Maxwell's Wheel

Maxwell's wheel is a massive disk with the axis hung on two threads which can wound around the axis (Fig. 1). If the thread is wound, the wheel goes upward; if the wheel is released, it performs reciprocating motion in the vertical plane, and the wheel rotates around its axis at the same time.

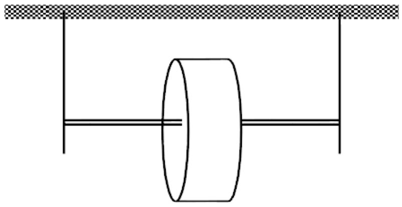

Figure 1.

Aim of the work

Calculating and measuring the moment of inertia of a cylindrical solid body with respect to its symmetry axis.

Moment of inertia of a material point of the mass *m* with respect to some rotation axis is the value $J = mr^2$, where r is the distance between the point and the axis. A rigid body can be regarded as the system of material points having the masses m_i and situated at the distances r_i from the axis, and thus its moment of inertia will be

$$J = \sum_i m_i r_i^2 \,. \qquad (1)$$

In the case of continuous distribution of mass the expression for *J* takes the following form:

$$J = \int r^2 dm \,. \qquad (2)$$

Limits of integration should include the whole body.
It can be calculated that for a cylinder having the radius *R* and the mass m, the moment of inertia with respect to its symmetry axis is:

$$J = \frac{1}{2} mR^2 , \qquad (3a)$$

and for the cylinder having a coaxial cavity drill out along its axis (in other words, a ring having the inner diameter R_1 and the outer diameter R_2), the moment of inertia is

$$J = \frac{1}{2} m(R_1^2 + R_2^2) \,. \qquad (3b)$$

Experimental determination of the moment of inertia

Consider one of the methods for determination of the moment of inertia using the Maxwell's wheel (pendulum). The forces acting in the system are shown in Fig. 2. To analyze the motion of the wheel, it is convenient to choose the reference frame centred in the center of masses A of the wheel. The centre of masses moves down with the linear acceleration **a**. The equation of motion of the wheel's centre of masses is written as

$$m\mathbf{a} = m\mathbf{g} + \mathbf{T}, \qquad (4)$$

where **T** is the resulting force of strain of both threads and m is the wheel's mass. In addition, the wheel performs rotational motion around the horizontal axis (which passes through the centre of masses) due to the moment of force of the thread strain: $M = R_0T$, where M is the moment of the force **T** and R_0 is the moment arm (radius of the rod).

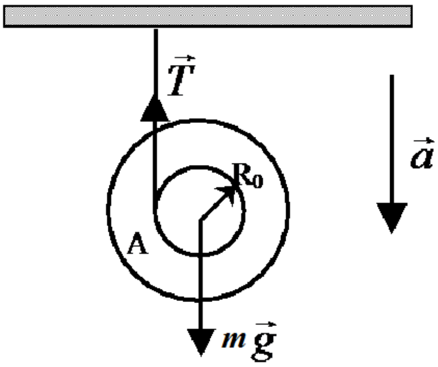

Figure 2.

The basic equation of rotational motion is

$$\mathbf{M} = J\boldsymbol{\varepsilon}, \qquad (5)$$

where $\boldsymbol{\varepsilon}$ is the angular acceleration of the wheel and J is its moment of inertia. Then we can write down expression for scalars:

$$ma = mg - T, \qquad (6)$$

$$M = J\varepsilon. \qquad (7)$$

The distance x which the centre goes down is equal to the length of the thread unwound from the rod:

$$x = \varphi R_0, \qquad (8)$$

where φ is the total rotation angle. By differentiating this expression twice with respect to time, we get

$$a = \frac{d^2x}{dt^2} = R_0\frac{d^2\varphi}{dt^2} = R_0\varepsilon, \qquad (9)$$

and taking into account Eq. (9), we can rewrite formula (7) as

$$R_0T = J\frac{a}{R_0} \text{ or } T = J\frac{a}{R_0^2}. \qquad (10)$$

Combining Eqs. (6) and (10), we get

$$a = \frac{mg}{m + J/R_0^2}, \qquad (11)$$

$$T = \frac{mg}{1 + \frac{mR_0^2}{J}}. \qquad (12)$$

Expressions (11) and (12) shown that acceleration and the thread strain force are constant (independent of time). Therefore, if the coordinate of the wheel is defined from the upper point of fastening, the time dependence of the coordinate will be

$$x = at^2/2. \qquad (13)$$

Substituting (13) into (12), we obtain the following expression for the moment of inertia of the Maxwell's wheel:

$$J = mR_0^2\left(\frac{gt^2}{2x} - 1\right) \qquad (14)$$

or

$$J = \frac{mD_0^2}{4}\left(\frac{gt^2}{2x} - 1\right), \qquad (15)$$

which contains parameters that are easy to measure. R_0 (D_0) is the outer radius (diameter) of the rod together with the thread wound on it, t is the time necessary for the wheel to go down by the distance x, and m is the wheel's mass. The mass is summed up from the mass of the rod m_0, the disk m_d, and a ring m_r which could be optionally put on the disk.

Experimental setup

Figure 3 shows the general look of the Maxwell's wheel. The basis 1 stands on regulating legs 2 used for levelling. The basis holds the column 3 with fixed upper arm 4 and movable lower arm 5. The upper arm contains the electromagnet 6, photoelectric sensor 7 and the tap wrench 8 for fixing and regulating the length of the thread 15. The lower arm with the photosensor 9 can be moved and fastened in different positions on the column.

The wheel 10 is set on the cylindrical rod 12; rings 11 can be put on the wheel to change the moment of inertia of the system.

The wheel (with a ring) is kept in the upper position by the electromagnet. The column has a millimeter gauge for determination of the displacement x of the wheel. Photoelectric sensors are connected to the timer 14.

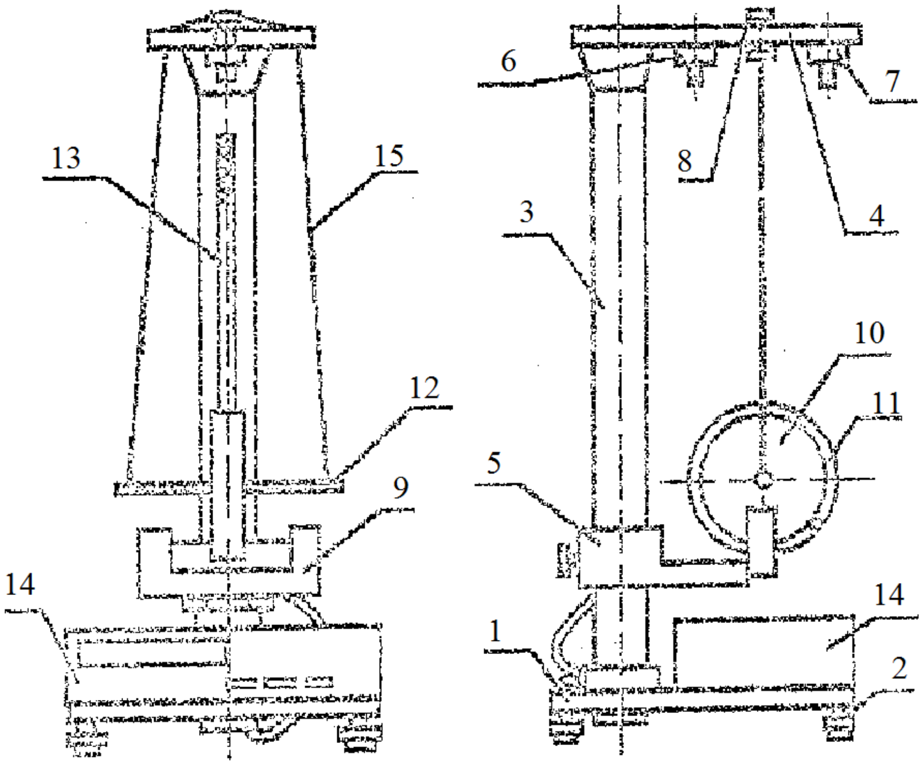

Figure 3.

The front panel of the timer is shown in Fig. 3. The button “Сеть”/“Power” switches the apparatus on; the lamps in the photosensors should start glowing. The button “Сброс”/“Clear” is used to set the timer to zero. The button “Пуск”/“Start” controls the electromagnet: upon pressing this button, the magnet is turned off and the timer goes to the time measurement regime.

Algorithm of measurements

The lower arm of the setup should be set in the lower position.

1. Put one of the rings on the wheel by pressing it on the side as far as possible.
2. Choose the thread length so as to the rime of the metallic ring in the lowest position is two millimeters lower than the photoelectric sensor. A the same time correct the position of the wheel so that its axis is parallel to the basis (horizontal). The regulation is made using the wrench 8 (Fig. 3).
3. Press the button “Сеть”/“Power”.
4. Wound the thread onto the rod. Check that the winding is uniform, turn by turn.
5. Fix the wheel at the magnet and rotate in at a small angle (~5°).
6. Press the button “Сброс”/“Clear”.
7. Press the button “Пуск”/“Start”.
8. Write down the measured value of the time of falling t into the table.
9. Make ten measurement of time and calculate the mean value t_m.

10. Find the distance *x* using the gauge on the column (in other words, find the length of the thread).
11. Measure the diameter of the thread and the wheel's rod in different cross-sections (orientations); calculate the mean value. Find the rod diameter allowing for the thread thickness: $D_0 = D + 2D_{thr}$ ($R_0 = D_0/2$).
12. Find the wheel's mass including the mass of the ring (all necessary data are written on the details).
13. Further work is made following either variant 1 or 2 (at the teacher's option).

Variant 1. Determination of the moment of inertia of the Maxwell's wheel

1. Calculate the moment of inertia theoretically using formulas (3a) and (3b). The net result is the sum of the moments of inertia of the rod J_r, the disk J_d and the additional ring J_{ring}. Abovementioned formulas show that
   $J_r = ½\, m_r R_0^2$, where m_r and R_0 is the mass and radius of the rod,
   $J_d = ½\, m_d (R_1^2 + R_0^2)$, where R_1 is the outer radius of the disk, and
   $J_{ring} = ½\, m_{ring} (R_1^2 + R_2^2)$, where R_2 is the outer radius of the ring.
   The masses are written of the details.
2. Find the moment of inertia experimentally using Eq. (14).
3. Compare the results.

Variant 2. Determination of the moment of inertia of the ring

1. Calculate the moment of inertia theoretically using formula (3b). The mass m_{ring} is written on the ring; R_1 and R_2 are the inner and outer radii of the ring.
   $J_{ring} = ½\, m_{ring} (R_1^2 + R_2^2)$.
2. Determine experimentally the moment of inertia J_1 of the Maxwell's wheel without the ring (following the steps 4-11 in the "Algorithm" section).
3. Determine experimentally the moment of inertia J_2 of the Maxwell's wheel with a ring put on it (steps 4-12).
4. Find the moment of inertia of the ring.
5. Compare theory and experiment.

Step 14. Determine the inaccuracy of the obtained value of the moment of inertia using the following formula:

$$\frac{\Delta J}{J} = \sqrt{\left(\frac{2\Delta D}{D + 2D_{thr}}\right)^2 + \left(\frac{4\Delta D_{thr}}{D + 2D_{thr}}\right)^2 + \left(\frac{2t\Delta t}{gt^2 - 2x}\right)^2 + \left(\frac{2}{gt^2 - 2x} + \frac{1}{x}\right)^2 \Delta x^2} ,$$

where ΔD, ΔD_{thr}, Δt, Δx are the confidence intervals for direct measurements of the values D, D_{thr}, t, x allowing for both random and systematic errors.

Table

№	m_r, kg	m_d, kg	m_{ring}, kg	R_0, m	R_1, m	R_2, m	x, m	t, s	t_m, s
…									

Questions

1. The theorem of the motion of the centre mass of a system of material points.
2. Definition of the moment of inertia of a material point and of a material points system.
3. Equation of motion of the Maxwell's wheel.
4. Dependence (behaviour) of the acceleration, velocity, and the thread strain force during the wheel's motion.
5. Dependence (behaviour) of the mechanical energy of the Maxwell's wheel during the motion.