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Two-Sided Estimate for the Torsional Rigidity of Convex Domain Generalizing the Polya–Szegö and Makai Inequalities

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Abstract—We establish generalizations of the classical Polya–Szegö and Makai inequalities which estimate the torsional rigidity of a convex domain. The main idea of the proof is to apply a new exact isoperimetric inequality for Euclidean moments of a domain with respect to its boundary. This inequality has a wide class of extremal domains and is of independent interest itself.

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1. INTRODUCTION

Let G be a simply connected planar domain with nondegenerate boundary. One of the important characteristics of G is the functional

$$\mathbf{P}(G) := 2\int_G \mathrm{u}(x,G)\mathrm{dA}, \tag{1}$$

called torsional rigidity in elasticity theory, and flow in hydrodynamics. Here $\mathrm{u}(x,G)$ is the stress function that satisfies the equation $\triangle u = -2$ in G and the boundary condition $u|_{\partial G} = 0$, while the differential area element is denoted by dA. It is well known that the stress function exists and is uniquely determined (see [1, 2]).

The problem of approximate calculation of torsional rigidity with the help of geometric characteristics of a given domain originates from the works by Lord Rayleigh and Cauchy; it was related to the accuracy of a torsion pendulum [3, 4]. The problem became widely known after the publications by B. de Saint-Venant. The mentioned names are associated with famous approximate formulas for calculating the functional (1), they are based on exact formulas for some domains. Later it was proved that some of these formulas, in fact, give one-sided estimates for $\mathbf{P}(G)$.

B. de Saint-Venant also determined another important area of research, the study of the properties of physical functionals of domains. This direction is often expressed in the form of isoperimetric inequalities. For example, the statement which claims that a disk has the maximal stiffness among all domains with a given area (B. de Saint-Venant) is equivalent to the Saint-Venant–Polya isoperimetric inequality [1, 6]

$$\mathbf{P}(G) \le \mathbf{A}(G)^2/(2\pi),$$

where $\mathbf{A}(G)$ is the area of G. Moreover, the equality here is only attained for the case of a disc.

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Denote by $\rho(x,G)$ the distance function from a point x to the boundary of G. The geometric functional defined by the equality

$$\mathbf{I}_p(G) = \int\limits_G \rho(x,G)^p \mathrm{dA}, \tag{2}$$

is called the Euclidean moment of G with respect to its boundary of order p ($p > -1$) (see [8]). Obviously, the functional (2) can be considered as a geometric integral functional of planar domain that generalizes the notion of area.

Avkhadiev [7, 9] showed that the torsional rigidity and the Euclidean moment of inertia are comparable in the class of simply connected domains. Moreover, the following two-sided estimates were proved

$$\mathbf{I}_2(G) \le \mathbf{P}(G) \le 64\,\mathbf{I}_2(G). \tag{3}$$

On the one hand, the two-sided inequality shows that the Saint-Venant problem does not have a simple solution in terms of classical geometric characteristics. On the other hand, the Saint-Venant problem was gradually reduced to the problem of finding approximate formulas with sharp bounds. The left-hand side of the inequality in (3) was strengthened in [10]. Namely, it was shown that

$$\frac{3}{2}\,\mathbf{I}_2(G) < \mathbf{P}(G). \tag{4}$$

As far as we know, the constants 64 in (3) and 3/2 in (4) are not optimal. The right-hand side of the inequality for the torsional rigidity in the class of simply connected domains has not been improved yet.

In practice, calculations of the torsional rigidity of a domain are performed with the use of various approximate methods and software products. After obtaining an approximate value of the torsional rigidity of a particular domain, accurate to a few decimal digits, it is often difficult to estimate its adequacy. Therefore, to estimate the obtained result, it is reasonable to use exact isoperimetric inequalities that give the upper and lower bounds for the functional $\mathbf{P}(G)$. From this point of view, isoperimetric inequalities are indispensable; it is desirable to include them in software products for testing the results of approximate calculations.

Further, if we restrict ourselves by consideration of the class of convex domains, the Saint-Venant problem can be solved with the use of the classical geometric characteristics of domains. G. Polya and G. Szegö [2] showed that for any convex domain the inequality

$$\frac{1}{2}\,\mathbf{A}(G)\boldsymbol{\rho}(G)^2 \le \mathbf{P}(G), \tag{5}$$

holds; here $\boldsymbol{\rho}(G)$ is the radius of the maximal circle contained in G. The equality in (5) is attained for a disk. E. Makai [11] obtained the opposite inequality

$$\mathbf{P}(G) < \frac{4}{3}\,\mathbf{A}(G)\boldsymbol{\rho}(G)^2. \tag{6}$$

The constant 4/3 is the best possible, and is attained at the limit, for example, on a sequence of rectangles degenerating into a "needle". Inequalities (5) and (6) already allow us to construct approximate formulas for the class of convex domains.

In fact, Makai [11] proved another inequality

$$\mathbf{P}(G) \le 4\,\mathbf{A}(G)\boldsymbol{\rho}(G)^2, \tag{7}$$

which is similar to (6), but the latter inequality is valid for simply connected domains of finite area.

Later in 1995, Avkhadiev [7, 9] showed that there are simply connected domains with infinite area and finite torsional rigidity. Therefore, an inequality similar to (5) does not take place in the class of simply connected domains. This fact also justified the necessity and the importance of introducing the functional $\mathbf{I}_2(G)$ in the class of simply connected domains.

In fact, it was shown [11] that inequality (6) is a consequence of another isoperimetric inequality

$$\mathbf{P}(G) < 4\,\mathbf{I}_2(G), \tag{8}$$

where G is a convex domain, the constant 4 is the best possible, and the extremal domains are degenerate.

Applying to (8) the inequality from [15]

$$\mathbf{I}_2(G) \leq \frac{\boldsymbol{\rho}(G)^2}{3}\,\mathbf{A}(G) - \frac{\pi\boldsymbol{\rho}(G)^4}{6}$$

we obtain that, for a convex domain of finite area, the following inequality holds

$$\mathbf{P}(G) < \frac{4}{3}\,\mathbf{A}(G)\,\boldsymbol{\rho}(G)^2 - \frac{2\pi}{3}\,\boldsymbol{\rho}(G)^4, \tag{9}$$

it improves Makai's inequality (6).

Further, narrowing the class of considered domains also allowed us to improve the inequality (4). In this way, in [12] the inequality

$$2\,\mathbf{I}_2(G) + \frac{\pi\boldsymbol{\rho}(G)^4}{6} \leq \mathbf{P}(G) \tag{10}$$

was proved for convex domains. As in the Polya–Szegö inequality (5), here the equality is attained for a disk. Note that the latter inequality, in spite of its accuracy, is not optimal in a certain sense (for more details see [12]). Further results and hypotheses on lower bounds for the torsional rigidity with the help of the Euclidean moment of inertia are presented in the same paper. However, inequalities (8) and (10), in contrast to (3), are more suitable for constructing approximate formulas.

The main goal of this article is to generalize some of the inequalities mentioned above to the class of convex domains.

Further attempts to generalize double inequality (3) to Euclidean moments of arbitrary order led only to one-sided inequalities of isoperimetric type (see [13, 14]). For example, using inequality (8) together with the inequality

$$\mathbf{I}_q(G) \leq \frac{p+1}{q+1}\,\mathbf{I}_p(G)\boldsymbol{\rho}(G)^{q-p} - \frac{2\pi(q-p)\boldsymbol{\rho}(G)^{q+2}}{(q+1)(q+2)(p+2)}\,, \tag{11}$$

proved in [15], we can easily obtain that

$$\mathbf{P}(G) < \frac{4(p+1)}{3}\,\mathbf{I}_p(G)\boldsymbol{\rho}(G)^{2-p} - \frac{2\pi\boldsymbol{\rho}(G)^4(2-p)}{3(p+2)}, \tag{12}$$

where G is a convex domain and $q > p$, $-1 \leq p \leq 2$. Note that, in contrast to inequality (8), inequality (11) is valid for simply connected domains. In particular, inequality (12) generalizes Makai's inequality (6) to the case of Euclidean moments of an arbitrary order. The reason for this behavior is that, in the case of simply connected domains, Euclidean moments of different orders are only connected with one another by one-sided isoperimetric inequalities. However, this is not true for the class of convex domains, namely, it is possible to bound Euclidean moments of various orders by two-sided inequalities.

In conclusion, we note that such a variety of inequalities is partly due to the fact that, in the class of convex domains with rectifiable boundary, the following conditions are equivalent

$$\mathbf{L}(G) < +\infty \Leftrightarrow \mathbf{A}(G) < +\infty \Leftrightarrow \mathbf{I}_p(G) < +\infty \Leftrightarrow \mathbf{P}(G) < +\infty;$$

here $-1 < p < +\infty$ and $\mathbf{L}(G)$ is the length of ∂G (see [5, 10, 13]). In particular, Euclidean moments of various orders with proper normalization are also comparable. Therefore, in the class of convex domains, the assumption on existence of inequalities similar to (12), but valid for $p > 2$, is natural.

The aim of this paper is to obtain a new inequality which forms a bridge between two torsional rigidity inequalities for convex domains [17].

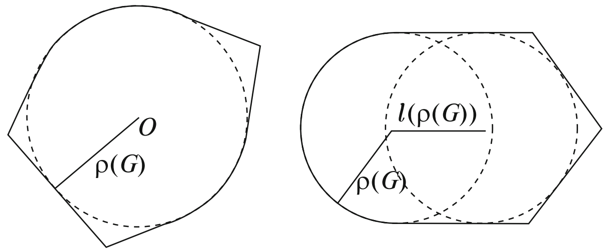

Fig. 1. Examples of a domains from the class Γ.

2. MAIN RESULTS

Let us introduce the following notation

$$G(\mu) := \{z \in G | \rho(z, G) > \mu\}, \quad \mathbf{a}(\mu) := \mathbf{A}(G(\mu)) := \int\limits_{G(\mu)} \mathrm{dA}.$$

We note that $G(\mu)$ is the level set of the distance function $\rho(x, G)$ and $\mathbf{a}(\mu)$ is the area of the level set $G(\mu)$, respectively. Denote by $\boldsymbol{l}(\mu)$ the length of the boundary curve of $G(\mu)$. Let

$$\boldsymbol{l}(\boldsymbol{\rho}(G)) := \lim_{\mu \to \boldsymbol{\rho}(G)} \boldsymbol{l}(\mu).$$

Further, if a domain G_0 can be obtained from G by cutting out a rectangular fragment and combining the remaining parts so that $\boldsymbol{\rho}(G_0) = \boldsymbol{\rho}(G)$, we will say that domain G is a stretching of G_0. On the other hand, it is natural to call the domain G_0 a compression of G. Note that not all domains can be stretched. Indeed, it is easy to see that it is not possible to stretch a triangle or any regular polygon with odd number of sides. If a domain G is not compressible, we put $G_0 \equiv G$. Obviously, a convex domain G is compressible if and only if $\boldsymbol{l}(\boldsymbol{\rho}(G)) \neq 0$. Thus, the above statements allow us to estimate how much a stretching of a convex domain affects the estimation of torsional rigidity.

Consider all convex domains which are either polygons circumscribed about a circle or circular polygons obtained from circumscribed polygons by replacing some of their sides or parts of the sides by arcs of the inscribed circle. Let Γ be the set obtained by adding to the set of domains, described above, all their possible stretching (see Fig. 1).

Theorem 1. *Let G be a convex planar domain of finite area. Then, for $0 \leq q \leq p < \infty$, the following inequality holds true*

$$\mathbf{I}_p(G) \geq \frac{\boldsymbol{\rho}(G)^{p-q}}{(p+1)(p+2)} \left[(q+1)(q+2)\mathbf{I}_q(G) + (p-q)\boldsymbol{l}(\boldsymbol{\rho}(G))\boldsymbol{\rho}(G)^{q+1}\right]. \tag{13}$$

The equality is attained if and only if $G \in \Gamma$.

As we will see, this theorem will play a key role in justifying the inequalities for the torsional rigidity of convex domains given below.

In [17], F.G. Avkhadiev and K.-J. Wirths obtain a new sharp estimate which builds a bridge between different Hardy-type inequalities. Similarly to the Avkhadiev—Wirths inequality, the following theorem allows us, with the help of introducing an additional parameter, to establish a connection between two estimates, (8) and (9), of the torsional rigidity for convex domains.

Theorem 2. *Let G be a convex planar domain of finite area and $0 \leq q \leq 2$. Then for $p \geq q$, the following inequality holds true*

$$\mathbf{P}(G) \leq \frac{4}{3(q+2)} \left[\frac{(p+1)(p+2)}{\boldsymbol{\rho}(G)^{p-2}}\mathbf{I}_p(G) - (p-q)\boldsymbol{l}(\boldsymbol{\rho}(G))\boldsymbol{\rho}(G)^3\right] - \frac{2\pi(2-q)\boldsymbol{\rho}(G)^4}{3(q+2)}. \tag{14}$$

The constants $4(p+1)(p+2)/(3(q+2))$ and $4(p-q)/3(q+2)$ are sharp.

The following statement can be proved by the methods similar to those used in the proof of Theorem 2.

Theorem 3. *Let G be a convex planar domain of finite area and $q > 0$. Then for $0 \le p \le q$, the following inequality holds true*

$$\mathbf{P}(G) \ge \frac{1}{2(q+2)} \left[\frac{(p+1)(p+2)}{\boldsymbol{\rho}(G)^{p-2}} \mathbf{I}_p(G) + (q-p)\boldsymbol{l}(\boldsymbol{\rho}(G))\boldsymbol{\rho}(G)^3 \right] + \frac{\pi q \boldsymbol{\rho}(G)^4}{2(q+2)}. \tag{15}$$

The equality in (15) is attained if G is a disk.

It should be noted that, in both the theorems, the functional enclosed in the square brackets consists of two terms, each of which is comparable to the torsional rigidity of the domain in the sense of Polya and Szegö (see [2]), while the remaining expressions in the right-hand sides of (14) and (15) are infinitesimal values of a higher order. Thus, the functionals in the square brackets are examples of more complicated geometric characteristics of domains comparable to the torsional rigidity.

Application of the concept of isoperimetric monotonicity [16] allows us to give an alternative formulation of Theorem 1 and to look at its statement from a different point of view.

Let us define the functional

$$\mathbf{H}(G;p) := \frac{(p+1)(p+2)}{\boldsymbol{\rho}(G)^{p+1}} \left(\mathbf{I}_p(G) - \frac{\boldsymbol{l}(\boldsymbol{\rho}(G))\boldsymbol{\rho}(G)^{p+1}}{p+1} \right), \tag{16}$$

where p is a parameter, $p > -1$.

First of all, we note that the construction of the functional $\mathbf{H}(G;p)$ is not accidental. Indeed, in [15] isoperimetric monotonicity with respect to a free parameter p was proved for the functional

$$\mathbf{F}(G;p) := \frac{p+1}{\boldsymbol{\rho}(G)^{p+1}} \left(\mathbf{I}_p(G) - \frac{2\pi\boldsymbol{\rho}(G)^{p+2}}{(p+1)(p+2)} \right). \tag{17}$$

Moreover, it was shown that extremal domains, i.e., the domains for which $\mathbf{F}(G;p)$ is a constant, form a two-parameter family of domains of Bonnesen type [15]. In terms of the above definitions, a Bonnesen type domain is a stretching of a circle. Finally, we can see some analogies in the construction of functionals (16) and (17) from the formula

$$\mathbf{I}_p(B) = \frac{2\pi\boldsymbol{\rho}(B)^{p+2}}{(p+1)(p+2)} + \frac{\boldsymbol{l}(\boldsymbol{\rho}(B))\boldsymbol{\rho}(B)^{p+1}}{p+1},$$

which is valid for an arbitrary domain B of Bonnesen type.

Theorem 4. *Let G be a convex planar domain of finite area and G_0 be a compression of G. Then,*

1) if G does not belong to the class of domains Γ, then $\mathbf{H}(G;p)$ is a strictly increasing function of p;

2) if G is a domain from the class Γ, then $\mathbf{H}(G;p) \equiv \mathbf{L}(G_0)$.

Note that domains of Bonnesen type are extremal both for the functional $\mathbf{F}(G;p)$ and for the functional $\mathbf{H}(G;p)$.

3. AUXILIARY STATEMENTS

First, we will prove the following two statements [13] which are necessary to justify Theorem 1.

Lemma 1. *Let G be a convex planar domain and $0 \le \mu \le \boldsymbol{\rho}(G)$. Then $G(\mu)$ is also a convex domain, and the set $G(\boldsymbol{\rho}(G)) := \{x \in G : \rho(x,G) = \boldsymbol{\rho}(G)\}$ is either a singleton or a segment of length $\boldsymbol{l}(\boldsymbol{\rho}(G))/2$.*

Lemma 2. *Let G be a convex planar domain and $\mathbf{A}(G) < +\infty$. Then the following inequality holds true*

$$\mathbf{a}(\mu) \ge \mathbf{A}(G) \left(1 - \frac{\mu}{\boldsymbol{\rho}(G)}\right)^2 + \boldsymbol{l}(\boldsymbol{\rho}(G))\mu \left(1 - \frac{\mu}{\boldsymbol{\rho}(G)}\right), \tag{18}$$

where $0 \le \mu \le \boldsymbol{\rho}(G(\mu))$. The equality in (18) is attained for all $\mu \Leftrightarrow G \in \Gamma$.

Proof of Lemma 2. Inequality (18) is proved in [13]. It remains to verify the last assertion of the lemma concerning the equality case.

First of all, we note that if $G \in \Gamma$, then $G(\mu) \in \Gamma$ $(0 \le \mu < \boldsymbol{\rho}(G))$. Now, let $G \in \Gamma$ and $\boldsymbol{l}(\boldsymbol{\rho}(G)) = 0$. Then, using the methods of elementary geometry, we can easily verify that, in fact, the function $\mathbf{l}(\mu)\boldsymbol{\rho}(G(\mu))^{-1}$ does not depend on μ in its domain of definition, in particular,

$$\frac{\mathbf{l}(\mu)}{\boldsymbol{\rho}(G(\mu))} = \frac{\mathbf{L}(G)}{\boldsymbol{\rho}(G)}.$$

Since G is circumscribed about a circle, the following equality holds true

$$\mathbf{A}(G) = \frac{\mathbf{L}(G)\boldsymbol{\rho}(G)}{2}. \tag{19}$$

From the latter two equalities we obtain

$$\frac{\mathbf{a}(\mu)}{\boldsymbol{\rho}(G(\mu))^2} = \frac{\mathbf{A}(G)}{\boldsymbol{\rho}(G)^2}. \tag{20}$$

Now, let $\boldsymbol{l}(\boldsymbol{\rho}(G)) \neq 0$. Then equality (20) takes the form

$$\frac{\mathbf{a}(\mu) - \boldsymbol{\rho}(G(\mu))\boldsymbol{l}(\boldsymbol{\rho}(G))}{\boldsymbol{\rho}(G(\mu))^2} = \frac{\mathbf{A}(G) - \boldsymbol{\rho}(G)\boldsymbol{l}(\boldsymbol{\rho}(G))}{\boldsymbol{\rho}(G)^2}.$$

With the use of simple algebraic transformations, it is easy to establish that the latter equality coincides with the equality case in inequality (18). This completes the proof of the sufficient condition of the assertion about the equality in (18).

To prove the necessary condition, we split the proof into several steps.

1. Let G be a circular sector of radius a and angle $\gamma = 2\pi\lambda$, $0 < \lambda \le 1/2$. We will verify that, in this case,

$$\mathbf{A}(G) > \frac{\mathbf{L}(G)\boldsymbol{\rho}(G)}{2}. \tag{21}$$

Indeed, we have $\mathbf{L}(G) = (2 + 2\pi\lambda)a$, $\mathbf{A}(G) = \pi\lambda a^2$, and $\boldsymbol{\rho}(G) = \dfrac{\sin \pi\lambda}{1 + \sin \pi\lambda}\, a$. Then, (21) is equivalent to the evident inequality $\pi\lambda > \sin \pi\lambda$.

2. Now, let G be a convex circular polygon such that only one of its sides is an arc of a circle of radius R.

Then we separate the polygon into triangles and consider the curvilinear triangle OA_1A_2 (see Fig. 2).

a) Consider the case where the triangle OA_1A_2 is isosceles. The area of the triangle OA_1A_2 and the length of the arc $\mathbf{L}(A_1A_2)$ are respectively equal to

$$\mathbf{A}(\Delta) = R^2 \arcsin(\boldsymbol{\rho}(G)/R) - R\boldsymbol{\rho}(G) + \boldsymbol{\rho}(G)^2, \quad \mathbf{L}(A_1A_2) = 2R \arcsin(\boldsymbol{\rho}(G)/R).$$

Let $g(\rho) := R \arcsin(\rho/R) - \rho$. Since $g(0) = 0$ and $g'(\rho) > 0$, we get

$$g(\boldsymbol{\rho}(G)) = R \arcsin(\boldsymbol{\rho}(G)/R) - \boldsymbol{\rho}(G) > 0.$$

The latter inequality is equivalent to (21).

b) Let OA_1A_2 be an arbitrary circular triangle with one circular arc. Then we draw the chord A_1A_2 and drop the altitude h of the rectilinear triangle ΔOA_1A_2 from the vertex O (see Fig. 3).

We will move the vertex O so that the length of the altitude h remains constant. Such a change in the vertex O keeps the area and the arc length. It is evident that with this movement we can achieve that the triangle becomes isosceles. But the case of isosceles triangle was considered in the case a). This completes the proof of the assertion about the equality cases. $\square$

Remark. There are convex domains whose level sets, starting with some μ, are domains from the set Γ. For example, consider a trapezoid, one side of which does not touch the maximal inscribed circle. Let b be the point of intersection of the bisectors, closest to the boundary of the domain. Then, for any $\mu \in [\rho(b, G), \boldsymbol{\rho}(G)]$, the level set of the trapezoid is a triangle (see Fig. 4).

Now, for an arbitrary convex domain of finite area, we will define a corresponding domain from the set Γ. We associate with a domain G a domain $G^{\Diamond}$ $(\in \Gamma)$ such that $\boldsymbol{\rho}(G^{\Diamond}) = \boldsymbol{\rho}(G)$, $\boldsymbol{l}(\boldsymbol{\rho}(G^{\Diamond})) = \boldsymbol{l}(\boldsymbol{\rho}(G))$,

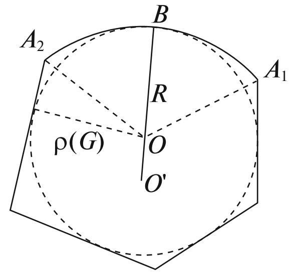

Fig. 2. A circumscribed circular polygon.

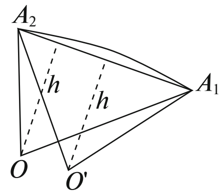

Fig. 3. Variations of a curvilinear triangle.

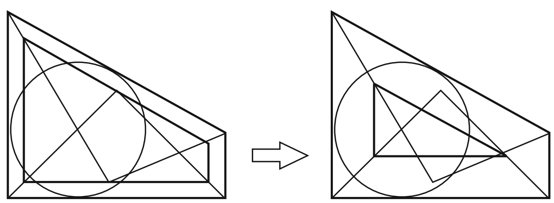

Fig. 4. Level sets of a trapezoid.

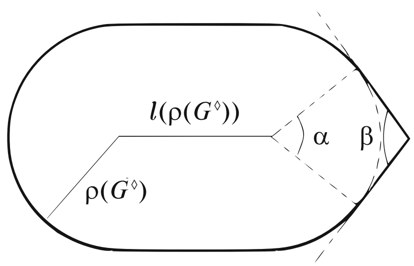

Fig. 5. An example of domain $G^\Diamond$.

and $\mathbf{A}(G^\Diamond) = \mathbf{A}(G)$. Note that, if there are several domains in Γ with the indicated properties, we select and fix one of them.

For example, as $G^\Diamond(\in \Gamma)$ we can take the domain that is obtained from a stretching of a disk by replacing a part of its boundary circular arc with the union of two segments tangent to the arc and forming an angle $\beta \in (0;\pi)$ (see Fig. 5). It is clear that we can assume that the values of functionals $\boldsymbol{\rho}(G^\Diamond)$ and $\boldsymbol{l}(\boldsymbol{\rho}(G^\Diamond))$ coincide with the corresponding values of the functionals for G.

Now will show that there exists a unique $\beta \in (0;\pi)$ such that $\mathbf{A}(G^\Diamond) = \mathbf{A}(G)$. For this, we will express the area of the domain in terms of the angle $\alpha = \pi - \beta$

$$\mathbf{A}(G^\Diamond) = \left(\pi + \frac{\alpha}{2} + \tan\frac{\alpha}{2}\right)\boldsymbol{\rho}(G^\Diamond)^2 + \boldsymbol{\rho}(G^\Diamond)\boldsymbol{l}(\boldsymbol{\rho}(G^\Diamond)).$$

The last equality shows that $\mathbf{A}(G^\Diamond)$, depending continuously on α, tends strictly monotonically to infinity as $\alpha \to \pi$, and for $\alpha = 0$ we have the area of a Bonnesen-type domain [15] which gives a

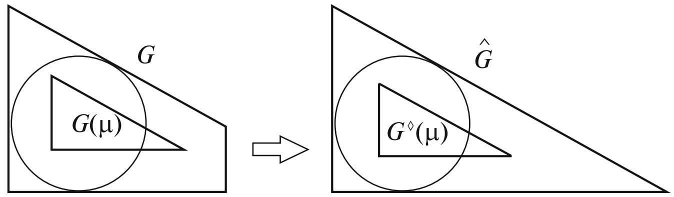

Fig. 6. An example of domain $\widehat{G}$.

lower bound for the area. Therefore, there is a unique domain $G^{\Diamond}$ of the described type such that $\mathbf{A}(G^{\Diamond}) = \mathbf{A}(G)$.

For the functionals of domain $G^{\Diamond}$, we keep the introduced notation, providing them with the rhombus symbol if necessary. For example, $G^{\Diamond}(\mu^{\Diamond}) = \{x \in G^{\Diamond} | \rho(x, G^{\Diamond}) > \mu^{\Diamond}\}$ and $\mathbf{a}^{\Diamond}(\mu) = \mathbf{A}(G^{\Diamond}(\mu))$. Since $\mathbf{a}(\mu)$ is strictly monotonically decreasing, the inverse function $\mu(\mathbf{a})$ is well defined. We will use the notation $G(\mathbf{a}) := G(\mu(\mathbf{a}))$.

Define a correspondence between subdomains $G(\mathbf{a})$ and $G^{\Diamond}(\mathbf{a}^{\Diamond})$: $G(\mathbf{a})$ corresponds to $G^{\Diamond}(\mathbf{a}^{\Diamond})$ if and only if they have the same levels, i.e. $\mu(\mathbf{a}) = \mu^{\Diamond}(\mathbf{a}^{\Diamond})$.

Applying Lemma 2 and the introduced correspondence, we obtain the estimation

$$\begin{aligned}\mathbf{a}(\mu) &\geq \mathbf{A}(G)\left(1 - \frac{\mu}{\boldsymbol{\rho}(G)}\right)^2 + \boldsymbol{l}(\boldsymbol{\rho}(G))\mu\left(1 - \frac{\mu}{\boldsymbol{\rho}(G))}\right) \\ &= \mathbf{A}(G^{\Diamond})\left(1 - \frac{\mu^{\Diamond}}{\boldsymbol{\rho}(G^{\Diamond})}\right)^2 + \boldsymbol{l}(\boldsymbol{\rho}(G^{\Diamond}))\mu^{\Diamond}\left(1 - \frac{\mu^{\Diamond}}{\boldsymbol{\rho}(G^{\Diamond})}\right) = \mathbf{a}^{\Diamond}(\mu^{\Diamond}),\end{aligned} \tag{22}$$

where $0 \leq \mu \leq \boldsymbol{\rho}(G)$.

Theorem 5. *Let G be a convex planar domain of finite area. Let $P(t)$ be a non-constant, absolutely continuous function on $(0, \boldsymbol{\rho}(G))$, and $dP(t) \geq 0$. Then, the following inequality holds true*

$$\int\limits_G P(\rho(x, G))\mathrm{dA} \geq \int\limits_{G^{\Diamond}} P(\rho(x, G^{\Diamond}))\mathrm{dA},$$

where $G^{\Diamond}$ is the polygon corresponding to G. The equality is attained for domains from the set Γ.

From the definition of the Lebesgue integral we obtain

$$\int\limits_G P(\rho(x, G))\mathrm{dA} = \int\limits_0^{\mathbf{A}(G)} P(\mu(\mathbf{a}))\mathrm{dA} = \mathbf{A}(G)P(0) + \int\limits_0^{\boldsymbol{\rho}(G)} \mathbf{a}(\mu)dP(\mu). \tag{23}$$

Then, from the introduced correspondence, (22), and (23), we obtain the needed inequality from Theorem 5. □

Let us recall the functional discussed in [15]. For a simply connected domain G with $p \geq 0$, we put

$$\mathbf{i}_p(\mu) := p \int\limits_{\mu}^{\boldsymbol{\rho}(G)} t^{p-1}\mathbf{a}(t)\mathrm{d}t, \tag{24}$$

Note that in the class of convex domains with finite area, there is no need in additional restrictions on the parameter p, in contrast to the case considered in [15]. For $\mu = 0$ this is the Euclidean moment of domain G with respect to its boundary, i.e., $\mathbf{i}_p(0) = \mathbf{I}_p(G)$.

Let us prove the following auxiliary statement which is, in a sense, inverse to the similar lemma proved in [15].

Lemma 3. *Let G be a convex planar domain of finite area. Then, for $0 \le \mu \le \boldsymbol{\rho}(G)$, the following inequality holds true*

$$\mathbf{i_q}(\mu) \ge \frac{\mathbf{I}_q(G) y_q(\mu)}{2\boldsymbol{\rho}(G)^{q+2}} + \frac{q\boldsymbol{l}(\boldsymbol{\rho}(G))\mu^q(\boldsymbol{\rho}(G)-\mu)^2}{2\boldsymbol{\rho}(G)}, \tag{25}$$

where $q \ge 0$ and

$$y_q(\mu) = q(q+1)(q+2)\int\limits_{\mu}^{\boldsymbol{\rho}(G)} t^{q-1}(\boldsymbol{\rho}(G)-t)^2 \mathrm{dt}.$$

The equality in (25), for all admissible μ, takes place only for domains from the class Γ.

Proof of Lemma 3. For $\mu = 0$ and $\mu = \boldsymbol{\rho}(G)$, inequality (25) turns into the equality. Therefore, we can assume that $0 < \mu < \boldsymbol{\rho}(G)$.

Consider the level sets of the function $\boldsymbol{\rho}(x, G(\mu))$. It is clear that

$$\rho(x, G(\mu)) = \rho(x, G) - \mu \quad (x \in G(\mu)), \quad \mathbf{b}(s) = \mathbf{a}(s+\mu) \quad (0 \le s \le \boldsymbol{\rho}(G(\mu))),$$

where $\mathbf{b}(s)$ is the area of a level set of the function $\rho(x, G(\mu))$. Using representation (23), we get

$$\int\limits_{G(\mu)} (\rho(x, G(\mu)) + \mu)^p \mathrm{dA} = \mathbf{a}(\mu)\mu^p + \int\limits_0^{\boldsymbol{\rho}(G(\mu))} \mathbf{b}(s)\mathrm{d}(s+\mu)^p = \mathbf{a}(\mu)\mu^p + \int\limits_{\mu}^{\boldsymbol{\rho}(G)} \mathbf{a}(t) dt^p.$$

Thus,

$$\mathbf{i}_p(\mu) = \int\limits_{G(\mu)} (\rho(x, G(\mu)) + \mu)^p \mathrm{dA} - \mathbf{a}(\mu)\mu^p. \tag{26}$$

Let $G^{\diamondsuit}(\mu)$ be the polygon corresponding to $G(\mu)$. We have

$$\mathbf{a}(\mu) = \mathbf{A}((G^{\diamondsuit}(\mu))). \tag{27}$$

Applying Theorem 5 to $G(\mu)$ and setting $P(t) = (t+\mu)^p$, we obtain

$$\int\limits_{G(\mu)} (\rho(x, G(\mu)) + \mu)^p \mathrm{dA} \ge \int\limits_{G^{\diamondsuit}(\mu)} (\rho(x, G^{\diamondsuit}(\mu)) + \mu)^p \mathrm{dA}. \tag{28}$$

From (26), (27), and (28), we get

$$\mathbf{i}_p(\mu) \ge \int\limits_0^{\boldsymbol{\rho}(G(\mu))} \mathbf{b}^{\diamondsuit}(s)\mathrm{d}(s+\mu)^p = \int\limits_{\mu}^{\boldsymbol{\rho}(G)} \mathbf{b}^{\diamondsuit}(t-\mu)\mathrm{d}t^p.$$

Note that in the general case, $\mathbf{b}^{\diamondsuit}(t-\mu)$ does not coincide with $\mathbf{a}^{\diamondsuit}(t)$ for $\boldsymbol{\rho}(G^{\diamondsuit}) \ge t$. Since $G^{\diamondsuit}(\mu)$ is the polygon corresponding to $G(\mu)$, we have $\mathbf{A}(G^{\diamondsuit}(\mu)) = \mathbf{a}(\mu) \geqslant \mathbf{a}^{\diamondsuit}(\mu)$. Denote by $\widehat{G}$ a polygon such that the boundary of $G^{\diamondsuit}(\mu)$ is the level curve of the domain $\widehat{G}$ located at a distance of μ from $\partial\widehat{G}$. Then $\boldsymbol{\rho}(\widehat{G}) = \boldsymbol{\rho}(G)$. The domain $\widehat{G}$ depends on μ. Lemma 3 implies that the area of $\widehat{G}$ does not coincide with the area of G (see Fig. 6).

Denote

$$\mathbf{i}_p^{\natural}(\mu) := \int\limits_{\mu}^{\boldsymbol{\rho}(G)} \mathbf{b}^{\diamondsuit}(t-\mu)\mathrm{d}t^p.$$

Then,

$$\mathbf{i}_p(\mu) \ge \mathbf{i}_p^{\natural}(\mu), \tag{29}$$

where $0 \le \mu \le \boldsymbol{\rho}((D))$. Lemma 2 implies the formula for calculation of the area of a level set for domain $\widehat{G} \in \Gamma$

$$\mathbf{a}(\widehat{G}(\mu)) = \mathbf{A}(\widehat{G})\left(1 - \frac{\mu}{\boldsymbol{\rho}(\widehat{G})}\right)^2 + \boldsymbol{l}(\boldsymbol{\rho}(\widehat{G}))\mu\left(1 - \frac{\mu}{\boldsymbol{\rho}(\widehat{G})}\right), \tag{30}$$

where $0 \le \mu \le \boldsymbol{\rho}(\widehat{G})$. Considering the functional $\mathbf{i}_q(\mu)$ and applying Lemma 2 for $q \ge 0$, we obtain

$$\mathbf{i}_p^{\natural}(\mu) = q \int\limits_{\mu}^{\boldsymbol{\rho}(G)} t^{q-1}\left(\mathbf{A}(\widehat{G})\left(1 - \frac{t}{\boldsymbol{\rho}(G)}\right)^2 + \boldsymbol{l}(\boldsymbol{\rho}(G))t\left(1 - \frac{t}{\boldsymbol{\rho}(G)}\right)\right)\mathrm{d}t. \tag{31}$$

Next, we express from (30) the area of domain $\widehat{G}$ and transform (31) to the form

$$\mathbf{i}_q(\mu) \ge \frac{\mathbf{A}(\widehat{G}(\mu))y_q(\mu)}{(q+1)(q+2)(\boldsymbol{\rho}(G)-\mu)^2} - \frac{\boldsymbol{l}(\boldsymbol{\rho}(G))(\mu y_q(\mu) - p(\boldsymbol{\rho}(G)-\mu)z(\mu))}{(q+1)(q+2)\boldsymbol{\rho}(G)(\boldsymbol{\rho}(G)-\mu)} \tag{32}$$

where $z(\mu) = \boldsymbol{\rho}(G)^{p+2} - (p+2)\boldsymbol{\rho}(G)\mu^{p+1} + (p+1)\mu^{p+2}$. The following equalities can be easily obtained from the definitions of the functional $\mathbf{i}_q(\mu)$ and the domain $\widehat{G}(\mu)$

$$[\mathbf{i}_q(\mu)]' = -q\mu^{q-1}\mathbf{a}(\mu) = -q\mu^{q-1}\mathbf{A}(\widehat{G}(\mu)).$$

Substituting the latter inequality into (32), we obtain the first order differential inequality

$$\mathbf{i}_q(\mu) \ge \frac{y_q(\mu)\mathbf{i}_q'(G)}{y'(\mu)} - \frac{\boldsymbol{l}(\boldsymbol{\rho}(G))(\mu y_q(\mu) - q(\boldsymbol{\rho}(G)-\mu)z(\mu))}{(q+1)(q+2)\boldsymbol{\rho}(G)(\boldsymbol{\rho}(G)-\mu)}.$$

Since $y'(\mu) = -p(p+1)(p+2)\mu^{p-1}(\boldsymbol{\rho}(G)-\mu)^2$, we see that $y'(\mu)$ is strictly negative for $0 \le \mu < \boldsymbol{\rho}(G)$. Multiplying the latter inequality by $y'(\mu)y_q(\mu)^{-2}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\left[\frac{\mathbf{i}_q(\mu)}{y_q(\mu)}\right] \ge \frac{\boldsymbol{l}(\boldsymbol{\rho}(G))(\mu y_q(\mu) - q(\boldsymbol{\rho}(G)-\mu)z(\mu))y'(\mu)}{(q+1)(q+2)\boldsymbol{\rho}(G)(\boldsymbol{\rho}(G)-\mu)}.$$

Integrating this inequality over $[0;\mu]$, we obtain

$$\frac{\mathbf{i}_q(\mu)}{y_q(\mu)} - \frac{\mathbf{I}_q(G)}{2\boldsymbol{\rho}(G)^{q+2}} \ge \frac{\boldsymbol{l}(\boldsymbol{\rho}(G))}{(q+1)(q+2)}\int\limits_0^{\mu}\frac{y'(\mu)(\mu y_q(\mu) - q(\boldsymbol{\rho}(G)-\mu)z(\mu))}{\boldsymbol{\rho}(G)(\boldsymbol{\rho}(G)-\mu)y_q(\mu)^2}\mathrm{d}\mu.$$

To calculate the last integral, we use the change of variables

$$h(\mu) = \frac{\mu y_q(\mu) - q(\boldsymbol{\rho}(G)-\mu)z(\mu)}{\boldsymbol{\rho}(G)(\boldsymbol{\rho}(G)-\mu)}$$

and take into account that $y_q(\mu) = h'(\mu)(\boldsymbol{\rho}(G)-\mu)^2$. We get

$$\int\limits_0^{\mu}\frac{y'(\mu)h(\mu)}{y(\mu)^2}\,\mathrm{d}\mu = \int\limits_0^{\mu}\frac{h(\mu)h''(\mu)}{(\boldsymbol{\rho}(G)-\mu)^2h'(\mu)^2}\,\mathrm{d}\mu - 2\int\limits_0^{\mu}\frac{h(\mu)}{y'(\mu)(\boldsymbol{\rho}(G)-\mu)^3}\,\mathrm{d}\mu$$

$$= -\int\limits_0^{\mu}\frac{h(\mu)}{(\boldsymbol{\rho}(G)-\mu)^2}\,\mathrm{d}\left(\frac{1}{h'(\mu)}\right) - 2\int\limits_0^{\mu}\frac{h(\mu)}{h'(\mu)(\boldsymbol{\rho}(G)-\mu)^3}\,\mathrm{d}\mu = -\frac{h(\mu)}{h'(\mu)(\boldsymbol{\rho}(G)-\mu)^2}$$

$$-\ \frac{q}{2\boldsymbol{\rho}(G)} + \int\limits_0^{\mu}\frac{1}{h'(\mu)}\frac{d}{d\mu}\left[\frac{h(\mu)}{(\boldsymbol{\rho}(G)-\mu)^2}\right]\mathrm{d}\mu - 2\int\limits_0^{\mu}\frac{h(\mu)}{h'(\mu)(\boldsymbol{\rho}(G)-\mu)^3}\,\mathrm{d}\mu$$

$$= -\frac{h(\mu)}{h'(\mu)(\boldsymbol{\rho}(G)-\mu)^2} - \frac{q}{2\boldsymbol{\rho}(G)} + \int\limits_0^{\mu}\frac{h'(\mu)(\boldsymbol{\rho}(G)-\mu)^2 + 2h(\mu)(\boldsymbol{\rho}(G)-\mu)}{h'(\mu)(\boldsymbol{\rho}(G)-\mu)^4}\,\mathrm{d}\mu$$

$$-2\int_0^{\mu}\frac{h(\mu)}{h'(\mu)(\boldsymbol{\rho}(G)-\mu)^3}\,\mathrm{d}\mu=-\frac{h(\mu)}{h'(\mu)(\boldsymbol{\rho}(G)-\mu)^2}-\frac{q}{2\boldsymbol{\rho}(G)}+\int_0^{\mu}\frac{\mathrm{d}\mu}{(\boldsymbol{\rho}(G)-\mu)^2}$$
$$=-\frac{h(\mu)}{h'(\mu)(\boldsymbol{\rho}(G)-\mu)^2}-\frac{q+2}{2\boldsymbol{\rho}(G)}+\frac{1}{\boldsymbol{\rho}(G)-\mu}.$$

Therefore,

$$\frac{\mathrm{i}_q(\mu)}{y_q(\mu)}-\frac{\mathrm{I}_q(G)}{2\boldsymbol{\rho}(G)^{q+2}}\geq\frac{\boldsymbol{l}(\boldsymbol{\rho}(G))}{(q+1)(q+2)}\left(-\frac{h(\mu)}{y_q(\mu)}-\frac{q+2}{2\boldsymbol{\rho}(G)}+\frac{1}{\boldsymbol{\rho}(G)-\mu}\right).$$

Whence the assertion of Lemma 3 follows by substitution of the expression for $h(\mu)$ in the last inequality. □

Proof of Theorem 1. Based on the proved result, we wish to obtain inequality (13). Using the representation of the Euclidean moment from [15] and integrating by parts, we have

$$\mathbf{I}_p(G)=-\frac{p}{q}\int_0^{\boldsymbol{\rho}(G)}\mu^{p-q}\mathrm{d}\mathbf{i}_q(\mu)=-\frac{p}{q}\left(\mu^{p-q}\mathbf{i}_q(\mu)\big|_0^{\boldsymbol{\rho}(G)}-\int_0^{\boldsymbol{\rho}(G)}\mathbf{i}_q(\mu)(p-q)\mu^{p-q-1}\mathrm{d}\mu\right)$$
$$\geq\frac{p}{q}\int_0^{\boldsymbol{\rho}(G)}\left(\frac{\mathbf{I}_q(G)y_q(\mu)}{2\boldsymbol{\rho}(G)^{q+2}}+\frac{q\boldsymbol{l}(\boldsymbol{\rho}(G))\mu^q(\boldsymbol{\rho}(G)-\mu)^2}{2\boldsymbol{\rho}(G)}\right)(p-q)\mu^{p-q-1}\mathrm{d}\mu$$
$$=\frac{\boldsymbol{\rho}(G)^{p-q}(\mathbf{I}_q(G)(q+1)(q+2)+\boldsymbol{l}(\boldsymbol{\rho}(G))(p-q)\boldsymbol{\rho}(G)^{q+1})}{(p+1)(p+2)}.$$

Thus, we obtained the inequality which coincides with (13).

It is easy to show that domains from the set Γ are extremal in inequality (13). Indeed, consider a domain $G\in\Gamma$ with homothetic level curves of the distance function $\mu=\boldsymbol{\rho}(G)(1-k)$, $0\leq\mu\leq\boldsymbol{\rho}(G)$, where k is the homothety coefficient and $\mathbf{l}(\boldsymbol{\rho}(G))=0$. Then, simple calculations leads to the equality

$$\mathbf{I}_p(G)=p\int_0^{\boldsymbol{\rho}(G)}t^{p-1}\mathbf{a}(t)\mathrm{d}tp\boldsymbol{\rho}(G)^p\mathbf{A}(G)\int_0^1(1-k)^{p-1}k^2\mathrm{d}k=p\boldsymbol{\rho}(G)^p\mathbf{A}(G)B(3,p)=\frac{2\boldsymbol{\rho}(G)^p\mathbf{A}(G)}{(p+1)(p+2)}.$$

Substituting the obtained value of the Euclidean moment of the domain G into (13), we obtain the equality.

Now, let G be an arbitrary domain from Γ. Using (19) and the definition of G_0, we can easily check that

$$\mathbf{I}_p(G)=\mathbf{I}_p(G_0)+\frac{\boldsymbol{l}(\boldsymbol{\rho}(G))\boldsymbol{\rho}(G)^{p+1}}{p+1}=\frac{\mathbf{L}(G_0)\boldsymbol{\rho}(G)^{p+1}}{(p+1)(p+2)}+\frac{\boldsymbol{l}(\boldsymbol{\rho}(G))\boldsymbol{\rho}(G)^{p+1}}{p+1}.$$

Thus, from the definition of functional $\mathbf{H}(G;p)$, it follows that $\mathbf{H}(G;p)=\mathbf{L}(G_0)$. Theorem 1 is completely proved. □

Corollary. *For every convex domain G with $\mathbf{A}(G)<+\infty$, we have the inequality*

$$\mathbf{I}_2(G)\geq\frac{1}{2}\mathbf{I}_1(G)\boldsymbol{\rho}(G)+\frac{\boldsymbol{l}(\boldsymbol{\rho}(G))\boldsymbol{\rho}(G)^3}{12}.$$

Proof of Theorem 2. Consider the Makai inequality (8). For $\mathbf{I}_2(G)$ and $0<q<2$, we apply estimate (11):

$$\mathbf{P}(G)\leq 4\mathbf{I}_2(G)\leq\frac{4(q+1)}{3}\boldsymbol{\rho}(G)^{2-q}\mathbf{I}_q(G)-\frac{2\pi(2-q)\boldsymbol{\rho}(G)^4}{3(q+2)}.\tag{33}$$

Let us estimate the functional $\mathbf{I}_q(G)$ in (33) for $p\geq q\geq 0$. From Theorem 1 we obtain

$$\mathbf{P}(G)\leq\frac{4}{3(q+2)}\left(\frac{(p+1)(p+2)\mathbf{I}_p(G)}{\boldsymbol{\rho}(G)^{p-2}}+(p-q)\boldsymbol{l}(\boldsymbol{\rho}(G))\boldsymbol{\rho}(G)^3\right)-\frac{2\pi(2-q)\boldsymbol{\rho}(G)^4}{3(q+2)}.\tag{34}$$

Now, we will verify the sharpness of the constants. Consider a rectangle G with side lengths a and b, $a > b$. The radius of its inscribed circle is $\boldsymbol{\rho}(G) = b/2$, the length of the level curve $\rho(z, G)$ located at a distance $\boldsymbol{\rho}(G)$ from the boundary ∂G is $\boldsymbol{l}(\boldsymbol{\rho}(G)) = 2(a - b)$. Direct calculations give

$$\mathbf{I}_p(G) = p \int\limits_0^{b/2} t^{p-1}(b - 2t)(a - 2t)\mathrm{d}t = \frac{b^{p+1}(a(p+2) - pb)}{2^p(p+1)(p+2)}.$$

Therefore,

$$\lim_{b/a\to 0} \left(\mathbf{P}(G) + \frac{2\pi(2-q)\boldsymbol{\rho}(G)^4}{3(q+2)} + \frac{4(p-q)\boldsymbol{l}(\boldsymbol{\rho}(G))\boldsymbol{\rho}(G)^3}{3(q+2)} \right) \frac{\boldsymbol{\rho}(G)^{p-2}}{\mathbf{I}_p(G)} = \frac{4(p+1)(p+2)}{3(q+2)}.$$

Thus, the constant for the functional $\mathbf{I}_p(G)\boldsymbol{\rho}(G)^{2-p}$ is sharp.

Now, we check the sharpness of the constant for the functional $\boldsymbol{l}(\boldsymbol{\rho}(G))\boldsymbol{\rho}(G)^3$ in (34). To do this, we calculate the following limit:

$$\lim_{b/a\to 0} \frac{1}{\boldsymbol{l}(\boldsymbol{\rho}(G))\boldsymbol{\rho}(G)^3} \left(-\mathbf{P}(G) - \frac{2\pi(2-q)\boldsymbol{\rho}(G)^4}{3(q+2)} + \frac{4(p+1)(p+2)}{3(q+2)\boldsymbol{\rho}(G)^{p-2}} \mathbf{I}_p(G) \right) = \frac{4(p-q)}{3(q+2)}.$$

However, the constant $(2\pi(2-q)\boldsymbol{\rho}(G)^4)/(3(q+2))$ is not sharp.

Proof of Theorem 3. For $p = 0$, from the Polya–Szegö inequality (5) and inequality (11), it follows that

$$\mathbf{A}(G) \geq \frac{(q+1)\mathbf{I}_q(G)}{\boldsymbol{\rho}(G)^q} + \frac{q\pi\boldsymbol{\rho}(G)^2}{q+2}.$$

Applying Theorem 1 to the latter inequality, we obtain Theorem 3. Considering G to be a disk, we obtain the equality in (15).

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