 18 (2011)).
Precise but degenerate test.


WMAP 5-year
-200
T(μK)
+200


TT
TE
6000
5000
4000
3000
2000
1000
0
2
1
0
–1
10
100
500
1000
l(l+1)C_l/2π [μK²]
(l+1)C_l/2π [μK²]
Multipole moment l

II. Baryon acoustic oscillations (BAO).

Large galaxy catalogs are needed. What is obtained is the following effective distance measure:

$$D_V(z) = H_0^{-1}\left[\frac{z}{h(z)}\left(\int_0^z \frac{dx}{h(x)}\right)^2\right]^{1/3}$$

Measured for 6 points by now:
$z = 0.106, 0.2, 0.35, 0.44, 0.6, 0.73$ – from SDSS DR7 (W. J. Persival *et al.*, MNRAS **401**, 2148 (2010)), WiggleZ (C. Blake *et al.*, MNRAS **415**, 2892 (2011); MNRAS **418**, 1707 (2011)) and 6dFGS (F. Beutler *et al.*, MNRAS **416**, 3017 (2011)) catalogs.

Null diagnostics

Aim: falsifying the cosmological constant with minimal assumptions.

The Om characteristic (V. Sahni, A. Shafieloo and A. A. Starobinsky, Phys. Rev. D **78**, 103502 (2008), see also C. Zunckel and C. Clarkson, Phys. Rev. Lett. **101**, 181301 (2008)):

$$Om(z_1, z_2) = \frac{h^2(z_1) - h^2(z_2)}{(1+z_1)^3 - (1+z_2)^3}$$

If Om considered as a function of one of its arguments (with the second one being fixed) is identically constant, then the model is the ΛCDM one and $Om = \Omega_{m0}$. Its calculation does not require the knowledge of the values of H_0 and Ω_{m0}.

Its variant customized for the usage of BAO data: the $Om3$ diagnostic (A. Shafieloo, V. Sahni and A. A. Starobinsky, Phys. Rev. D **86**, 103527 (2012)):

$$Om3(z_1, z_2, z_3) = \frac{Om(z_1, z_2)}{Om(z_2, z_3)}$$

where z_2 lies between z_1 and z_3. If $Om3$ considered as a function of z_2 for fixed z_1 and z_3 is identically equal to unity, then the model is the ΛCDM one once more.

Outcome of all observations

$T^{\nu}_{\mu\,(DE)}$ is very close to $\Lambda\delta^{\nu}_{\mu}$ for the concrete solution describing our Universe;

$$|< w_{DE} > +1| \lesssim 0.1$$

at 1σ confidence level. E.g., $w_{DE} = -1.008 \pm 0.085$ assuming $w_{DE} = const$ and $\Omega_k = 0$ (D. Parkinson *et al.*, arXiv:1210.2130). Thus, at the present level of knowledge only one constant is needed for quantitative description of present DE.

In the language of "coincidences" – present DE introduces only one new coincidence as yet.

Four fundamental cosmological constants

One-to-one relation to the four epochs of the history of the Universe.

A fundamental theory beyond each of these constants.

- Characteristic amplitude of primordial scalar (adiabatic) perturbations.

$$\Delta_\zeta^2 = 2.2 \times 10^{-9} \; , \quad P_s(k) = \int \frac{\Delta_\zeta^2}{k} dk$$

  Theory of initial conditions – inflation.

- Baryon to photon ratio.

$$\frac{n_b}{n_\gamma} = 6.01 \times 10^{-10} \, \frac{\Omega_b h^2}{0.0022} \, \left(\frac{2.725}{T_\gamma(\mathrm{K})} \right)^3$$

  Theory of baryogenesis.

- Baryon to total non-relativistic matter density.

$$\frac{\rho_b}{\rho_m} = 0.167 \, \frac{\Omega_b}{0.05} \, \frac{0.3}{\Omega_m}$$

Theory of dark matter.

- Energy density of present dark energy.

$$\rho_{DE} = \frac{\epsilon_{DE}}{c^2} = 6.44 \times 10^{-30} \, \frac{\Omega_{DE}}{0.70} \, \left(\frac{H_0}{70}\right)^2 \, \mathrm{g/cm^3}$$

$$\frac{G^2 \hbar \epsilon_{DE}}{c^7} = 1.25 \times 10^{-123} \, \frac{\Omega_{DE}}{0.70} \, \left(\frac{H_0}{70}\right)^2$$

Theory of present dark energy (of a cosmological constant).

The minimal present standard cosmological model

ΛCDM + $(\mathcal{K} = 0)$+(scale-invariant adiabatic perturbations) contains two more parameters:

- H_0 – not a constant, but a present value of $H(t)$;
- $\tau \approx 0.09$ – optical width after recombination – a constant, but not fundamental.

4 fundamental cosmological constants $\Longrightarrow$ no more than 4 cosmological "coincidences", all other "coincidences" exist already at the level of usual laboratory physics.

New constant discovered

Observations tend to increase the number of fundamental constants, but theory can counteract it by "unification", by expressing these new constants through already existing ones. With the Planck and WMAP9 CMB data:

$$n_s(k) - 1 \equiv \frac{d \ln P_s(k)}{d \ln k} = -0.040 \pm 0.007$$

There exist inflationary models of the early Universe including the pioneer $R + R^2/(6M^2)$ one (Starobinsky, 1980) which predict just this value theoretically, so no need in the increase of the number of fundamental constants.

Scalar-tensor models of dark energy

Still no need to go beyond Einstein (General Theory of Relativity + a cosmological constant + hydrodynamic matter (dust, radiation)) to describe dark matter and present dark energy. However, primordial dark energy driving the de Sitter (= inflationary) stage in the early Universe may not be described by a cosmological constant since it should be unstable (though sufficiently metastable). Thus, we have to go beyond Einstein at least in this place.

A new scalar degree of freedom is needed, both for an exit from the first de Sitter stage and for quantum generation of scalar (adiabatic) perturbations. Natural and long well known generic extension: scalar-tensor gravity.

In the Jordan (physical) frame

$$L = \frac{1}{2}\left(F(\Phi)R + g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi\right) - U(\Phi) + L_m(g_{\mu\nu})$$

Equations for the spatially flat FLRW model with the scale factor $a(t)$:

$$3FH^2 = \rho_m + \frac{\dot{\Phi}^2}{2} + U - 3H\dot{F}$$

$$-2F\dot{H} = \rho_m + \dot{\Phi}^2 + \ddot{F} - H\dot{F}$$

$$\ddot{\Phi} + 3H\dot{\Phi} + \frac{dU}{d\phi} - 3(\dot{H} + 2H^2)\frac{dF}{d\Phi} = 0$$

where $H = \frac{\dot{a}}{a}$ (only two of them are independent).

Particular cases

1. $F = \frac{1}{\kappa^2} = const, \quad \kappa^2 = 8\pi G.$
Einstein gravity sourced by a minimally coupled scalar field with some potential.

2. $F = \frac{1}{\kappa^2} - \xi\phi^2.$
Non-minimally coupled scalar field.

3. The limiting case of a very large F when the scalar field kinetic term may be neglected: $f(R)$ gravity.

$$L = \frac{f(R)}{2\kappa^2}, \quad f(R) = \kappa^2(FR - 2U)$$

where $\phi(R)$ is determined from the equation

$$F'(\phi)R = 2U'(\phi)$$

.

Conformal duality of vacuum scalar-tensor gravity to the Einstein gravity with a minimally coupled scalar field

Conformal transformation to the Einstein frame:

$$\hat{g}_{\mu\nu} = \kappa^2 F g_{\mu\nu}, \quad \hat{U} = \frac{U}{(\kappa^2 F)^2}$$

$$\left(\frac{d\hat{\phi}}{d\phi}\right)^2 = \frac{2F + 3F'^2}{2\kappa^2 F^2}$$

However, free particles in the Einstein frame do not follow space-time geodesics, they are coupled to the scalar field, too.

FRW dynamics with a scalar field

In the absence of spatial curvature and other matter, it can be reduced to the first order Hamilton-Jacobi-like equation for $H(\phi)$:

$$\frac{2}{3\kappa^2}\left(\frac{dH}{d\phi}\right)^2 = H^2 - \frac{\kappa^2}{3}U(\phi)$$

Time dependence is determined using the relation

$$t = -\frac{\kappa^2}{2}\int \left(\frac{dH}{d\phi}\right)^{-1} d\phi$$

However, during oscillations of ϕ, $H(\phi)$ acquires non-analytic behaviour of the type $const + \mathcal{O}(|\phi - \phi_1|^{3/2})$ at the points where $\dot{\phi} = 0$, and then the correct matching with another solution is needed.

Due to conformal duality, the same refers to FRW dynamics in generic scalar-tensor gravity.

Inflationary slow-roll dynamics

Slow-roll occurs if: $|\ddot{\phi}| \ll H|\dot{\phi}|, \;\; \dot{\phi}^2 \ll U$, and then $|\dot{H}| \ll H^2$.

Necessary conditions: $|U'| \ll \kappa U, \;\; |U''| \ll \kappa^2 U$. Then

$$H^2 \approx \frac{\kappa^2 U}{3}, \quad \dot{\phi} \approx \frac{U'}{3H}, \quad N \equiv \ln \frac{a_f}{a} \approx \kappa^2 \int_{\phi_f}^{\phi} \frac{U}{U'}\, d\phi$$

First obtained in A.A. Starobinsky, Sov. Astron. Lett. 4, 82 (1978) in the $V = \frac{m^2\phi^2}{2}$ case and for a bouncing model.

General scheme of generation of perturbations

A genuine quantum-gravitational effect: a particular case of the effect of particle-antiparticle creation by an external gravitational field. Requires quantization of a space-time metric. Similar to electron-positron creation by an electric field. From the diagrammatic point of view: an imaginary part of a one-loop correction to the propagator of a gravitational field from all quantum matter fields including the gravitational field itself, too.

One spatial Fourier mode $\propto e^{i\mathbf{kr}}$ is considered.

For scales of astronomical and cosmological interest, the effect occurs at the primordial de Sitter (inflationary) stage when $k \sim a(t)H(t)$ where $k \equiv |\mathbf{k}|$ (the first Hubble radius crossing).

After that, for a very long period when $k \ll aH$ until the second Hubble radius crossing (which occurs rather recently at the FRWRD or FRWMD stages), there exist one mode of scalar (adiabatic, density) perturbations and two modes of tensor perturbations (primordial gravitational waves) for which metric perturbations are constant (in some gauge) and independent of (unknown) local microphysics due to the causality principle.
In this regime in the coordinate representation:

$$ds^2 = dt^2 - a^2(t)(\delta_{lm} + h_{lm})dx^l dk^m, \quad l, m = 1, 2, 3$$

$$h_{lm} = 2\zeta(\mathbf{r})\delta_{lm} + g(\mathbf{r})e_{lm}, \quad e_l^l = 0, \; g_{,l}e_m^l = 0, \; e_{lm}e^{lm} = 1$$

Classical-to-quantum transition

Quantum-to-classical transition: in fact, metric perturbations h_{lm} are quantum (operators in the Heisenberg representation) and remain quantum up to the present time. But, after omitting of a very small part, decaying with time, they become commuting and, thus, equivalent to classical (c-number) stochastic quantities with the Gaussian statistics (up to small terms quadratic in ζ, g).

Spectral predictions of the one-field inflationary scenario in GR

Scalar (adiabatic) perturbations:

$$P_\zeta(k) = \frac{H_k^4}{4\pi^2\dot{\phi}^2} = \frac{GH_k^4}{\pi|\dot{H}|_k} = \frac{128\pi G^3 V_k^3}{3V_k'^2}$$

where the index k means that the quantity is taken at the moment $t = t_k$ of the Hubble radius crossing during inflation for each spatial Fourier mode $k = a(t_k)H(t_k)$. Through this relation, the number of e-folds from the end of inflation back in time $N(t)$ transforms to $N(k) = \ln \frac{k_f}{k}$.

The spectral slope

$$n_s(k) - 1 \equiv \frac{d \ln P_\zeta(k)}{d \ln k} = \frac{1}{8\pi G}\left(2\frac{V_k''}{V_k} - 3\left(\frac{V_k'}{V_k}\right)^2\right)$$

Tensor perturbations - primordial gravitational waves (A.A. Starobinsky, JETP Lett. 50, 844 (1979)):

$$P_g(k) = \frac{16GH_k^2}{\pi}; \quad n_g(k) \equiv \frac{d\ln P_g(k)}{d\ln k} = -\frac{1}{8\pi G}\left(\frac{V'_k}{V_k}\right)^2$$

The consistency relation:

$$r(k) \equiv \frac{P_g}{P_\zeta} = \frac{16|\dot{H}_k|}{H_k^2} = 8|n_g(k)|$$

Tensor perturbations are always suppressed by at least the factor $\sim 8/N$ compared to scalar ones where $N = (50 - 60)$ is the number of e-folds between the first Hubble radius crossing during inflation of the present Hubble scale and the end of inflation.

Combined results from Planck and other experiments

P. A. R. Ade et al., arXiv:1303.5082

Model	Parameter	*Planck*+WP	*Planck*+WP+lensing	*Planck* + WP+high-ℓ	*Planck*+WP+BAO
ΛCDM + tensor	n_s	0.9624 ± 0.0075	0.9653 ± 0.0069	0.9600 ± 0.0071	$0.9643 + 0.0059$
	$r_{0.002}$	< 0.12	< 0.13	< 0.11	< 0.12
	$-2\Delta \ln \mathcal{L}_{max}$	0	0	0	-0.31

Table 4. Constraints on the primordial perturbation parameters in the ΛCDM+r model from *Planck* combined with other data sets. The constraints are given at the pivot scale $k_* = 0.002$ Mpc^{-1}.

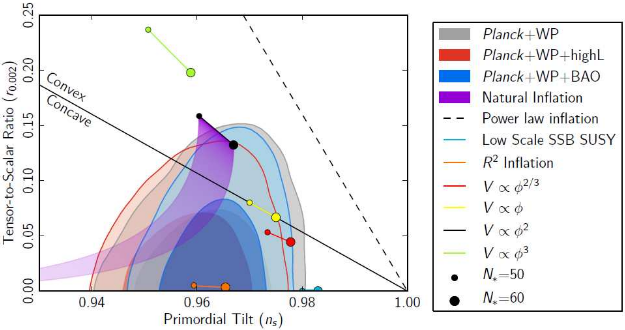


Fig. 1. Marginalized joint 68% and 95% CL regions for n_s and $r_{0.002}$ from *Planck* in combination with other data sets compared to the theoretical predictions of selected inflationary models.

Remaining models

I. Disfavoured at 95% and more CL.

1. Scale-free (or, the Harrison-Zeldovich) spectrum $n_s = 1$.
2. Power-law inflation (exponential $V(\phi)$).
3. Power-law $V(\phi) \propto \phi^n$ with $n \geq 2$.

II. Lying between 68% and 95% CL.

1. Other monomial potentials.
2. New inflation (or, the hill-top model with $V(\phi) = V_0 - \frac{\lambda\phi^4}{4}$).
3. Natural inflation.

III. Most favoured: models with $n_s - 1 = \frac{2}{N} \approx 0.04$ and $r \ll 8|n_s - 1|$.

1. $R + R^2$ model (A.A. Starobinsky, Phys. Lett. B 91, 99 (1980)).

2. A scalar field model with $V(\phi) = \frac{\lambda\phi^4}{4}$ at large ϕ and strong non-minimal coupling to gravity $\xi R\phi^2$ with $\xi < 0,\ |\xi| \gg 1$, including the Higgs inflationary model.

3. Minimally coupled (GR) models with a very flat $V(\phi)$: if $n_s - 1 = \frac{2}{N}$ and $r \ll 8|n_s - 1|$ for all N, then:

$$V(\phi) = V_0 + V_1 \exp(-\alpha\kappa\phi),\ \ \kappa = \sqrt{8\pi G}$$

with α not very small.

All these models have $r \sim 10/N^2$, namely $r = \frac{12}{N^2} \approx 0.4\%$ for the models 1 and 2, and $r = \frac{8}{\alpha^2 N^2}$ for the third model.

$f(R)$ gravity

The simplest model of modified gravity (= geometrical dark energy) considered as a phenomenological macroscopic theory in the fully non-linear regime and non-perturbative regime.

$$S = \frac{1}{16\pi G} \int f(R)\sqrt{-g}\, d^4x + S_m$$

$$f(R) = R + F(R), \quad R \equiv R^{\mu}_{\mu} \ .$$

One-loop corrections depending on R only (not on its derivatives) are assumed to be included into $f(R)$. The normalization point: at laboratory values of R where the scalaron mass (see below) $m_s \approx \mathrm{const}$.

Field equations

$$\frac{1}{8\pi G}\left(R_{\mu}^{\nu}-\frac{1}{2}\,\delta_{\mu}^{\nu}R\right)=-\left(T_{\mu\,(vis)}^{\nu}+T_{\mu\,(DM)}^{\nu}+T_{\mu\,(DE)}^{\nu}\right)\ ,$$

where $G = G_0 = const$ is the Newton gravitational constant measured in laboratory and the effective energy-momentum tensor of DE is

$$8\pi G T_{\mu\,(DE)}^{\nu}=F'(R)\,R_{\mu}^{\nu}-\frac{1}{2}\,F(R)\delta_{\mu}^{\nu}+\left(\nabla_{\mu}\nabla^{\nu}-\delta_{\mu}^{\nu}\nabla_{\gamma}\nabla^{\gamma}\right)F'(R)\,.$$

Because of the need to describe DE, de Sitter solutions in the absence of matter are of special interest. They are given by the roots $R = R_{dS}$ of the algebraic equation

$$Rf'(R)=2f(R)\ .$$

Degrees of freedom

I. In quantum language: particle content.
1. Graviton – spin 2, massless, transverse traceless.
2. Scalaron – spin 0, massive, mass - R-dependent:
$m_s^2(R) = \frac{1}{3f''(R)}$ in the WKB-regime.

II. Equivalently, in classical language: number of free functions of spatial coordinates at an initial Cauchy hypersurface.
Six, instead of four for GR – two additional functions describe massive scalar waves.

Thus, $f(R)$ gravity is a non-perturbative generalization of GR.
It is equivalent to scalar-tensor gravity with $\omega_{BD} = 0$ (if $f''(R) \neq 0$).

Background FRW equations in $f(R)$ gravity

$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

$$H \equiv \frac{\dot{a}}{a}\,, \quad R = 6(\dot{H} + 2H^2)$$

The trace equation (4th order)

$$\frac{3}{a^3}\frac{d}{dt}\left(a^3\frac{df'(R)}{dt}\right) - Rf'(R) + 2f(R) = 8\pi G(\rho_m - 3p_m)$$

The 0-0 equation (3d order)

$$3H\frac{df'(R)}{dt} - 3(\dot{H} + H^2)f'(R) + \frac{f(R)}{2} = 8\pi G\rho_m$$

Most favoured inflationary models in $f(R)$ gravity

1. The simplest one (Starobinsky, 1980):

$$f(R) = R + \frac{R^2}{6M^2}$$

with small one-loop quantum gravitational corrections producing the scalaron decay via the effect of particle-antiparticle creation by gravitational field (so all present matter is created in this way).
During inflation ($H \gg M$): $H = \frac{M^2}{6}(t_f - t), \quad |\dot{H}| \ll H^2$.

The only parameter M is fixed by observations – by the primordial amplitude of adiabatic (density) perturbations in the gravitationally clustered matter component:

$$M = 3.0 \times 10^{-6} M_{Pl} \, (50/N) \, ,$$

where $N \sim (50 - 55)$, $M_{Pl} = \sqrt{G} \approx 10^{19}$ GeV.

$$n_s = 1 - \frac{2}{N} \approx 0.96$$

$$r = \frac{12}{N^2} \approx 0.004$$

.

2. Generic $f(R)$ inflationary model having $n_s = 1 - \frac{2}{N}$; $r \sim \frac{10}{N^2}$. For large R,

$$f(R) = \frac{R^2}{6M^2} + CR^{2-\alpha\sqrt{3/2}}$$

. Less natural, has one more free parameter, but still possible.

One viable microphysical model leading to such form of $f(R)$

A non-minimally coupled scalar field with a large negative coupling ξ (for this choice of signs, $\xi_{conf} = \frac{1}{6}$):

$$L = \frac{R}{16\pi G} - \frac{\xi R\phi^2}{2} + \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi), \quad \xi < 0, \quad |\xi| \gg 1 \; .$$

Leads to $f' > 1$.
Recent development: the Higgs inflationary model (F. Bezrukov and M. Shaposhnikov, 2008). In the limit $|\xi| \gg 1$, the Higgs scalar tree level potential $V(\phi) = \frac{\lambda(\phi^2 - \phi_0^2)^2}{4}$ just produces $f(R) = \frac{1}{16\pi G}\left(R + \frac{R^2}{6M^2}\right)$ with $M^2 = \lambda/24\pi\xi^2 G$ and $\phi^2 = |\xi|R/\lambda$ (for this model, $|\xi|G\phi_0^2 \ll 1$).

SM loop corrections to the tree potential leads to $\lambda = \lambda(\phi)$, then the same expression for $f(R)$ follows with

$$M^2 = \frac{\lambda(\phi(R))}{24\pi\xi^2 G}\left(1 + \mathcal{O}\left(\frac{d\ln\lambda(\phi(R))}{d\ln\phi}\right)^2\right).$$

.

The approximate shift invariance $\phi \to \phi + c, \;\; c = \mathrm{const}$ permitting slow-roll inflation for a minimally coupled inflaton scalar field transforms here to the approximate scale (dilatation) invariance
$\phi \to c\phi, \;\; R \to c^2 R, \;\; x^\mu \to x^\mu / c, \;\; \mu = 0, ..3$
in the physical (Jordan) frame. Of course, this symmetry needs not be fundamental, i.e. existing in some more microscopic model at the level of its action.

Generality of inflation

Theorem. In these models, there exists an open set of classical solutions with a non-zero measure in the space of initial conditions at curvatures much exceeding those during inflation which have a metastable inflationary stage with a given number of e-folds.

For the GR inflationary model this follows from the generic late-time asymptotic solution for GR with a cosmological constant found in A. A. Starobinsky, JETP Lett. 37, 55 (1983). For the $R + R^2$ model, this was proved in A. A. Starobinsky and H.-J. Schmidt, Class. Quant. 4, 695 (1987).

Generic initial conditions near a curvature singularity in these models: anisotropic and inhomogeneous (though quasi-homogeneous locally).

1. Modified gravity models (the $R+R^2$ and Higgs ones). Structure of the singularity in terms of a local Bianchi I type metric:

$$ds^2 = dt^2 - \sum_{i=1}^{3} a_i |t|^{2p_i} dx_i^2, \; 0 < s < 3/2, \; u = s(2-s)$$

where $s = \sum_i p_i, \; u = \sum_i p_i^2$. Here $R \propto |t|^{1-s} \to \infty$ (for $1 < s < 3/2$, otherwise it approaches a constant) and $R^2 \ll R_{\alpha\beta}R^{\alpha\beta}$. No infinite number of BKL oscillations.

2. GR model with a very flat potential.

A similar behaviour but with $s = 1, \; u < 1$ and with negligible potential.

In both cases, spatial gradients may become important for some period before the beginning of inflation.

Conclusions

- At present, cosmology requires the introduction of four fundamental constants to describe observational data, additional to those known from laboratory physics.
- One new constant has been discovered recently $n_s - 1 \approx -0.04$ but this value has been predicted by some inflationary models including the pioneer one (1980).
- Regarding the present dark energy:
  a) still no statistically significant deviation from an exact cosmological constant;
  b) one constant is sufficient to describe its properties;
  c) no more than one new "coincidence problem".
- Regarding the primordial dark energy driving inflation in the early Universe:
  a number of inflationary models having only one free parameter can explain all existing observational data.

- Namely, there exists a class of inflationary models having $n_s - 1 = \frac{2}{N}$ and $r \sim \frac{10}{N^2}$ which is most favoured by the Planck and other recent observational data. This class includes the one-parametric pioneer $R + R^2$ and Higgs inflationary models in modified (scalar-tensor) gravity, and more general two-parametric models including a GR model with a very flat inflaton potential.
- Inflation is generic in this models.
- Non-Gaussianity of primordial perturbations is small, as in all one-field slow-roll inflationary models.
- The most critical observational test for these models is small, but not too small value of r.