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Limitwise Monotonic Reducibility on Sets and on Pairs of Sets

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Abstract—We study limitwise monotonic sets and pairs of sets. We investigate the properties of limitwise monotonic reducibility between sets and pairs of sets defined in terms of Σ-reducibility corresponding to initial segment of sets. In addition, we obtain a description of Σ-reducibility of families of a special form in terms of lm-reducibility. At the same time we show the relationship of concepts of lm-reducibility and Σ-reducibility between the pairs of sets.

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INTRODUCTION

This article investigates limitwise monitonic sets and pairs of sets properties and limitwise monotonic reducibility (which for the sake of brevity will be also denoted by lm-reducibility) between sets and pairs of sets. Moreover, we consider the relationship between the sets lm-reducibility and Σ-reducibility for the families of special type.

In [1] the authors introduced the concept of Σ-reducibility for the natural numbers subsets families. This notion allows us to consider these sets in their own, i.e., without their representations in the set of natural numbers. Also the study of limitwise monotonic set and a pair of sets in [2] gave rise to the notion of lm-reducibility on the sets and lm-reducibility between pairs of sets. Here we consider lm-reducibility between sets in terms of Σ-definability for the relative initial segment families of these sets.

We give the notion of Σ-reducibility for the natural number subset families in the first part of the article according to [1]. Moreover we prove that the concept of Σ-reducibility is equivalent to that of lm-reducibility between two sets, provided the latter is given by limitwise monotonic operator. Then we consider lm-reducibility and Σ-reducibility of the set to pair of sets. Here we also prove the equivalence of these two notions. Moreover, we give results on lm-reducibility between pairs of sets. The basic notions on the limitwise monotonic functions and sets study can be found in [2–4]. We also borrow the notation from [1] and [5].

1. Σ-REDUCIBILITY AND lm-REDUCIBILITY ON SETS

The definitions of limitwise monotonic function, limitwise monotonic operator and lm-reducibility between two sets belong to [5]. Moreover, certain interesting results on lm-reducibility can be found in [6]. Here we give a detailed study of Σ-*reducibility* for the set families.

Definition 1. Consider the family $\mathcal{S} \subseteq \mathcal{P}(\mathbb{N})$. An $\mathcal{S}$-tuple is the set of type $\{\langle n, k\rangle\} \oplus (X_1 \oplus \cdots \oplus X_n)$, here $n, k \in \mathbb{N}$ and $X_i \in \mathcal{S}$ for all $i = \overline{1, n}$. We denote by $K_{\mathcal{S}}$ the set of all $\mathcal{S}$-tuples.

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Definition 2. We say that a family $\mathcal{S}_0 \subseteq \mathcal{P}(\mathbb{N})$ is Σ-*reducible* to a family $\mathcal{S}_1 \subseteq \mathcal{P}(\mathbb{N})$ (and denote it by $\mathcal{S}_0 \sqsubseteq_\Sigma \mathcal{S}_1$) if

$$\mathcal{S}_0 \cup \{\varnothing\} = \{\Phi(X \oplus Y \oplus E(\mathcal{S}_1)) \mid X \in K_{\mathcal{S}_1}\}$$

for some enumeration operator Φ and a set $Y \in K_{\mathcal{S}_1}$. Here $E(\mathcal{S}) = \{n \in \mathbb{N} \mid (\exists F \in \mathcal{S})[D_n \subseteq F]\}$ and D_n stands for a finite set with the canonical index n.

Consider two arbitrary sets A and B. We fix relative to A and B initial segment families as the necessary families, namely, $\mathcal{S}(A) = \{\mathbb{N} \upharpoonright a : a \in A\}$ and $\mathcal{S}(B) = \{\mathbb{N} \upharpoonright b : b \in B\}$, respectively. Now for the given set $\mathcal{S}(B)$ the set $E(\mathcal{S}(B))$ of Definition 2 is computably enumerable. Moreover, the family $\mathcal{S}(B)$ consists only of computably enumerable sets. Thus we may omit the sets Y and $E(\mathcal{S}(B))$ of $\mathcal{S}$-tuple for the family $\mathcal{S}(B)$ of Definition 2. So we are able to write down Σ-reducibility between the initial segment families of the sets A and B with the help of the enumeration operator Φ as follows:

$$\{\mathbb{N} \upharpoonright a : a \in A\} = \{\Phi(\{\langle n, k\rangle\} \oplus X_1 \oplus \cdots \oplus X_n) \mid n, k \in \mathbb{N}\}, \tag{1}$$

here $X_i = \{\mathbb{N} \upharpoonright b_i : b_i \in B\}$ for $i = \overline{1, n}$.

Now we rewrite equality (1) as

$$\{\mathbb{N} \upharpoonright a : a \in A\} = \{\Phi(\{n\} \oplus \mathbb{N} \upharpoonright b_1 \oplus \cdots \oplus \mathbb{N} \upharpoonright b_n) \mid n \in \mathbb{N} \;\&\; b_1, \ldots, b_n \in B\}.$$

The following statement establishes a connection between notions of lm-reducibility and Σ-reducibility for two sets.

Theorem 1. $A \leqslant_{lm} B \Longleftrightarrow \mathcal{S}(A) \sqsubseteq_\Sigma \mathcal{S}(B)$.

2. Σ-REDUCIBILITY AND lm-REDUCIBILITY OF A SET TO A PAIR OF SETS

Now we consider the concepts of Σ-reducibility and lm-reducibility of a set to a pair of sets. The main result of this Section is the answer to the following question: What is the relation between the notion of lm-reducibility for a set and a pair of sets and the notion of Σ-reducibility for initial segment families of a set and a pair of sets?

Definition 3. A function $\overline{\theta} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is *limitwise monotonic* if there exists partially computable function $\theta : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that the relations

(i) $\forall t \geqslant s\, [\theta(x, y, s)\downarrow \Rightarrow \theta(x, y, s) \leqslant \theta(x, y, t)\downarrow]$;

(ii) $\overline{\theta}(x, y) = \max_t \theta(x, y, t) < \infty$ (we guess $\max \varnothing = 0$)

hold for all x, y, and s.

Definition 4. A mapping $\Theta : \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is a *limitwise monotonic operator* if there exists a partially computable function

$$\theta : \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}^{<\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$$

such that conditions

(i) $\forall \zeta \geqslant \rho\, \forall \eta \geqslant \tau\, \forall t \geqslant s\, [\theta(\rho, \tau, s)\downarrow \Rightarrow \theta(\rho, \tau, s) \leqslant \theta(\zeta, \eta, t)\downarrow]$;

(ii) $\overline{\theta}(\rho, \tau) = \max_t \theta(\rho, \tau, t) < \infty$;

(iii) $\forall X \subseteq \mathbb{N}\, \forall Y \subseteq \mathbb{N}\, [\Theta\langle X, Y\rangle = \{\overline{\theta}(\rho, \tau) : \rho \in X^{<\mathbb{N}}, \tau \in Y^{<\mathbb{N}}\}]$;

hold for all $\rho \in \mathbb{N}^{<\mathbb{N}}$, $\tau \in \mathbb{N}^{<\mathbb{N}}$ and $s \in \mathbb{N}$

Let $\{\theta_e\}_{e\in\mathbb{N}}$ be an effective enumeration of all triple partially computable functions meeting condition (i) of Definition 4. Now we denote by Θ_e for any e the limitwise monotonic operator acting from $\mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N} \cup \{\infty\})$ and defined for any sets $X \subseteq \mathbb{N}$ and $Y \subseteq \mathbb{N}$ by the equality

$$\Theta_e\langle X, Y\rangle = \{\overline{\theta}_e(\rho, \tau) : \rho \in X^{<\mathbb{N}},\ \tau \in Y^{<\mathbb{N}}\}.$$

Definition 5. A set A is *limitwise monotonic reducible* (or, alternatively, is *lm-reducible*) to a pair of sets (B, C) (we denote this by $A \leqslant_{lm} (B, C)$) if $A = \varnothing$ or $A = \Theta_e\langle B, C\rangle$ for some limitwise monotonic operator Θ_e.

We now pass to consider Σ-reducibility between special initial segment families for a set and a pair of sets. Consider an arbitrary set A and a pair of sets (B, C). We then define the following initial segment families:

$$\mathcal{S}(A) = \{\mathbb{N} \upharpoonright a : a \in A\}$$

and

$$\mathcal{S}(B, C) = \{\{0\} \oplus \mathbb{N} \upharpoonright b : b \in B\} \oplus \{\{1\} \oplus \mathbb{N} \upharpoonright c : c \in C\},$$

respectively.

Now we are able to define the notion of Σ-reducibility between the set $\mathcal{S}(A)$, given for an arbitrary set A and the set $\mathcal{S}(B, C)$ given for a pair of sets (B, C).

Definition 6. We say that the family $\mathcal{S}(A)$ is *Σ-reducible* to the family $\mathcal{S}(B, C)$ (and denote this by $\mathcal{S}(A) \sqsubseteq_\Sigma \mathcal{S}(B, C)$) if

$$\{\mathbb{N} \upharpoonright a : a \in A\} = \{\Phi(\{n\} \oplus (\{x_1\} \oplus \mathbb{N} \upharpoonright d_1) \oplus \cdots \oplus (\{x_n\} \oplus \mathbb{N} \upharpoonright d_n))|,$$

$$|n \in \mathbb{N}\ \&\ x_i = \{0, 1\},\ i = \overline{1, n}\}$$

for some enumeration operator Φ and for the function

$$d_i = \begin{cases} b_i \in B, & \text{if } x_i = 0; \\ c_i \in C, & \text{if } x_i = 1, \end{cases} \quad i = \overline{1, n}.$$

The following statement asserts that a set is lm-reducible to a pair of sets if and only if the initial segment family of this set is Σ-reducible to the initial segment family of the pair of sets.

Theorem 2. $A \leqslant_{lm} (B, C) \iff \mathcal{S}(A) \sqsubseteq_\Sigma \mathcal{S}(B, C)$.

Now we define limitwise monotonic reducibility between two pairs of sets through Σ-definability of the special type families for these sets. Consider two arbitrary pairs of sets (A, B) and (C, D). We fix the following initial segment families for these two pairs of sets:

$$\mathcal{S}(A, B) = \{\{0\} \oplus \mathbb{N} \upharpoonright a : a \in A\} \oplus \{\{1\} \oplus \mathbb{N} \upharpoonright b : b \in B\}$$

and

$$\mathcal{S}(C, D) = \{\{0\} \oplus \mathbb{N} \upharpoonright c : c \in C\} \oplus \{\{1\} \oplus \mathbb{N} \upharpoonright d : d \in D\},$$

respectively.

It seems clear that $\mathcal{S}(A, B) \sqsubseteq_\Sigma \mathcal{S}(C, D) \iff \mathcal{S}(A) \sqsubseteq_\Sigma \mathcal{S}(C, D)$ and $\mathcal{S}(B) \sqsubseteq_\Sigma \mathcal{S}(C, D)$. Then Theorem 2 yields $\mathcal{S}(A) \sqsubseteq_\Sigma \mathcal{S}(C, D)$ and $\mathcal{S}(B) \sqsubseteq_\Sigma \mathcal{S}(C, D) \iff A \leqslant_{lm} (C, D)$ and $B \leqslant_{lm} (C, D) \iff (A, B) \leqslant_{lm} (C, D)$.

Hence we immediately obtain

Definition 7. We say that a pair (A, B) is *lm-reducible* to a pair (C, D) (and denote this by $(A, B) \leqslant_{lm} (C, D)) if and only if \mathcal{S}(A, B) \sqsubseteq_\Sigma \mathcal{S}(C, D)$.

In [5] the authors established the existence of a pair of sets (A, B) such that the inequality $(A, B) \nleqslant_{lm} C$ holds for any set C. Using this result one can see that the class of the set families we consider is proper. So we have

Theorem 3. *There exists a pair of sets (A, B) such that the inequality $\mathcal{S}(A, B) \neq_\Sigma \mathcal{S}(C)$ holds for any set C.*

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