Lecture 3. Work and Energy

Work and Kinetic Energy

If you integrate a constant acceleration of an object twice, you obtain:

$$v(t) = at + v_0$$

$$x(t) = \frac{1}{2}at^2 + v_0 t + x_0$$

where v_0 is the initial speed and x_0 is the initial x position at time t = 0.

Now, suppose you want to find the speed v_1 the object will have when it reaches position x_1. One can algebraically, once and for all note that this must occur at some time t_1 such that:

$$v(t_1) = at_1 + v_0 = v_1$$

$$x(t_1) = \frac{1}{2}a{t_1}^2 + v_0 t_1 + x_0 = x_1$$

We can algebraically solve the first equation once and for all for t_1:

$$t_1 = \frac{v_1 - v_0}{a}$$

and substitute the result into the second equation, eliminating time altogether from the solutions:

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Work and Kinetic Energy

$$\frac{1}{2}a\left(\frac{v_1 - v_0}{a}\right)^2 + v_0\left(\frac{v_1 - v_0}{a}\right) + x_0 = x_1$$

$$\frac{1}{2}a({v_1}^2 - 2v_0v_1 + {v_0}^2) + \left(\frac{v_0v_1 - v_0^2}{a}\right) = x_1 - x_0$$

$${v_1}^2 - 2v_0v_1 + {v_0}^2 + 2v_0v_1 - v_0^2 = 2a(x_1 - x_0)$$

$$\text{or } {v_1}^2 - {v_0}^2 = 2a(x_1 - x_0)$$

Lets consider a constant acceleration in one dimension only:

$${v_1}^2 - {v_0}^2 = 2a\Delta x$$

If we multiply by *m* (the mass of the object) and move the annoying 2 over to the other side, we can make the constant acceleration *a* into a constant force $F_x = ma$:

$$(ma)\Delta x = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2$$

$$F_x\Delta x = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2$$

We now define the **work done by the constant force F_x on the mass** *m* as it moves through the distance Δx to be: $\Delta W = F_x\Delta x$

Work is a form of energy.

$$1 \text{ Joule} = 1 \text{ Newton} \cdot \text{meter} = 1\,\frac{\text{kilogram} \cdot \text{meter}^2}{\text{second}^2}$$

Kinetic Energy

Let's define the quantity changed by the work to be the kinetic energy and will use the symbol K to represent it in this work:

$$K = \frac{1}{2}mv^2$$

Work-Kinetic Energy Theorem:

The work done on a mass by the total force acting on it is equal to the change in its kinetic energy.

and as an equation that is correct for constant one dimensional forces only:

$$\Delta W = F_x \Delta x = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \Delta K$$

Lecture 3. Work and Energy

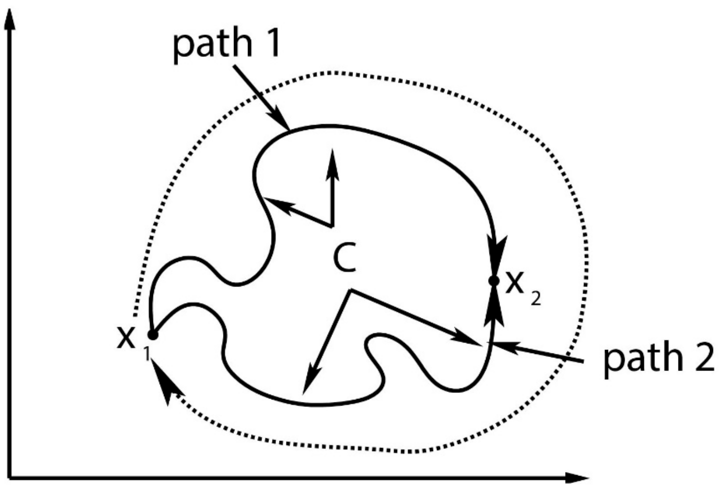
Conservative Forces: Potential Energy

We define a ***conservative force*** to be one such that the work done by the force as you move a point mass from point $\vec{x}_1$ to point $\vec{x}_2$ is independent of the path used to move between the points:

$$W_{loop} = \oint_{\vec{x}_1(\text{path 1})}^{\vec{x}_2} \vec{F} \cdot d\vec{l} = \oint_{\vec{x}_1(\text{path 2})}^{\vec{x}_2} \vec{F} \cdot d\vec{l}$$

In this case (only), the work done going around an arbitrary closed path (starting and ending on the same point) will be identically zero!

$$W_{loop} = \oint_C \vec{F} \cdot d\vec{l} = 0$$



The work done going around an arbitrary loop by a conservative force is zero. This ensures that the work done going between two points is *independent* of the path taken, its defining characteristic.

Conservative Forces: Potential Energy

Since the work done moving a mass *m* from an arbitrary starting point to any point in space is the **same** independent of the path, we can assign each point in space a numerical value: the work done by *us* on mass *m*, *against* the conservative force, to reach it.

This is the *negative* of the work done by the force. We do it with this sign for reasons that will become clear in a moment. We call this function the ***potential energy*** of the mass *m* associated with the conservative force $\vec{F}$:

$$U(\vec{x}) = -\int_{x_0}^{x} \vec{F} \cdot d\vec{x} = -W$$

Note Well: that only one limit of integration depends on *x*; the other depends on where you choose to make the potential energy zero. This is a *free choice*. No physical result that can be measured or observed can uniquely depend on where you choose the potential energy to be zero.

Conservation of Mechanical Energy

The principle of the ***Conservation of Mechanical Energy***:

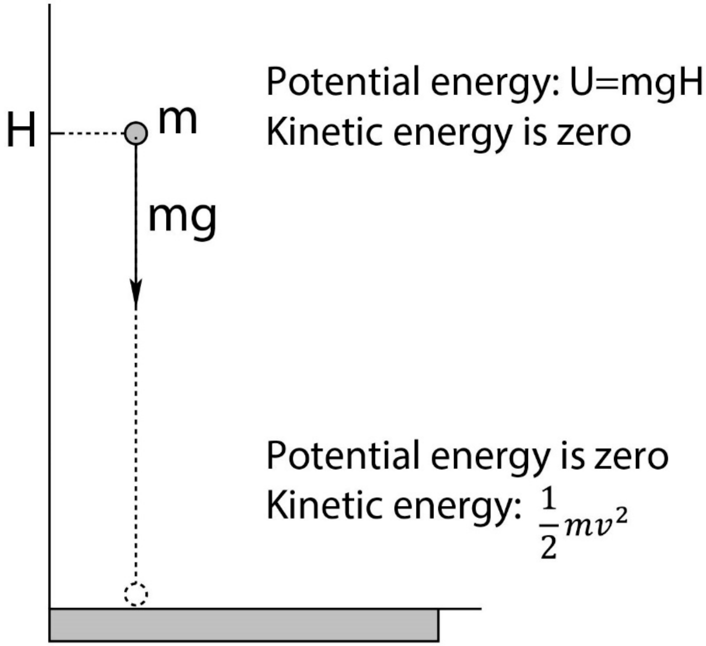
The total mechanical energy (defined as the sum of its potential and kinetic energies) of a particle being acted on by only conservative forces is constant.

Or, if only conservative forces act on an object and U is the potential energy function for the total conservative force, then

$$E_{mech} = K + U = A \quad \text{scalar constant}$$

The fact that the force is the negative derivative of the potential energy of an object ***means*** that ***the force points in the direction the potential energy decreases in***.

Example: Falling Ball Reprise



To see how powerful this is, let us look back at a falling object of mass *m* (neglecting drag and friction). First, we have to determine the gravitational potential energy of the object a height *y* above the ground (where we will choose to set $U(0) = 0$):

$$U(y) = -\int_0^y (-mg)dy = mgy$$

Now, suppose we have our ball of mass *m* at the height *H* and drop it from rest. How fast is it going when it hits the ground? This time we simply write the total energy of the ball at the top (where the potential is *mgH* and the kinetic is zero) and the bottom (where the potential is zero and kinetic is $\frac{1}{2}mv^2$ and set the two equal! Solve for *v*, done:

$$E_i = mgH + (0) = (0) + \frac{1}{2}mv^2 = E_f$$

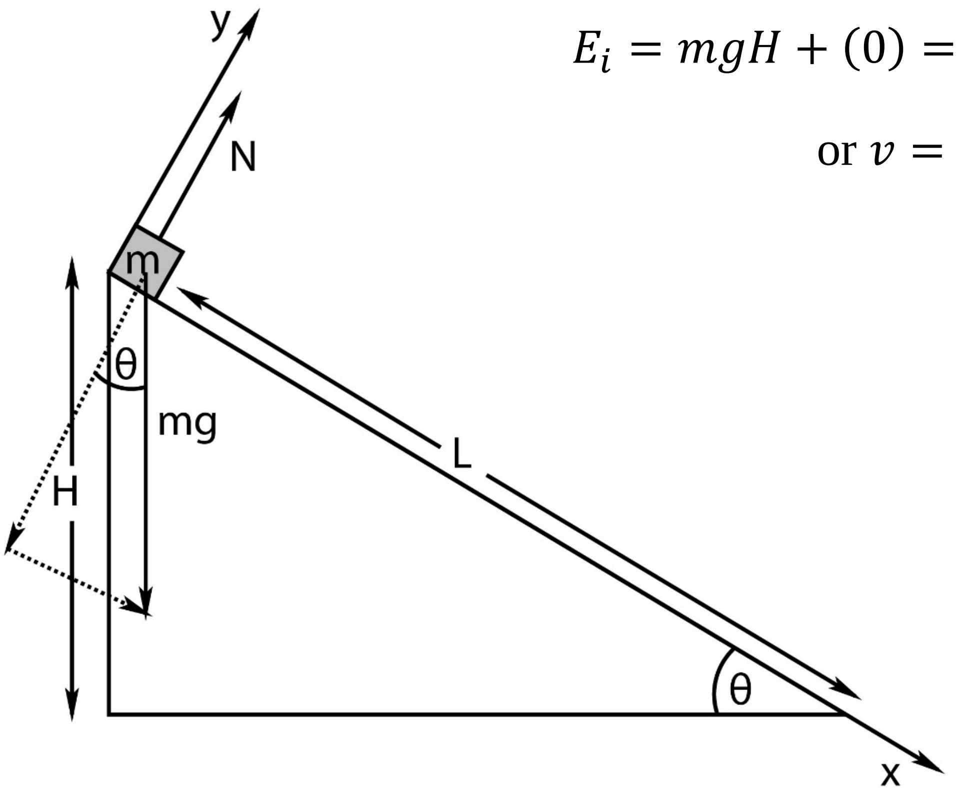
$$\text{or } v = \sqrt{2gH}$$

Example: Block Sliding Down Frictionless Incline Reprise

The block starts out a height H above the ground, with potential energy mgH and kinetic energy of 0. It slides to the ground (no non-conservative friction!) and arrives with no potential energy and kinetic energy $\frac{1}{2}mv^2$

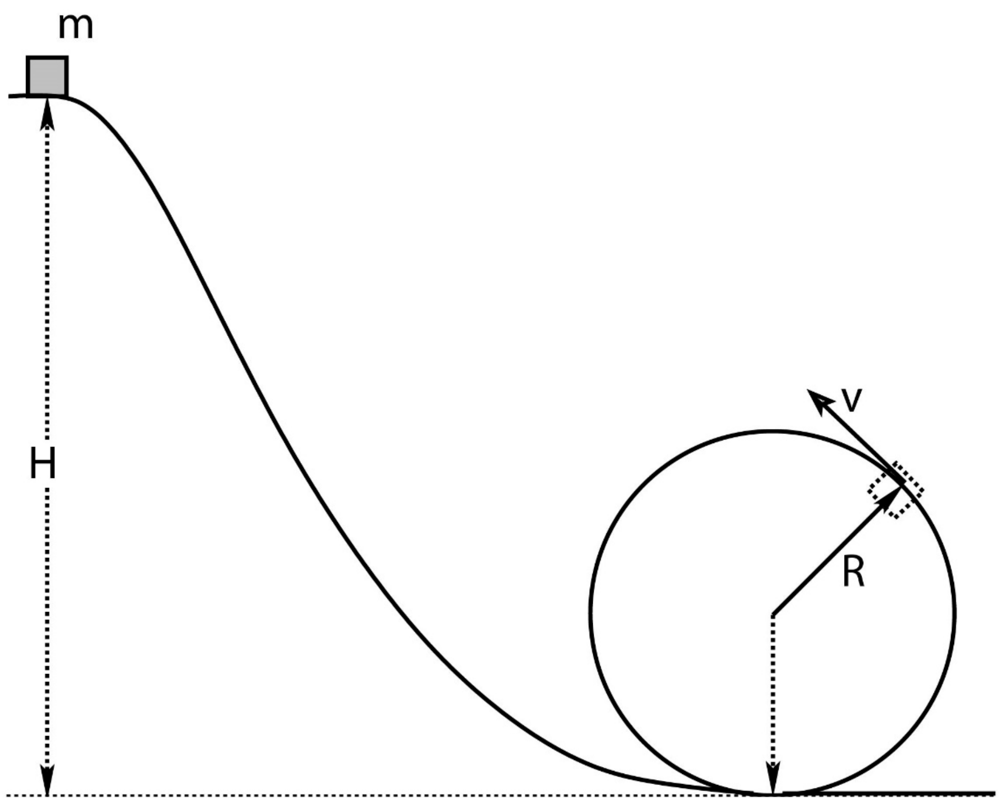
$$E_i = mgH + (0) = (0) + \frac{1}{2}mv^2 = E_f$$

$$\text{or } v = \sqrt{2gH}$$



Example: Looping the Loop

What is the minimum height H such that a block of mass m loops-the-loop (stays ***on the frictionless track all the way around the circle***)?



Example: Looping the Loop

Here we need two physical principles: ***Newton's Second Law*** and ***the kinematics of circular motion*** since the mass is undoubtedly moving in a circle if it stays on the track. Here's the way we reason:

"If the block moves in a circle of radius R at speed v, then its acceleration towards the center must be $a_c = v^2/R$. Newton's Second Law then tells us that the *total force component* in the direction of the center must be mv^2/R. That force can only be made out of (a component of) gravity and the normal force, which points towards the center. So we can relate the normal force to the speed of the block on the circle at any point."

At the top (where we expect v to be at its minimum value, assuming it stays on the circle) gravity points straight towards the center of the circle of motion, so we get:

$$mg + N = \frac{mv^2}{R}$$

and in the limit that $N \to 0$ ("barely" looping the loop) we get the condition:

$$mg = \frac{mv_t^2}{R}$$

where v_t is the (minimum) speed at the top of the track needed to loop the loop.

Example: Looping the Loop

Now we need to relate the speed at the top of the circle to the original height H it began at. This is where we need our third principle – *Conservation of Mechanical Energy*!
With energy we don't care about the shape of the track, only that the track do no work on the mass which (since it is frictionless and normal forces do no work) is in the bag. Thus:

$$E_i = mgH = mg2R + \frac{1}{2}mv_t^2 = E_f$$

If you put these two equations together (e.g. solve the first for mv_t^2 and substitute it into the second, then solve for H in terms of R) you should get

$$H_{min} = 5R/2.$$

Example: Generalized Work-Mechanical Energy Theorem

Let's consider what happens if ***both*** conservative and nonconservative forces are acting ***on a particle***. In that case the argument above becomes:

$$W_{rot} = W_C + W_{NC} = \Delta K$$

$$\text{or } W_{NC} = \Delta K - W_C = \Delta K + \Delta U = \Delta E_{mech}$$

which we state as the **Generalized Non-Conservative Work-Mechanical Energy Theorem:**

The work done by all the non-conservative forces acting on a particle equals the change in its total mechanical energy.

Example: Heat and Conservation of Energy

The important empirical law is the ***Law of Conservation of Energy***. Whenever we examine a physical system and try very hard to keep track of all of the mechanical energy exchanges within that system and between the system and its surroundings, we find that we can always account for them all without any gain or loss.

In other words, we find that the total mechanical energy of an ***isolated*** system never changes, and if we add or remove mechanical energy to/from the system, it has to come from or go to somewhere outside of the system. This result, applied to well defined systems of particles, can be formulated as the ***First Law of Thermodynamics***:

$$\Delta Q_{in} = \Delta E_{of} + W_{by}$$

In words, the heat energy flowing *in* to a system equals the change in the internal total mechanical energy *of* the system plus the external work (if any) done *by* the system on its surroundings.

Example: Heat and Conservation of Energy

When a block slides down a rough table from some initial velocity to rest, kinetic friction turns the bulk organized kinetic energy of the collectively moving mass into ***disorganized microscopic energy*** – heat.

As the rough microscopic surfaces bounce off of one another and form and break chemical bonds, it sets the actual molecules of the block bounding, increasing the internal microscopic mechanical energy of the block and *warming it up*.

Power

The energy in a given system is not, of course, usually constant in time. Energy is added to a given mass, or taken away, at some rate.

There are many times when we are given the **rate** at which energy is added or removed in time, and need to find the total energy added or removed. This rate is called the **power**.

Power: The rate at which work is done, or energy released into a system.

$$dW = \vec{F} d\vec{x} = \vec{F} \cdot \frac{dx}{dt} dt$$

$$P = \frac{dW}{dt} = \vec{F} \cdot \vec{v}$$

so that $$\Delta W = \Delta E_{tot} = \int P dt$$

The units of power are clearly Joules/sec = Watts. Another common unit of power is "Horsepower", 1 HP = 746 W.

Equilibrium

The **force** is given by the **negative gradient of the potential energy**:

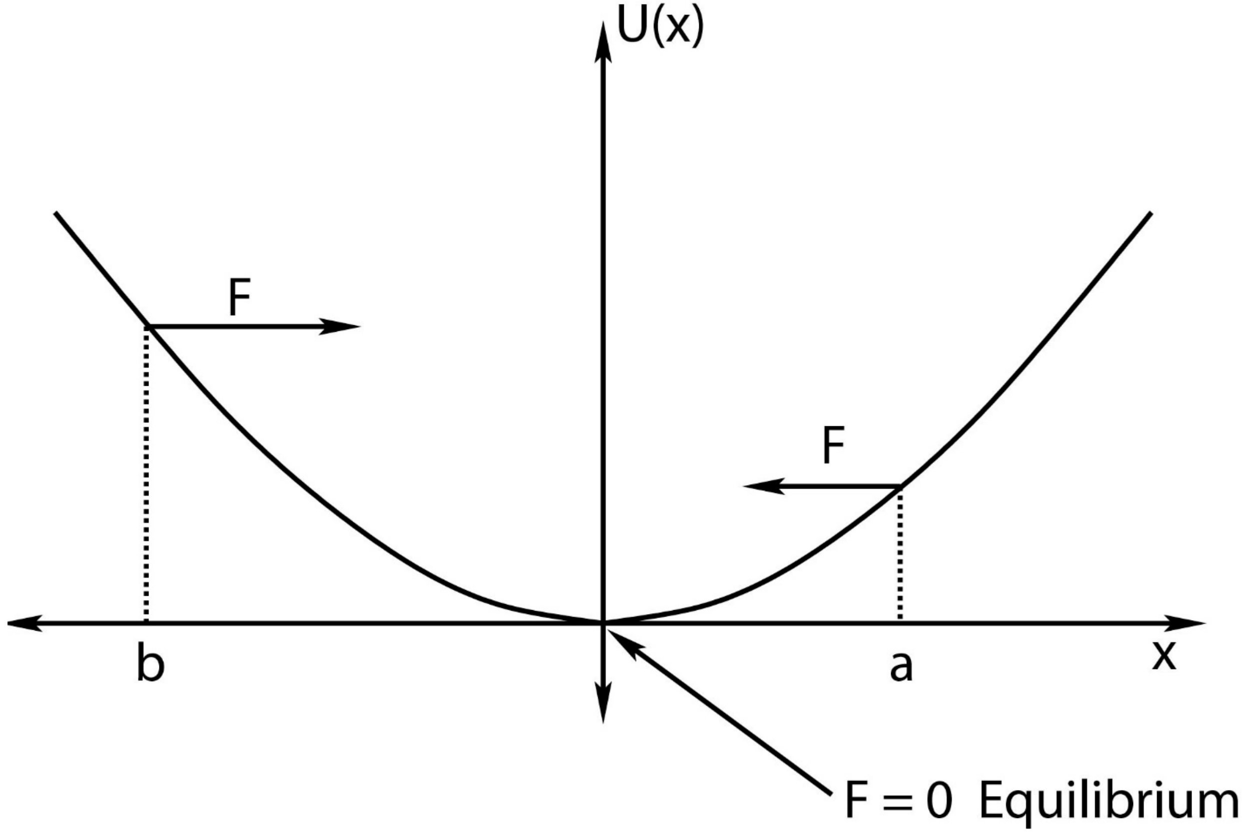
$$\vec{F} = -\vec{\nabla} U$$

or (in each direction): $F_x = -\frac{dU}{dx}, F_y = -\frac{dU}{dy}, F_z = -\frac{dU}{dz},$

or the force is the negative **slope** of the potential energy function in this direction.

The *meaning* of this is that if a particle moves in the direction of the (conservative) force, it speeds up. If it speeds up, its kinetic energy increases. If its kinetic energy increases, its potential energy must *decrease*. The force (component) acting on a particle is thus the rate *at which the potential energy* decreases (the negative slope) in any given direction

Equilibrium



A one-dimensional potential energy curve $U(x)$.

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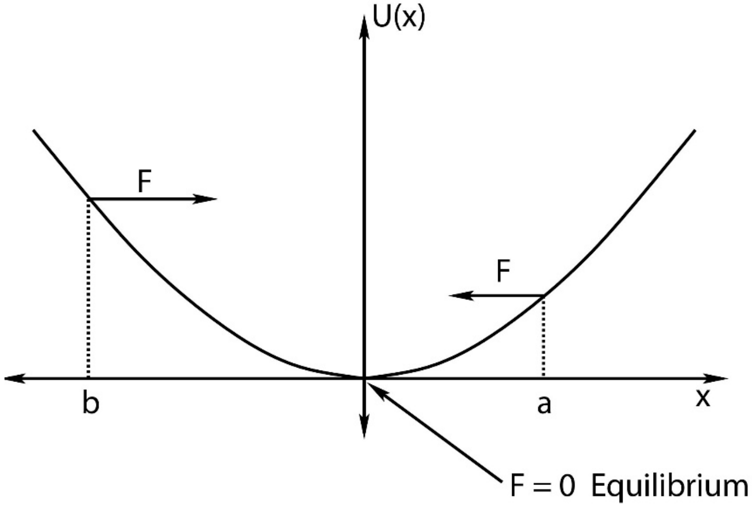
Equilibrium

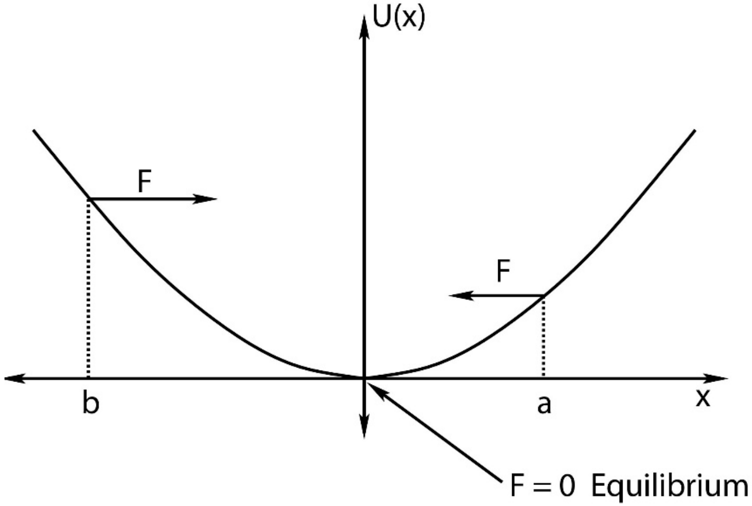


A one-dimensional potential energy curve $U(x)$.

At the point labelled *a*, the *x*-slope of $U(x)$ is *positive*. The *x* (component of the) force, therefore, is in the negative *x* direction. At the point *b*, the *x*-slope is *negative* and the force is correspondingly positive. Note well that the force gets larger as the slope of $U(x)$ gets larger (in magnitude).

The point in the middle, at $x = 0$, is *special*. Note that this is a *minimum* of $U(x)$ and hence the *x*-slope is zero. Therefore the *x*-directed force *F* at that point is zero as well. A point at which the force on an object is zero is, as we previously noted, a point of ***static force equilibrium*** – a particle placed there at rest will remain there at rest.

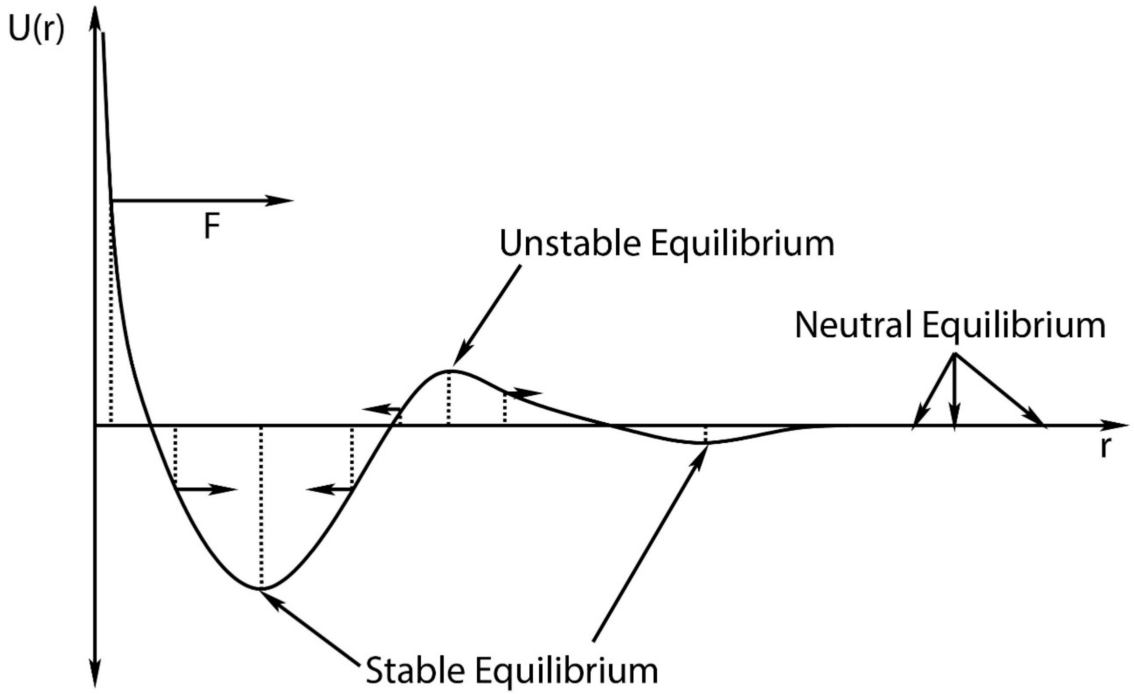
Equilibrium



A one-dimensional potential energy curve $U(x)$.

In this particular figure, if one moves the particle a small distance to the right or the left of the equilibrium point, the force *pushes the particle back towards equilibrium*. Points where the force is zero and small displacements cause a restoring force in this way are called *stable equilibrium points*. As you can see, the ***isolated minima*** of a potential energy curve (or surface, in higher dimensions) are all ***stable equilibria***.

Equilibrium



A fairly generic potential energy shape for microscopic (atomic or molecular) interactions, drawn to help exhibits features one might see in such a curve more than as a realistically scaled potential energy in some set of units. In particular, the curve exhibits stable, unstable, and neutral equilibria for a radial potential energy as a function of *r*, the distance between two e.g. atoms.

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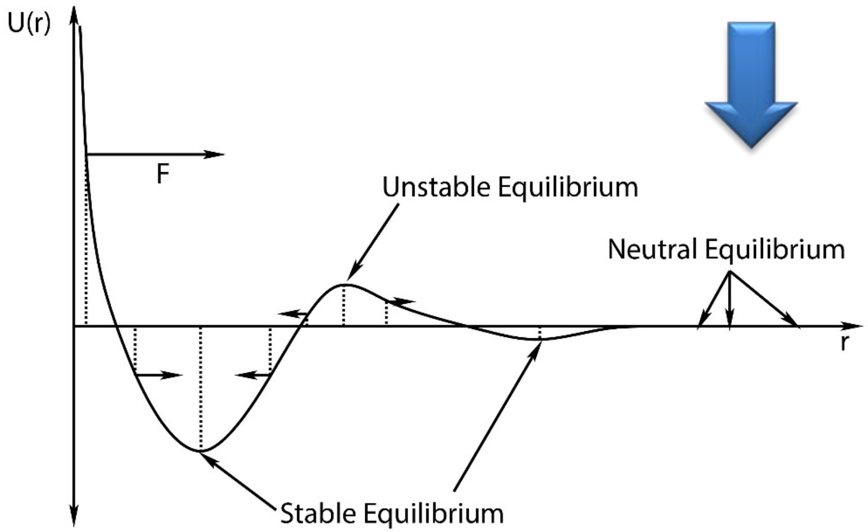
Equilibrium



At very long ranges, the forces between neutral atoms are extremely small, effectively *zero*. This is illustrated as an extended region where the potential energy is *flat* for large *r*. Such a range is called ***neutral equilibrium*** because there are no forces that either restore or repel the two atoms. Neutral equilibrium is *not stable* in the specific sense that a particle placed there with *any nonzero velocity* will move freely (according to Newton's First Law). Since it is nearly impossible to prepare an atom at absolute rest relative to another particle, one basically "never" sees two unbound microscopic atoms with a large, perfectly constant spatial orientation.

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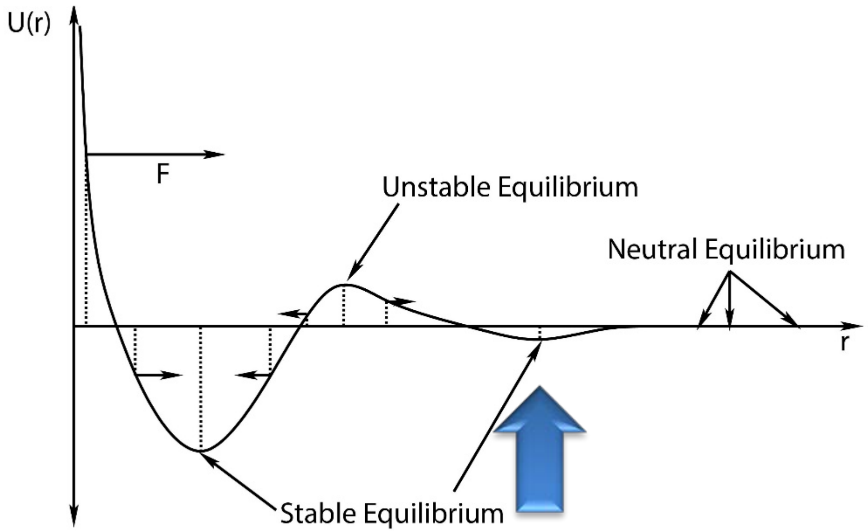
Equilibrium



As the two atoms near one another, their interaction becomes first *weakly attractive* due to e.g. quantum dipole-induced dipole interactions and then *weakly repulsive* as the two atoms start to “touch” each other. There is a potential energy minimum in between where two atoms separated by a certain distance can be in stable equilibrium without being chemically bound.

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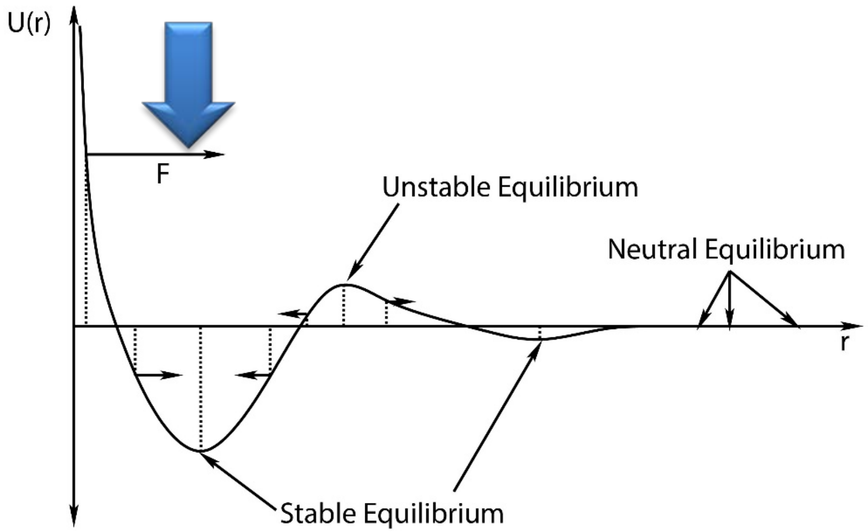
Equilibrium



Atoms that approach one another still more closely encounter a second potential energy well that is at first strongly attractive followed by a hard core repulsion as the electron clouds are prevented from interpenetrating by e.g. the Pauli exclusion principle. This second potential energy well is often modelled by a Lennard-Jones potential energy It also has a point of stable equilibrium.

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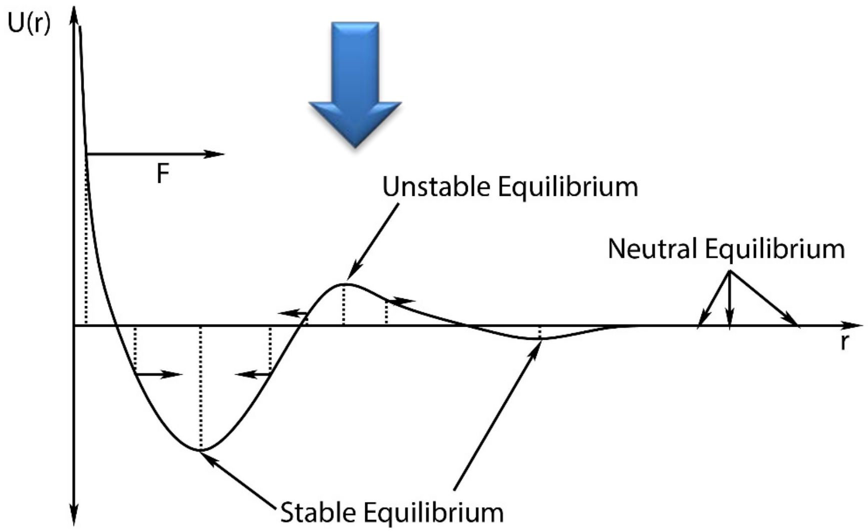
Equilibrium



In between, there is a point where the growing attraction of the inner potential energy well and the growing repulsion of the outer potential energy well *balance*, so that the potential energy function has a *maximum*. At this maximum the slope is zero (so it is a position of force equilibrium) but because the force on either side of this point pushes the particle *away* from it, this is a point of ***unstable equilibrium***. Unstable equilibria occur at ***isolated maxima*** in the potential energy function, just as stable equilibria occur at *isolated minima*.