

# Novel Approach to Calculation of Box Dimension of Fractal Functions

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## Abstract

We introduce new method of calculation of box dimension of fractal functions' graphs, which are based on fractal interpolation functions. Provide a comparison of the effectiveness of the traditional method of calculating the box dimension to our new approach. On the example of the Weierstrass function we show that the new method almost 3 times more effective than classical.

**Mathematics Subject Classification:** 28A80

**Keywords:** box dimension, fractal, approximation, Weierstrass function, iterated function system, fractal interpolation, Minkowski dimension

## 1 Introduction

Box dimension is one of the widely used fractal dimensions. It is also called Minkowski-Bouligand dimension, or Minkowski dimension. The classical method [1] of finding the box dimension is approximate and gives inaccurate results. Bedford [2] calculated the box dimension of a class of self-affine curves. These curves appear as attractors of hyperbolic iterated function systems (HIFS) of affine mappings. Box dimension is used for bounds and separation of Minkowski contents. Upper and lower box dimension are used for analysis of Lebesgue integrals and for creation of gauge functions of Minkowski contents [3].

In this paper we consider a new method of calculating box dimension of fractal functions, which are used for fractal approximation and compare the obtained results with the result of a classical algorithm of finding box dimension.

## 2 Method of fractal approximation.

The method is in finding free parameters of affine iterated function systems (IFS). This common approach was developed by P. Massopust [4]. Let  $[a, b] \subset \mathbb{R}$  be a nonempty interval,  $1 < N \in \mathbb{N}$  and  $(x_i, y_i) \in [a, b] \times \mathbb{R} \mid a = x_0 < x_1 < \dots < x_{N-1} < x_N = b$  - are points of interpolation. For all  $i = \overline{1, N}$  consider affine transformations of the plane

$$A_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad A_i \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} a_i & 0 \\ c_i & d_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \end{pmatrix}$$

Following two conditions hold true for all  $i$ :

$$A_i(x_0, y_0) = (x_{i-1}, y_{i-1}), \quad A_i(x_N, y_N) = (x_i, y_i).$$

in this case

$$a_i = \frac{x_i - x_{i-1}}{b - a}, \quad c_i = \frac{y_i - y_{i-1} - d_i(y_N - y_0)}{b - a},$$

$$e_i = \frac{bx_{i-1} - ax_i}{b - a}, \quad f_i = \frac{by_{i-1} - ay_i - d_i(by_0 - ay_N)}{b - a},$$

where  $\{d_i\}_{i=1}^N$  operate like family of parameters [5]. We approximate discrete data  $Z = \{(z_m, \omega_m)\}_{m=1}^M$ ,  $a = z_1 < z_2 < \dots < z_m = b$ . Discrete formulas for  $d$  have the form:

$$d_i = \frac{\sum_{z_m \in [x_{i-1}, x_i]} (\alpha_i(z_m) - \omega_m)(\beta_i(z_m) - g \circ \gamma_i(z_m))}{\sum_{z_m \in [x_{i-1}, x_i]} (\beta_i(z_m) - g \circ \gamma_i(z_m))^2} \quad (1)$$

In this equation  $g(z) = \omega_m$ , where  $(z_m, \omega_m) \in Z$  and  $z_m$  is a nearest neighbour of  $z$ . Other functions can be expressed as follows:

$$\alpha_i(x) = \frac{(y_i - y_{i-1})x + (x_i y_{i-1} - x_{i-1} y_i)}{x_i - x_{i-1}},$$

$$\beta_i(x) = \frac{(y_N - y_0)x + (x_i y_0 - x_{i-1} y_N)}{x_i - x_{i-1}}, \quad \gamma_i(x) = \frac{(b - a)x + (x_i a - x_{i-1} b)}{x_i - x_{i-1}}.$$

After finding  $d_i$  we obtain formulas of affine transformations  $A_i$  and we are able to construct fractal interpolation function.

### 3 Calculation box dimension of a random graph.

As a basis for the use of fractal approximation in finding bound box we use the following theorem. Suppose that  $f \in G^2[a, b]$  and that the interpolation set  $Y := \{(x_\nu, f(x_\nu)) \mid \nu = 0, 1, \dots, N\}$  is not collinear and  $\gamma = \sum_{i=1}^N |d_i| > 1$ . Then the box dimension of graph  $f$  is the unique positive solution of

$$\sum_{i=1}^N |d_i| a_i^{dim-1} = 1. \tag{2}$$

where  $a_i = (x_i - x_{i-1}) / (b - a)$ ,  $i=1, \dots, N$ , the box dimension of an affine fractal function depends only on the knot spacing and the scaling factors  $d_i$ . When using the method of fractal approximation,  $d_i$  from (1) equal to  $d_i$  from equation (2). Calculate the value of the following graph. This graph (Fig. 1) was made by fractal algorithm approximation of 3 points:  $(0, 0)$ ,  $(0.5, 0.5)$ ,  $(1, 0)$ ; and a constant value of  $d_i = 0.65$  for all  $i$ .

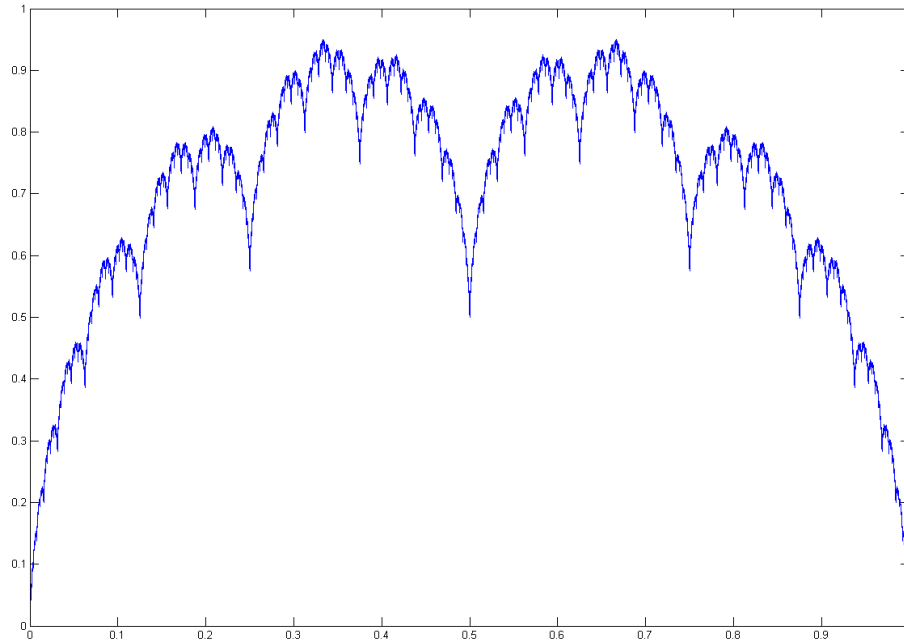


Figure 1: Fractal interpolation function

We apply the method of fractal approximation with the same points of interpolation as when creating this graph. To simplify the calculation of the

distance between interpolation points we take identical ones. As an approximation we take 16385 equidistant points, including the first and the last point of the graph. In such a way obtain 2 values of the coefficient  $d_i$  and, accordingly, 2 matrices of affine transformation. Various combinations on  $K$  times these transformations and the inclusion of the last point will  $2^K + 1$  points. Next, using the function (2), we express dimension, knowing that all the variables  $x_i - x_{i-1}$  are the same, so  $a_i$  are the same too and equal  $a = x_2 - x_1$ .

$$\dim = \frac{\log(a/\sum_{i=1}^N |\lambda_i|)}{\log(a)}. \quad (3)$$

All  $|d_i|$  are equal 0.65 like when we create a graph. The sum of  $|d_i|$  is equal 1.3 and  $a = 0.5$ , so by (3)  $\dim = 1.378511623253732$ . This is the real box dimension of the graph. If we take a different number of interpolation points obtained value is close to the real. For example, when number of interpolation points is 5. Box dimension will be 1.378511623253729.

To find a different way, consider the following definition of box dimension. For any  $k_1, k_2, \dots, k_n \in \mathbb{Z}$  and  $\varepsilon > 0$ , we call  $\prod_{i=1}^n [k_i\varepsilon, (k_i+1)\varepsilon]$  an  $\varepsilon$ -coordinate cube in  $\mathbb{R}^n$ . Let  $M$  be a bounded set in  $\mathbb{R}^n$  and  $N_M(\varepsilon)$  the number of  $\varepsilon$ -coordinate cubes intersecting  $M$ . If

$$\lim_{\varepsilon \rightarrow 0} \frac{\log N_M(\varepsilon)}{\log 1/\varepsilon} \quad (4)$$

exists, then it is called the box dimension of  $M$  and denoted by  $\dim_B(M)$  [1]. Then this algorithm will consistently reduce the value of  $\varepsilon$  and  $N_M(\varepsilon)$  for each case. Getting values  $\log 1/\varepsilon$  and  $\log N_M(\varepsilon)$  approximate a straight line. Find a tangent, it will be box dimension of this graph. Dividing the x-axis at 4, 8, 16, ..., 8192 parts and approximating obtain  $\dim = 1.363349283183837$ .

## 4 Box dimension of Weierstrass function. Numerical results.

Weierstrass function is an example of a pathological real-valued function on the real line. The function has the property to be continuous everywhere but nowhere differentiable. Function was defined as

$$W_{\lambda,b}(x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x). \quad (5)$$

If the function has this form then box dimension is expressed as follows [6]:

$$D = 2 + \frac{\log \lambda}{\log b}. \quad (6)$$

Construct a graph of the Weierstrass function, taking the parameters  $\lambda = 0.5$  and  $b = 4$  on the interval  $[-1, 1]$  by using (5). It is shown below and is painted blue (Fig. 2). Using (6) box dimension of this graph is 1.5. Apply the fractal approximation of Weierstrass function. We divide the interval to 4 equidistant parts, receiving 5 interpolation points. To approximate and locate  $d_i$  use 50001 equidistant points on this interval including the first and the last point. After approximation 4 variables  $d_i$  are equal 0.486 and their sum is 1.9443. All  $a_i$  are 0.25. Using (3) get  $dim = 1.479633469572568$ . Graph of fractal approximation presented in red (Fig. 2).

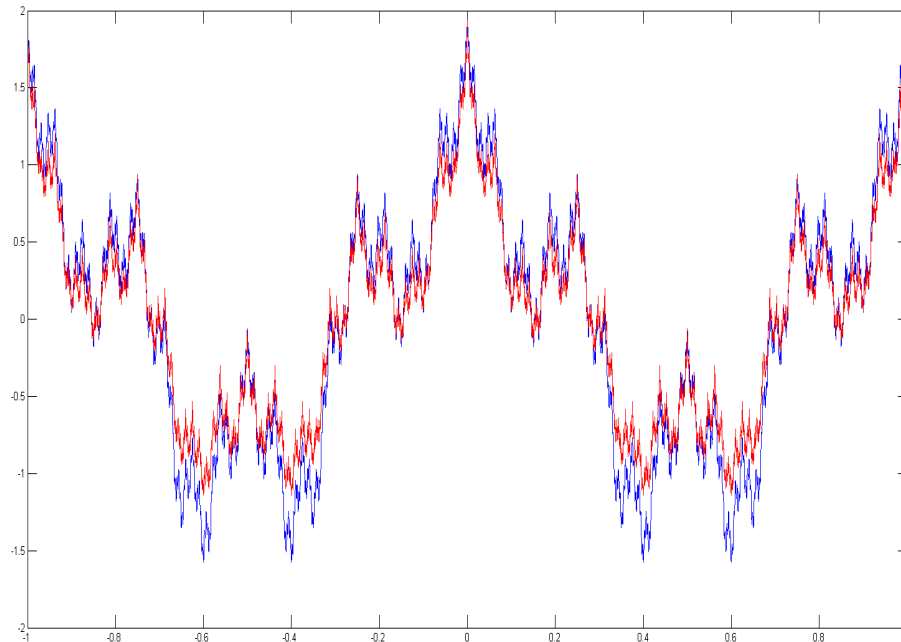


Figure 2: Weierstrass function (blue) and its approximation (red)

The classical algorithm (4) will consistently reduce the value of  $\varepsilon$  and for each case to calculate  $N_M(\varepsilon)$ . Getting values  $\log 1/\varepsilon$  and  $\log N_M(\varepsilon)$  approximate a straight line. Find a tangent, it will be box dimension of this graph. Dividing the x-axis at 2, 4, 8, ...,  $2^{14}$  parts and approximating obtain  $dim = 1.393773147208170$ . Test algorithm of fractal approximation and the results of the formula (4) and compare the results for different parameters of the Weierstrass function.

Value $\lambda, b$	Real dimension	Classical algorithm	Fractal algorithm	Error of Classical algorithm (%)	Error of Fractal algorithm (%)
$\lambda = 0.5$ $b = 4$	1.5	1.393773	1.479633	7.0818	1.3578
$\lambda = 0.5$ $b = 8$	1.666667	1.673349	1.653089	0.4009	0.8146
$\lambda = 0.25$ $b = 4$	1	1.130163	0.998369	13.0163	0.1631
$\lambda = 0.75$ $b = 2$	1.584962	1.226558	1.692899	22.6128	6.8101
$\lambda = 0.5$ $b = 6$	1.613147	1.569272	1.646773	2.7198	2.0845
$\lambda = 0.7$ $b = 3$	1.675340	1.357324	1.764921	18.9822	5.3470
$\lambda = 0.3$ $b = 5$	1.251929	1.247206	1.188171	0.3773	5.0928
$\lambda = 0.6$ $b = 7$	1.737487	1.660821	1.693850	4.4125	2.5115
$\lambda = 0.6$ $b = 3$	1.535026	1.297207	1.591265	15.4928	3.6637
$\lambda = 0.4$ $b = 7$	1.529120	1.485881	1.551931	2.8277	1.4918

The average error of the classical algorithm is 8.79241%. Average error of fractal algorithm is 2.93369%. So, the method of fractal approximation has shown result close to the real box dimension by 2.997 times. But remains the problem of finding the optimal number of points of interpolation for fractal algorithm.

**Acknowledgements.** The work was performed according to the Russian Government Program of Competitive Growth of Kazan Federal University as part of the OpenLab Raduga.

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**Received: September 10, 2014**