Siberian Mathematical Journal, Vol. 59, *No.* 2, *pp.* 243–251, 2018

IDEAL SPACES OF MEASURABLE OPERATORS AFFILIATED TO A SEMIFINITE VON NEUMANN ALGEBRA

© **A. M. Bikchentaev** UDC 517.983:517.986

Abstract: Suppose that $\mathscr{M}$ is a von Neumann algebra of operators on a Hilbert space $\mathscr{H}$ and τ is a faithful normal semifinite trace on $\mathscr{M}$. Let $\mathscr{E}$, $\mathscr{F}$ and $\mathscr{G}$ be ideal spaces on $(\mathscr{M},\tau)$. We find when a τ-measurable operator X belongs to $\mathscr{E}$ in terms of the idempotent P of $\mathscr{M}$. The sets $\mathscr{E}+\mathscr{F}$ and $\mathscr{E}\cdot\mathscr{F}$ are also ideal spaces on $(\mathscr{M},\tau)$; moreover, $\mathscr{E}\cdot\mathscr{F}=\mathscr{F}\cdot\mathscr{E}$ and $(\mathscr{E}+\mathscr{F})\cdot\mathscr{G}=\mathscr{E}\cdot\mathscr{G}+\mathscr{F}\cdot\mathscr{G}$. The structure of ideal spaces is modular. We establish some new properties of the $L_1(\mathscr{M},\tau)$ space of integrable operators affiliated to the algebra $\mathscr{M}$. The results are new even for the $*$-algebra $\mathscr{M}=\mathscr{B}(\mathscr{H})$ of all bounded linear operators on $\mathscr{H}$ which is endowed with the canonical trace $\tau=\mathrm{tr}$.

DOI: 10.1134/S0037446618020064

Keywords: Hilbert space, linear operator, von Neumann algebra, normal semifinite trace, measurable operator, compact operator, integrable operator, commutator, ideal space

Introduction

The section of functional analysis, called noncommutative integration theory, is an important part of the theory of operator algebras. This article is devoted to noncommutative analogs of the classical methods for constructing function spaces. The beginning of the development of the corresponding aspect of noncommutative integration theory is related to the names of Segal and Dixmier, who in the early 1950s created a theory of integration with respect to a trace on a semifinite von Neumann algebra [1]. The results of these investigations found spectacular applications in the duality theory for unimodular groups and stimulated the progress of "noncommutative probability theory." The theory of algebras of measurable and locally measurable operators is rapidly developing and has interesting applications in various areas of functional analysis, mathematical physics, statistical mechanics, and quantum field theory.

In [2–4], Muratov introduced and investigated ideal spaces of measurable operators on a finite von Neumann algebra. They were also studied by Chilin in [5]. In the above-mentioned works, the ideal spaces serve primarily as the object of investigation. Recently, there have appeared publications in which they act as a tool. The foregoing demonstrates the relevance of (1), the search for new methods for constructing ideal spaces of measurable operators; (2), the development of a general theory of these spaces; and (3), the consideration of new particular examples.

Suppose that a von Neumann algebra $\mathscr{M}$ of operators acts on a Hilbert space $\mathscr{H}$, while τ is a faithful semifinite trace on $\mathscr{M}$. Let $\mathscr{E}$, $\mathscr{F}$, and $\mathscr{G}$ be ideal spaces on $(\mathscr{M},\tau)$. Let us list the obtained results. Given a normal τ-measurable operator X and an idempotent $P\in\mathscr{M}$, we show that $X\in\mathscr{E}\Leftrightarrow XP+P^{\perp}X\in\mathscr{E}\Leftrightarrow PXP+P^{\perp}X\in\mathscr{E}\Leftrightarrow XP+P^{\perp}XP^{\perp}\in\mathscr{E}$ (Theorem 1). The condition of normality for X is substantial in Theorem 1 (Example 3). The sets $\mathscr{E}+\mathscr{F}$ and $\mathscr{E}\cdot\mathscr{F}$ are also ideal spaces on $(\mathscr{M},\tau)$; moreover, $\mathscr{E}\cdot\mathscr{F}=\mathscr{F}\cdot\mathscr{E}$ and $(\mathscr{E}+\mathscr{F})\cdot\mathscr{G}=\mathscr{E}\cdot\mathscr{G}+\mathscr{F}\cdot\mathscr{G}$. The structure of ideal spaces is modular: if $\mathscr{E}\subset\mathscr{G}$ then $(\mathscr{E}+\mathscr{F})\cap\mathscr{G}=\mathscr{E}+(\mathscr{F}\cap\mathscr{G})$ (Theorems 2 and 3).

Let τ-measurable operators X, Y, and an idempotent $P\in\mathscr{M}$ be such that $XP-PY\in L_1(\mathscr{M},\tau)$. Then $\tau(XP-PY)=\tau(PXP-PYP)$ and for $X=Y$ we have $\tau([X,P])=0$ (Theorem 4). Let $\mathrm{Re}\,Y$

The author was financially supported by the Russian Foundation for Basic Research and the Government of the Republic of Tatarstan (Grant 15–41–02433) and by the subsidies allocated to Kazan Federal University for the state task in science (1.1515.2017/4.6 and 1.9773.2017/8.9).

Kazan. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 59, No. 2, pp. 309–320, March–April, 2018; DOI: 10.17377/smzh.2018.59.206. Original article submitted July 14, 2017.

and $\Im Y$ be the real and imaginary components of the τ-measurable operator Y and assume that $p \geq 1$ and $X \in L_{2p}(\mathscr{M}, \tau)$. Then (1) $\operatorname{Re} X^2, \Im X^2 \in L_p(\mathscr{M}, \tau)$ and $\max\{\|\operatorname{Re} X^2\|_p, \|\Im X^2\|_p\} \leq \|X^*X\|_p$; if $\tau(I) = 1$ then $\max\{\|\operatorname{Re} X\|_p, \|\Im X\|_p\} \leq \sqrt{\|X^*X\|_p}$ (Corollary 6); (2) $\|\operatorname{Re} X^2\|_p \leq 2^{\max\{0,2-2p\}}\|X\|_{2p}$ (Corollary 7). The results are new even for the $*$-algebra $\mathscr{M} = \mathscr{B}(\mathscr{H})$ of all bounded linear operators in $\mathscr{H}$ endowed with the canonical trace $\tau = \operatorname{tr}$.

1. Notations and Definitions

Suppose that $\mathscr{M}$ is the von Neumann algebra of operators on a Hilbert space $\mathscr{H}$, while $\mathscr{M}^{\mathrm{id}}$ and $\mathscr{M}^{\mathrm{pr}}$ are the subset of idempotents ($P = P^2$) and the lattice of projections ($P = P^2 = P^*$) in $\mathscr{M}$ respectively, I is the identity of $\mathscr{M}$, $P^\perp = I - P$ for $P \in \mathscr{M}^{\mathrm{id}}$, and $\mathscr{M}^+$ is the cone of positive elements in $\mathscr{M}$. The formula $S_P = 2P - I$ establishes a bijection between $\mathscr{M}^{\mathrm{id}}$ and the set of symmetries ($S^2 = I$).

A mapping $\varphi : \mathscr{M}^+ \to [0, +\infty]$ is called a *trace* if $\varphi(X + Y) = \varphi(X) + \varphi(Y)$, $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathscr{M}^+$, $\lambda \geq 0$ (here $0 \cdot (+\infty) \equiv 0$) and $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathscr{M}$. A trace φ is called *faithful* if $\varphi(X) > 0$ for all $X \in \mathscr{M}^+$, $X \neq 0$; *semifinite* if $\varphi(X) = \sup\{\varphi(Y) : Y \in \mathscr{M}^+,\ Y \leq X,\ \varphi(Y) < +\infty\}$ for every $X \in \mathscr{M}^+$; *normal* if $X_i \nearrow X$ ($X_i, X \in \mathscr{M}^+$) $\Rightarrow \varphi(X) = \sup \varphi(X_i)$ (see [6, Chapter V, § 2]). For a trace φ, put $\mathfrak{M}_\varphi^+ = \{X \in \mathscr{M}^+ :\ \varphi(X) < +\infty\}$, and $\mathfrak{M}_\varphi = \operatorname{lin}_{\mathbb{C}} \mathfrak{M}_\varphi^+$.

An operator $\mathscr{H}$ (not necessarily bounded or densely defined) is called *affiliated to a von Neumann algebra* $\mathscr{M}$ if $\mathscr{H}$ commutes with every unitary operator in the commutant $\mathscr{M}'$ of $\mathscr{M}$. From now on, τ is a faithful normal semifinite trace on $\mathscr{M}$. A closed operator X affiliated to $\mathscr{M}$ and having everywhere dense domain of definition $\mathscr{D}(X)$ in $\mathscr{H}$ is called τ*-measurable* if for every $\varepsilon > 0$ there exists $P \in \mathscr{M}^{\mathrm{pr}}$ such that $P\mathscr{H} \subset \mathscr{D}(X)$ and $\tau(P^\perp) < \varepsilon$. The set $\widetilde{\mathscr{M}}$ of all τ-measurable operators is a $*$-algebra with respect to passing to the adjoint operator, multiplication by a scalar, and the operations of strong addition and multiplication that are obtained by closing the usual operations [1, 7]. Given a family $\mathscr{L} \subset \widetilde{\mathscr{M}}$, denote by $\mathscr{L}^+$ and $\mathscr{L}^{\mathrm{sa}}$ its positive and Hermitian parts respectively. The partial order in $\widetilde{\mathscr{M}}^{\mathrm{sa}}$, generated by the proper cone $\widetilde{\mathscr{M}}^+$, will be denoted by $\leq$. If $X \in \widetilde{\mathscr{M}}$ and $X = U|X|$ is the polar decomposition of X then $U \in \mathscr{M}$ and $|X| \in \widetilde{\mathscr{M}}^+$.

Denote by $\mu_t(X)$ the *rearrangement* of $X \in \widetilde{\mathscr{M}}$, i.e., the nonincreasing right continuous function $\mu(X) : (0, \infty) \to [0, \infty)$ defined by the formula

$$\mu_t(X) = \inf\{\|XP\|_\infty : P \in \mathscr{M}^{\mathrm{pr}},\ \tau(P^\perp) \leq t\}, \quad t > 0,$$

where $\|\cdot\|_\infty$ is the uniform operator norm on the $*$-algebra $\mathscr{B}(\mathscr{H})$ of all bounded linear operators in $\mathscr{H}$. The set of τ-compact operators $\widetilde{\mathscr{M}}_0 = \{X \in \widetilde{\mathscr{M}} : \lim_{t\to\infty} \mu_t(X) = 0\}$ is an ideal in $\widetilde{\mathscr{M}}$ (see [8]).

Let m be the linear Lebesgue measure on $\mathbb{R}$. The noncommutative Lebesgue L_p-space ($0 < p < \infty$) associated with $(\mathscr{M}, \tau)$ can be defined as $L_p(\mathscr{M}, \tau) = \{X \in \widetilde{\mathscr{M}} : \mu_t(X) \in L_p(\mathbb{R}^+, m)\}$ with the F-norm (the norm for $1 \leq p < \infty$) $\|X\|_p = \|\mu_t(X)\|_p$, $X \in L_p(\mathscr{M}, \tau)$. The restriction $\tau|\mathfrak{M}_\tau^+$ extends to a bounded linear functional on $L_1(\mathscr{M}, \tau)$, which we will denote by the same symbol τ. We have $\mathfrak{M}_\tau = \mathscr{M} \cap L_1(\mathscr{M}, \tau)$, $L_p(\mathscr{M}, \tau) \subset \widetilde{\mathscr{M}}_0$, and $\|X\|_p = \tau(|X|^p)^{1/p}$ for all $0 < p < \infty$.

A subspace $\mathscr{E}$ in $\widetilde{\mathscr{M}}$ is called an *ideal space on* $(\mathscr{M}, \tau)$ (see [9, 10]) if (1) $X \in \mathscr{E}$ implies that $X^* \in \mathscr{E}$; (2) $X \in \mathscr{E}$, $Y \in \widetilde{\mathscr{M}}$, and $|Y| \leq |X|$ imply that $Y \in \mathscr{E}$. Such are, for example, the algebra $\mathscr{M}$, the set of elementary operators $\mathscr{F}(\mathscr{M})$, $\widetilde{\mathscr{M}}_0$, $(L_1 + L_\infty)(\mathscr{M}, \tau)$, and $L_p(\mathscr{M}, \tau)$ for $0 < p < \infty$. The real and imaginary components $\operatorname{Re} X = (X + X^*)/2$ and $\Im X = (X - X^*)/(2i)$ of $X \in \mathscr{E}$ lie in $\mathscr{E}$ as well.

If $\mathscr{M} = \mathscr{B}(\mathscr{H})$ and $\tau = \operatorname{tr}$ is the canonical trace then $\widetilde{\mathscr{M}}$ and $\widetilde{\mathscr{M}}_0$ coincide with $\mathscr{B}(\mathscr{H})$ and the ideal of compact operators in $\mathscr{H}$ respectively. We have

$$\mu_t(X) = \sum_{n=1}^{\infty} s_n(X)\chi_{[n-1,n)}(t), \quad t > 0,$$

where $\{s_n(X)\}_{n=1}^\infty$ is the sequence of the s-numbers of a compact operator X [11, Chapter 2, § 2]; χ_A is the indicator of a set $A \subset \mathbb{R}$. Then the $L_p(\mathscr{M}, \tau)$ space is the Schatten–von Neumann ideal $\mathfrak{S}_p$, $0 < p < \infty$.

Let (Ω, ν) be a measure space and let $\mathscr{M}$ be the von Neumann algebra of multiplication by functions in $L_\infty(\Omega, \nu)$ in $L_2(\Omega, \nu)$. The algebra $\mathscr{M}$ contains no nonzero compact operators if and only if the measure ν has no atoms [12, Theorem 8.4].

2. Lemmas and Examples

Lemma 1 [13, p. 720]. *If $X, Y \in \widetilde{\mathscr{M}}^{\mathrm{sa}}$ and $Z \in \widetilde{\mathscr{M}}$ then from $X \leq Y$ it follows that $ZXZ^* \leq ZYZ^*$.*

Lemma 2 [14, p. 261]. *If $X, Y \in \widetilde{\mathscr{M}}^{+}$ and $X \leq Y$ then there exists $Z \in \mathscr{M}$ with $\|Z\|_\infty \leq 1$ such that $\sqrt{X} = Z\sqrt{Y}$ and $X = ZYZ^*$.*

Lemma 3 [15, Theorem 17]. *If $X, Y \in \widetilde{\mathscr{M}}$ and $XY, YX \in L_1(\mathscr{M}, \tau)$ then $\tau(XY) = \tau(YX)$.*

Lemma 4. *If $X \in L_1(\mathscr{M}, \tau)$ then $\tau(X) = \tau(S_P X S_P)$ for all $P \in \mathscr{M}^{\mathrm{id}}$.*

Proof. If $X \in L_1(\mathscr{M}, \tau)$ then $AXB \in L_1(\mathscr{M}, \tau)$ for all $A, B \in \mathscr{M}$. Since $X \in L_1(\mathscr{M}, \tau)$, we have $XS_P \in L_1(\mathscr{M}, \tau)$. Since the operators $S_P \cdot XS_P$ and $XS_P \cdot S_P = X$ lie in $L_1(\mathscr{M}, \tau)$, by Lemma 3, $\tau(X) = \tau(S_P X S_P)$. The lemma is proved. □

Example 1. For $P \in \mathscr{M}^{\mathrm{id}}$ and $X \in \mathscr{M}^{\mathrm{pr}}$, the equality $\mu_t(X) = \mu_t(S_P X S_P)$ in general fails. In $\mathbb{M}_2(\mathbb{C})^{\mathrm{id}}$, choose

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $s_1(X) = 1 < \sqrt{5} = s_1(S_P X S_P)$.

Lemma 5. *If $\mathscr{E}$ is an ideal space on $(\mathscr{M}, \tau)$, $X \in \mathscr{E}$, and $Y, Z \in \mathscr{M}$ then $YXZ \in \mathscr{E}$. Therefore, $\mathscr{E} \cap \mathscr{M}$ is an ideal in $\mathscr{M}$. If $A \in \widetilde{\mathscr{M}}$ and $A^*A \in \mathscr{E}$ then $AA^* \in \mathscr{E}$.*

Proof. If $Y \in \mathscr{M}$ and $X \in \mathscr{E}$ then $Y^*Y \leq \|Y\|_\infty^2 \cdot I$ and $|YX| = \sqrt{X^*Y^*YX} \leq \|Y\|_\infty \cdot |X|$ by Lemma 2 and the operator monotonicity of $t \mapsto \sqrt{t}$ $(t \geq 0)$. Consequently, $YX \in \mathscr{E}$. Using this, the equality $(XZ)^* = Z^*X^*$, and the definition of ideal space, we see that $XZ \in \mathscr{E}$ for all $Z \in \mathscr{M}$ and $X \in \mathscr{E}$. If $A = U|A|$ is the polar decomposition of A then $U \in \mathscr{M}$ and $AA^* = UA^*AU^*$. The lemma is proved. □

Lemma 6. *Let $\mathscr{E}$ be an ideal space on $(\mathscr{M}, \tau)$. The following are equivalent for $X \in \widetilde{\mathscr{M}}$ and $P \in \mathscr{M}^{\mathrm{id}}$:*

(i) $XP + P^\perp X \in \mathscr{E}$;

(ii) $PXP + P^\perp X \in \mathscr{E}$;

(iii) $XP + P^\perp XP^\perp \in \mathscr{E}$.

Proof. (i)⇒(ii): We have

$$XP = \frac{1}{2}((XP + P^\perp X)P + P(XP + P^\perp X)) \in \mathscr{E}, \tag{1}$$

and so $PXP, P^\perp X \in \mathscr{E}$ by Lemma 5 and $PXP + P^\perp X \in \mathscr{E}$.

(ii)⇒(i): The operators $PXP = P(PXP + P^\perp X)$ and $P^\perp X = P^\perp(PXP + P^\perp X)$ lie in $\mathscr{E}$ by Lemma 5; therefore, $XP = P^\perp XP + PXP \in \mathscr{E}$. Thus, $XP + P^\perp X \in \mathscr{E}$.

(iii)⇒(i): The operators $XP = (XP + P^\perp XP^\perp)P$ and $P^\perp XP^\perp = (XP + P^\perp XP^\perp)P^\perp$ lie in $\mathscr{E}$ by Lemma 5; therefore, $PXP \in \mathscr{E}$ and

$$P^\perp X = X - PX = P^\perp XP^\perp + XP - PXP \in \mathscr{E}.$$

Thus, $XP + P^\perp X \in \mathscr{E}$.

(i)⇒(iii): We have $P^\perp XP^\perp = (XP + P^\perp X)P^\perp \in \mathscr{E}$ by Lemma 5. Now, (1) gives $XP \in \mathscr{E}$. The lemma is proved. □

Under the equivalent conditions of Lemma 6, we have $PXP + P^\perp XP^\perp \in \mathscr{E}$.

Lemma 7. *Let $\mathscr{E}$ be an ideal space on $(\mathscr{M}, \tau)$. The following are equivalent for $X \in \widetilde{\mathscr{M}}^{+}$ and $P \in \mathscr{M}^{\mathrm{pr}}$:*
(i) $X \in \mathscr{E}$;
(ii) $PXP + P^{\perp}XP^{\perp} \in \mathscr{E}$.

PROOF. (ii)⇒(i): For the selfadjoint symmetry $S_P = 2P - I$, by Lemma 1, we have $S_P X S_P \geq 0$ and

$$0 \leq \frac{1}{2}X \leq \frac{1}{2}(X + S_P X S_P) = PXP + P^{\perp}XP^{\perp}. \tag{2}$$

The implication (i)⇒(ii) follows, for example, from Lemma 5. The lemma is proved. □

In [16, Theorem 4.8], it was proved that if $\tau(I) = 1$ then the following are equivalent for $X \in L_1(\mathscr{M}, \tau)$: (i) $\tau(X) = 0$; and (ii) $\|I + zX\|_1 \geq 1$ for all $z \in \mathbb{C}$. In particular, if $\tau(I) = 1$ and $A, B \in \mathscr{M}$ then $\|I + z[A, B]\|_1 \geq 1$ for all $z \in \mathbb{C}$. If $\tau(I) = 1$ and $X \in L_1(\mathscr{M}, \tau)$ then $\|I + z(X - S_P X S_P)\|_1 \geq 1$ for all $P \in \mathscr{M}^{\mathrm{id}}$ and $z \in \mathbb{C}$ (see Lemma 4).

Lemma 8 [17, Theorem 2.23]. *Let $P = P^2 \in \widetilde{\mathscr{M}}$. There exists a unique decomposition $P = \widetilde{P} + Z$, where $\widetilde{P} \in \mathscr{M}^{\mathrm{pr}}$ is the range projection of the idempotent P and a nilpotent $Z \in \widetilde{\mathscr{M}}$ with $Z^2 = 0$ and $Z\widetilde{P} = 0$, $\widetilde{P}Z = Z$.*

EXAMPLE 2 [18, Example 1]. Suppose that $0 < p, q < \infty$ and $a_n = 2^{n+1}n^{-q}$, $n \in \mathbb{N}$. Endow the von Neumann algebra $\mathscr{M} = \bigoplus_{n=1}^{\infty} \mathbb{M}_2(\mathbb{C})$ with a faithful normal finite trace $\tau = \bigoplus_{n=1}^{\infty} 2^{-n} \operatorname{tr}_2$ and put $A = \bigoplus_{n=1}^{\infty} \begin{pmatrix} 1 & a_n \\ 0 & 0 \end{pmatrix}$. We have $A = A^2$ and $A \in L_p(\mathscr{M}, \tau)$ for $pq > 1$ and $A \notin L_p(\mathscr{M}, \tau)$ for $pq \leq 1$.

Lemma 9. *The inequality $Z|T|Z^* \leq |TZ^*|$ is fulfilled for all $T \in \widetilde{\mathscr{M}}$ and $Z \in \mathscr{M}$ with $\|Z\|_\infty \leq 1$.*

PROOF. Since $t \mapsto \sqrt{t}$ $(t \geq 0)$ is operator monotone, Hansen's Theorem of [19] gives $Z|T|Z^* = Z\sqrt{T^*T}Z^* \leq \sqrt{ZT^*TZ^*} = |TZ^*|$. □

2. The Main Results

Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathscr{M}$.

Theorem 1. *Let $\mathscr{E}$ be an ideal space on $(\mathscr{M}, \tau)$. The following are equivalent for a normal $X \in \widetilde{\mathscr{M}}$ and $P \in \mathscr{M}^{\mathrm{id}}$:*
(i) $X \in \mathscr{E}$;
(ii) $XP + P^{\perp}X \in \mathscr{E}$;
(iii) $PXP + P^{\perp}X \in \mathscr{E}$;
(iv) $XP + P^{\perp}XP^{\perp} \in \mathscr{E}$.

PROOF. (ii)⇒(i): $XP \in \mathscr{E}$ by (1); therefore,

$$P^{\perp}X \in \mathscr{E}. \tag{3}$$

STEP 1. Let $X \in \widetilde{\mathscr{M}}^{+}$. Consider the decomposition $P = \widetilde{P} + Z$ described in Lemma 8 with $Z\widetilde{P} = 0$ and $\widetilde{P}Z = Z$, $Z^2 = 0$. From (1) we obtain $XZ = (X\widetilde{P} + XZ)Z = XP \cdot Z \in \mathscr{E}$ by Lemma 5; thus, $X\widetilde{P} = XP - XZ \in \mathscr{E}$. By (3),

$$ZX = Z(\widetilde{P}^{\perp} - Z)X = Z \cdot P^{\perp}X \in \mathscr{E};$$

consequently, $\widetilde{P}^{\perp}X = P^{\perp}X + ZX \in \mathscr{E}$. Thus, the operators $\widetilde{P}X\widetilde{P}$ and $\widetilde{P}^{\perp}X\widetilde{P}^{\perp}$ lie in $\mathscr{E}$ and $\widetilde{P}X\widetilde{P} + \widetilde{P}^{\perp}X\widetilde{P}^{\perp} \in \mathscr{E}$. Now, $X \in \mathscr{E}$ by Lemma 7.

STEP 2. Suppose that $X \in \widetilde{\mathscr{M}}$ is normal. Consider the polar decomposition $X = V|X|$, where $V \in \mathscr{M}$ and $V|X| = |X|V$ (see [20]). We infer that

$$|X|P = V^*V|X|P = V^* \cdot XP \in \mathscr{E} \tag{4}$$

on using (1) and Lemma 6. For $P_1 = P^{*\perp}$ and $X_1 = X^*$, we obtain $|X_1| = |X|$ and

$$X_1P_1 + P_1^{\perp}X_1 = (XP + P^{\perp}X)^* \in \mathscr{E}.$$

By the above (see (4)), we have $|X_1|P_1 \in \mathscr{E}$; therefore, $P^\perp|X| = (|X_1|P_1)^* \in \mathscr{E}$. Now, $|X| \in \mathscr{E}$ by step 1. Consequently, $X \in \mathscr{E}$.

The equivalence of (ii)–(iv) is established in Lemma 6. The theorem is proved. $\square$

Corollary 1. *Let a normal operator $X \in \widetilde{\mathscr{M}}$ and $P \in \mathscr{M}^{\mathrm{id}}$ be such that $XP + P^\perp X \in L_1(\mathscr{M}, \tau)$. Then $X \in L_1(\mathscr{M}, \tau)$ and $\tau(XP + P^\perp X) = \tau(X)$.*

Example 3. The normality of $X \in \widetilde{\mathscr{M}}$ is substantial in Theorem 1 and Corollary 1. Endow the von Neumann algebra $\mathscr{M} = \bigoplus_{n=1}^\infty \mathbb{M}_2(\mathbb{C})$ with the faithful normal semifinite trace $\tau = \bigoplus_{n=1}^\infty \operatorname{tr}_2$ and put

$$X = \bigoplus_{n=1}^\infty \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \quad P = \bigoplus_{n=1}^\infty \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

Then $XP = P^\perp X = 0$ but $X \notin L_1(\mathscr{M}, \tau)$.

Theorem 2. *Let $\mathscr{E}$ and $\mathscr{F}$ be ideal spaces on $(\mathscr{M}, \tau)$ and take $p \in \{2^k : k \in \mathbb{N}\}$. Then the sets $\mathscr{E} \cap \mathscr{F}$, $\mathscr{E} + \mathscr{F} = \{A + B : A \in \mathscr{E},\ B \in \mathscr{F}\}$, $\mathscr{E} \cdot \mathscr{F} = \big\{X \in \widetilde{\mathscr{M}} : \exists \{A_k\}_{k=1}^n \subset \mathscr{E}$, and $\{B_k\}_{k=1}^n \subset \mathscr{F}$ such that $|X| \le \sum_{k=1}^n |A_k B_k|\big\}$ as well as $\mathscr{E}_p = \{A \in \widetilde{\mathscr{M}} : |A|^p \in \mathscr{E}\}$ are ideal spaces on $(\mathscr{M}, \tau)$ too. Moreover, $\mathscr{E} \cdot \mathscr{F} = \mathscr{F} \cdot \mathscr{E}$, $(\mathscr{E} \cap \mathscr{F})^{\mathrm{sa}} = \mathscr{E}^{\mathrm{sa}} \cap \mathscr{F}^{\mathrm{sa}}$, $(\mathscr{E} \cap \mathscr{F})^+ = \mathscr{E}^+ \cap \mathscr{F}^+$, and $(\mathscr{E} + \mathscr{F})^{\mathrm{sa}} = \mathscr{E}^{\mathrm{sa}} + \mathscr{F}^{\mathrm{sa}}$, $(\mathscr{E} + \mathscr{F})^+ = \mathscr{E}^+ + \mathscr{F}^+$.*

Proof. If $A, B \in \widetilde{\mathscr{M}}$ and $c > 0$ then

$$|A + B|^2 \le (1 + c)|A|^2 + \left(1 + \frac{1}{c}\right)|B|^2,$$

where equality holds if and only if $B = cA$. This follows since $\big(\sqrt{c}A - \frac{1}{\sqrt{c}}B\big)^*\big(\sqrt{c}A - \frac{1}{\sqrt{c}}B\big) \ge 0$. Therefore, $A + B \in \mathscr{E}_2$ for all $A, B \in \mathscr{E}_2$. Let $A = U|A|$ be the polar decomposition of $A \in \mathscr{E}_2$. Then $|A^*|^2 = U|A|^2U^* \in \mathscr{E}_2$ by Lemma 5; thus, $A^* \in \mathscr{E}_2$. Let $A \in \mathscr{E}_2$, $B \in \widetilde{\mathscr{M}}$, and $|B| \le |A|$. Owing to Lemma 2, there is $Z \in \mathscr{M}$ with $\|Z\|_\infty \le 1$ with $|B| = Z|A|Z^*$. Hence, $Z^*Z \le I$ and, by Lemmas 1 and 5, we infer

$$|B|^2 = Z|A|Z^*Z|A|Z^* \le Z|A|^2Z^* \in \mathscr{E}.$$

Thus, $|B|^2 \in \mathscr{E}$ and $\mathscr{E}_2$ are ideal spaces on $(\mathscr{M}, \tau)$. Note also that

$$\mathscr{E}_4 = (\mathscr{E}_2)_2,\ \mathscr{E}_8 = (\mathscr{E}_4)_2, \dots,\ \mathscr{E}_{2^k} = (\mathscr{E}_{2^{k-1}})_2$$

for all $k \in \mathbb{N}$.

Assume that $A \in \mathscr{E} + \mathscr{F}$, $B \in \widetilde{\mathscr{M}}$, and $|B| \le |A|$. Then $A = A_1 + A_2$ with $A_1 \in \mathscr{E}$, $A_2 \in \mathscr{F}$, and there are partial isometries $V, W \in \mathscr{M}$ such that $|B| \le |A_1 + A_2| \le V|A_1|V^* + W|A_2|W^*$ [21, Theorem 2.2]. Consequently, $|B| = ZV|A_1|V^*Z^* + ZW|A_2|W^*Z^*$ for some $Z \in \mathscr{M}$ with $\|Z\|_\infty \le 1$ by Lemma 2. If $B = U|B|$ is the polar decomposition then $B = U|B| = UZV|A_1|V^*Z^* + UZW|A_2|W^*Z^*$, where $UZV|A_1|V^*Z^* \in \mathscr{E}$ and $UZW|A_2|W^*Z^* \in \mathscr{F}$ by Lemma 5. Thus, $B \in \mathscr{E} + \mathscr{F}$.

Let $X = U|X|$ be the polar decomposition of $X \in \mathscr{E} \cdot \mathscr{F}$. Then, by Lemmas 1 and 9, we have

$$|X^*| = U|X|U^* \le \sum_{k=1}^n U|A_k B_k|U^* \le \sum_{k=1}^n |A_k \cdot B_k U^*|,$$

where $\{B_k U^*\}_{k=1}^n \subset \mathscr{F}$ by Lemma 5. Thus, $X^* \in \mathscr{E} \cdot \mathscr{F}$.

Suppose that $X, Y \in \mathscr{E} \cdot \mathscr{F}$ and $|Y| \le \sum_{j=1}^l |C_j D_j|$ with some $\{C_j\}_{j=1}^l \subset \mathscr{E}$ and $\{D_j\}_{j=1}^l \subset \mathscr{F}$. Then there are partial isometries $V, W \in \mathscr{M}$ with $|X + Y| \le V|X|V^* + W|Y|W^*$ [21, Theorem 2.2]. Lemmas 1 and 9 yield

$$|X + Y| \le \sum_{k=1}^n V|A_k B_k|V^* + \sum_{j=1}^l W|C_j D_j|W^* \le \sum_{k=1}^n |A_k \cdot B_k V^*| + \sum_{j=1}^l |C_j \cdot D_j W^*|,$$

where $\{B_k V^*\}_{k=1}^n, \{D_j W^*\}_{j=1}^l \subset \mathscr{F}$ by Lemma 5. Hence, $X + Y \in \mathscr{E} \cdot \mathscr{F}$.

For checking that $\mathscr{E}\cdot\mathscr{F}=\mathscr{F}\cdot\mathscr{E}$, it suffices to show that $\mathscr{E}\cdot\mathscr{F}\subset\mathscr{F}\cdot\mathscr{E}$. Suppose that $X\in\mathscr{E}\cdot\mathscr{F}$ and $|X|\le\sum_{k=1}^n|A_kB_k|$ with some $\{A_k\}_{k=1}^n\subset\mathscr{E}$ and $\{B_k\}_{k=1}^n\subset\mathscr{F}$. Let $B_k^*A_k^*=U_k|B_k^*A_k^*|$ be the polar decomposition of $B_k^*A_k^*=(A_kB_k)^*$, $k=1,2,\dots,n$. Then, by Lemma 9,

$$|A_kB_k|=|(B_k^*A_k^*)^*|=U_k|B_k^*A_k^*|U_k^*\le|B_k^*\cdot A_k^*U_k^*|,\quad k=1,2,\dots,n.$$

Thus, $|X|\le\sum_{k=1}^n|B_k^*\cdot A_k^*U_k^*|$. Since $\{A_k^*U_k^*\}_{k=1}^n\subset\mathscr{E}$, by Lemma 5 we obtain $X\in\mathscr{F}\cdot\mathscr{E}$.

If $X\in(\mathscr{E}+\mathscr{F})^{\mathrm{sa}}$ then $X=X^*=A+B$ with $A\in\mathscr{E}$ and $B\in\mathscr{F}$. Then $X=2^{-1}(A+A^*)+2^{-1}(B+B^*)$, where $2^{-1}(A+A^*)\in\mathscr{E}^{\mathrm{sa}}$ and $2^{-1}(B+B^*)\in\mathscr{F}^{\mathrm{sa}}$.

If $X\in(\mathscr{E}+\mathscr{F})^+$ then $X=|X|=A+B$ with $A\in\mathscr{E}$ and $B\in\mathscr{F}$. By [21, Theorem 2.2], $X=|A+B|\le U|A|U^*+V|B|V^*$ with some partial isometries $U,V\in\mathscr{M}$. By Lemma 2, there exists $Z\in\mathscr{M}$ with $\|Z\|_\infty\le1$ such that $X=ZU|A|U^*Z^*+ZV|B|V^*Z^*$. Now, $ZU|A|U^*Z^*\in\mathscr{E}^+$ and $ZV|B|V^*Z^*\in\mathscr{F}^+$ by Lemma 5. The theorem is proved. □

Theorem 3. *The structure of ideal spaces is modular: if $\mathscr{E}$, $\mathscr{F}$, and $\mathscr{G}$ are ideal spaces on $(\mathscr{M},\tau)$ and $\mathscr{E}\subset\mathscr{G}$ then $(\mathscr{E}+\mathscr{F})\cap\mathscr{G}=\mathscr{E}+(\mathscr{F}\cap\mathscr{G})$; moreover, $(\mathscr{E}+\mathscr{F})\cdot\mathscr{G}=\mathscr{E}\cdot\mathscr{G}+\mathscr{F}\cdot\mathscr{G}$.*

Proof. Prove the inclusion $\supset$. If $A\in\mathscr{E}$ and $B\in\mathscr{F}\cap\mathscr{G}$ then $A+B\in\mathscr{G}$ and $A+B\in\mathscr{E}+\mathscr{F}$.

Prove the inclusion $\subset$. If $X\in(\mathscr{E}+\mathscr{F})\cap\mathscr{G}$ then $X=A+B$ with some $A\in\mathscr{E}$ and $B\in\mathscr{F}$. Therefore, $B=X-A\in\mathscr{G}$ (since $\mathscr{E}\subset\mathscr{G}$); i.e., $B\in\mathscr{F}\cap\mathscr{G}$.

Since $\mathscr{E}\cdot\mathscr{G},\mathscr{F}\cdot\mathscr{G}\subset(\mathscr{E}+\mathscr{F})\cdot\mathscr{G}$ and $(\mathscr{E}+\mathscr{F})\cdot\mathscr{G}$ is a subspace of $\widetilde{\mathscr{M}}$, we have $\mathscr{E}\cdot\mathscr{G}+\mathscr{F}\cdot\mathscr{G}\subset(\mathscr{E}+\mathscr{F})\cdot\mathscr{G}$. Let $X\in(\mathscr{E}+\mathscr{F})\cdot\mathscr{G}$, i.e., $|X|\le\sum_{k=1}^n|(A_k+B_k)C_k|$ with some $\{A_k\}_{k=1}^n\subset\mathscr{E}$, $\{B_k\}_{k=1}^n\subset\mathscr{F}$ and $\{C_k\}_{k=1}^n\subset\mathscr{G}$. By [21, Theorem 2.2], there are partial isometries $V_k,W_k\in\mathscr{M}$ such that

$$|(A_k+B_k)C_k|\le V_k|A_kC_k|V_k^*+W_k|B_kC_k|W_k^*,\quad k=1,2.\dots,n.$$

Then $|X|\le\sum_{k=1}^nV_k|A_kC_k|V_k^*+W_k|B_kC_k|W_k^*$ and, by Lemma 2, there exists $Z\in\mathscr{M}$ with $\|Z\|_\infty\le1$ such that

$$|X|=\sum_{k=1}^nZV_k|A_kC_k|V_k^*Z^*+ZW_k|B_kC_k|W_k^*Z^*.$$

Let $X=U|X|$ be the polar decomposition of X. Then $X=A+B$, where $A=\sum_{k=1}^nUZV_k|A_kC_k|V_k^*Z^*\in\mathscr{E}\cdot\mathscr{G}$ and $B=\sum_{k=1}^nUZW_k|B_kC_k|W_k^*Z^*\in\mathscr{F}\cdot\mathscr{G}$ by Lemma 5. The theorem is proved. □

Remark 1. We have $\mathscr{E}+\mathscr{F}=\widetilde{\mathscr{M}}$ for $\mathscr{E}=\mathscr{M}$ and $\mathscr{F}=\widetilde{\mathscr{M}}_0$ [22]. If $0<p<\infty$ and $\mathscr{E}=L_p(\mathscr{M},\tau)$ then $\mathscr{E}_2=L_{2p}(\mathscr{M},\tau)$. In [23], for operators $X,Y\in\widetilde{\mathscr{M}}$, sufficient conditions were established that $XY,YX\in L_1(\mathscr{M},\tau)$. For such operators, $\tau([X,Y])=0$ by Lemma 3.

Proposition 1. *Let $P=P^2\in\widetilde{\mathscr{M}}$ and let $P=\widetilde{P}+Z$ be the decomposition described in Lemma 8. The following are equivalent:*

(i) $P\in\widetilde{\mathscr{M}}_0$;

(ii) $\widetilde{P}\in\mathfrak{M}_\tau^+$ *and* $Z\in\widetilde{\mathscr{M}}_0$.

Proof. (i)⇒(ii): We have $Z=\widetilde{P}Z=PZ\in\widetilde{\mathscr{M}}_0$ and

$$\widetilde{P}=P-Z\in\widetilde{\mathscr{M}}_0. \tag{5}$$

Since $\widetilde{P}\in\mathscr{M}^{\mathrm{pr}}$, we have $\mu_t(\widetilde{P})\in\{0,1\}$ for all $t>0$. Now, $\widetilde{P}\in\mathfrak{M}_\tau^+$ by (5), and the proposition is proved. □

Theorem 4. *Let operators $X,Y\in\widetilde{\mathscr{M}}$ and $P\in\mathscr{M}^{\mathrm{id}}$ be such that $XP-PY\in L_1(\mathscr{M},\tau)$. Then $\tau(XP-PY)=\tau(PXP-PYP)$ and for $X=Y$ we get $\tau([X,P])=0$.*

Proof. Owing to Lemma 5,

$$PXP-PYP=\frac{1}{2}(S_P(XP-PY)S_P+(XP-PY))\in L_1(\mathscr{M},\tau).$$

By Lemma 7,

$$\tau(XP - PY) = \tau(S_P(XP - PY)S_P) = \tau(2(PXP - PYP) - (XP - PY));$$

therefore, $\tau(XP - PY) = \tau(PXP - PYP) = \tau(P(X - Y)P)$, and the theorem is proved. $\square$

Theorem 4 and Lemma 1 imply

Corollary 2. *Let $X, Y \in \widetilde{\mathscr{M}}^{\mathrm{sa}}$ with $Y \le X$ and $P \in \mathscr{M}^{\mathrm{pr}}$ be such that $XP - PY \in L_1(\mathscr{M}, \tau)$. Then $\tau(XP - PY) \ge 0$.*

Proposition 2. *Suppose that $\mathscr{E}$ is an ideal space on $(\mathscr{M}, \tau)$, $P, Q \in \mathscr{M}^{\mathrm{id}}$, and $P = \widetilde{P} + Z$, $Q = \widetilde{Q} + T$ are the decomposition described in Lemma 8. The following are equivalent:*
(i) $P - Q \in \mathscr{E}$;
(ii) $\widetilde{P} - \widetilde{Q}, Z - T \in \mathscr{E}$;
(iii) $Q^{\perp}P, QP^{\perp} \in \mathscr{E}$.

Proof. (i)$\Rightarrow$(ii): We have

$$\widetilde{P} - \widetilde{Q}\widetilde{P} + T\widetilde{P} = (P - Q)\widetilde{P} \in \mathscr{E} \tag{6}$$

and $T\widetilde{P} = \widetilde{Q}(\widetilde{P} - \widetilde{Q}\widetilde{P} + T\widetilde{P}) \in \mathscr{E}$. Now (6) yields $\widetilde{P} - \widetilde{Q}\widetilde{P} \in \mathscr{E}$. Similarly, $\widetilde{Q} - \widetilde{P}\widetilde{Q} \in \mathscr{E}$. Therefore, $\widetilde{P} - \widetilde{Q} = (\widetilde{P} - \widetilde{Q}\widetilde{P})^* - (\widetilde{Q} - \widetilde{P}\widetilde{Q}) \in \mathscr{E}$.

The equivalence (i)$\Leftrightarrow$(iii) follows from the equalities

$$P - Q = Q^{\perp}P - QP^{\perp}, \quad Q^{\perp}P = Q^{\perp}(P - Q), \quad QP^{\perp} = -(P - Q)P^{\perp}$$

and Lemma 4. The proposition is proved. $\square$

Proposition 3. *Let $\mathscr{E}$ be an ideal space on $(\mathscr{M}, \tau)$. If $P, Q \in \mathscr{M}^{\mathrm{id}}$ and $\{P - Q, P + Q - I\} \cap \mathscr{E} \ne \varnothing$ then $[P, Q] \in \mathscr{E}$.*

Proof. The proposition follows from Lemma 5 and

$$(P + Q - I)(P - Q) = QP - PQ = (Q - P)(P + Q - I). \quad \square$$

Proposition 4. *Suppose that $\mathscr{E}$ is an ideal space on $(\mathscr{M}, \tau)$, $X \in \widetilde{\mathscr{M}}$, $P \in \mathscr{M}^{\mathrm{id}}$, and $P = \widetilde{P} + Z$ is the decomposition described in Lemma 8. The following are equivalent:*
(i) $P^{\perp}XP \in \mathscr{E} \Leftrightarrow \widetilde{P}^{\perp}X\widetilde{P}, ZX\widetilde{P}, ZXZ, \widetilde{P}^{\perp}XZ \in \mathscr{E}$;
(ii) $P^{\perp}XP = 0 \Leftrightarrow \widetilde{P}^{\perp}X\widetilde{P} = ZX\widetilde{P} = ZXZ = \widetilde{P}^{\perp}XZ = 0$.

Proof. (i)$\Rightarrow$(ii): By Lemma 5,

$$\widetilde{P}^{\perp}X\widetilde{P} - ZX\widetilde{P} = (P^{\perp}XP)\widetilde{P} \in \mathscr{E}. \tag{7}$$

Therefore, $ZX\widetilde{P} = Z(\widetilde{P}^{\perp}X\widetilde{P} - ZX\widetilde{P}) \in \mathscr{E}$ and (7) gives $\widetilde{P}^{\perp}X\widetilde{P} \in \mathscr{E}$. Now,

$$V = P^{\perp}XP - \widetilde{P}^{\perp}X\widetilde{P} = \widetilde{P}^{\perp}XZ - ZX\widetilde{P} - ZXZ \in \mathscr{E}$$

and $ZXZ = ZV \in \mathscr{E}$. Consequently, $\widetilde{P}^{\perp}XZ - ZX\widetilde{P} = V - VZ \in \mathscr{E}$ and $ZA\widetilde{P} = -(V - VZ)\widetilde{P} \in \mathscr{E}$. The rest is obvious. $\square$

Corollary 3. *Let $\mathscr{E} = L_1(\mathscr{M}, \tau)$ under the conditions of Proposition 4. Then $\tau(ZXZ) = 0$ and $\tau(\widetilde{P}^{\perp}XZ) = \tau(ZXP)$.*

Proof. The operators $\widetilde{P} \cdot ZXZ = ZXZ$ and $ZXZ \cdot \widetilde{P} = 0$ lie in $L_1(\mathscr{M}, \tau)$; therefore, $\tau(ZXZ) = \tau(\widetilde{P} \cdot ZXZ) = \tau(ZXZ \cdot \widetilde{P}) = \tau(0) = 0$ by Lemma 3. Similarly, $\tau(P^{\perp}XP) = \tau(\widetilde{P}^{\perp}X\widetilde{P}) = 0$, and the proposition ensues from the equality $\tau(P^{\perp}XP) = \tau(\widetilde{P}^{\perp}X\widetilde{P}) - \tau(ZXZ) + \tau(\widetilde{P}^{\perp}XZ) - \tau(ZX\widetilde{P})$. $\square$

Remark 2. If $\mathscr{E} = \widetilde{\mathscr{M}}_0$ then Propositions 2–4 are carried over (with similar proofs) to unbounded idempotents $P, Q \in \widetilde{\mathscr{M}}$.

Theorem 5. *Suppose that $\lambda > 0$, $X \in \widetilde{\mathscr{M}}$, and $Y = X^*X + XX^*$. Then there exist unitary operators $S_\lambda, S \in \mathscr{M}^{\mathrm{sa}}$ such that*

$$|\operatorname{Re} X| \le \frac{1}{4\lambda}(X^*X + S_\lambda X^*XS_\lambda) + \frac{\lambda}{2}I, \tag{8}$$

$$|\operatorname{Re} X^2| \le \frac{1}{4}(Y + SYS). \tag{9}$$

Proof. The inequality $(X \pm \lambda I)^*(X \pm \lambda I) \ge 0$ yields $\mp\lambda(X + X^*) \le X^*X + \lambda^2 I$. Therefore,

$$-\left(\frac{1}{\lambda}X^*X + \lambda I\right) \le X^* + X \le \frac{1}{\lambda}X^*X + \lambda I$$

and by [24, Theorem 1] there exists a unitary operator $S_\lambda \in \mathscr{M}^{\mathrm{sa}}$ such that

$$2|X + X^*| \le \frac{1}{\lambda}X^*X + \lambda I + S_\lambda\left(\frac{1}{\lambda}X^*X + \lambda I\right)S_\lambda = \frac{1}{\lambda}(X^*X + S_\lambda X^*XS_\lambda) + 2\lambda I.$$

Thus, (8) holds

Since $Z = i(X - X^*)$ is selfadjoint, $Z^2 \ge 0$; therefore, $Y \ge 2\operatorname{Re} X^2$. Since $(X + X^*)^2 \ge 0$, we have $2\operatorname{Re} X^2 \ge -Y$. Thus, $-Y \le 2\operatorname{Re} X^2 \le Y$, and (9) follows from [24, Theorem 1]. Note that if $X = I$ then (8) for $\lambda = 1$ and (9) become equalities. The theorem is proved. $\square$

Since $\Im X = \operatorname{Re}(iX)$, $(iX)^*(iX) = X^*X$, $(iX)(iX)^* = XX^*$, and $\Im X^2 = \operatorname{Re}(iX^2) = \operatorname{Re}((zX)^2)$ with $z = e^{\frac{\pi}{4}i}$, $(zX)^*(zX) = X^*X$, and $(zX)(zX)^* = XX^*$ for all $X \in \widetilde{\mathscr{M}}$, we have

Corollary 4. *Suppose that $\lambda > 0$, $X \in \widetilde{\mathscr{M}}$, and $Y = X^*X + XX^*$. Then there exist unitary operators $T_\lambda, T \in \mathscr{M}^{\mathrm{sa}}$ such that*

$$|\Im X| \le \frac{1}{4\lambda}(X^*X + T_\lambda X^*XT_\lambda) + \frac{\lambda}{2}I, \tag{10}$$

$$|\Im X^2| \le \frac{1}{4}(Y + TYT). \tag{11}$$

Corollary 5 (cf. [25, Corollary 3.4]). *Let $X = X^2 \in \widetilde{\mathscr{M}}$ and $Y = X^*X + XX^*$. Then there exist unitary operators $U, V \in \mathscr{M}^{\mathrm{sa}}$ such that $4|\operatorname{Re} X| \le Y + UYU$ and $4|\Im X| \le Y + VYV$.*

Corollary 6. *Suppose that $p \ge 1$ and $X \in L_{2p}(\mathscr{M}, \tau)$. Then*

(i) *$\operatorname{Re} X^2, \Im X^2 \in L_p(\mathscr{M}, \tau)$ and $\max\{\|\operatorname{Re} X^2\|_p, \|\Im X^2\|_p\} \le \|X^*X\|_p$;*

(ii) *if $\tau(I) = 1$ then $\max\{\|\operatorname{Re} X\|_p, \|\Im X\|_p\} \le \sqrt{\|X^*X\|_p}$.*

Proof. (i) By (9), (11), the triangle inequality for $\|\cdot\|_p$, the equality $\|XX^*\|_p = \|X^*X\|_p$, and the unitary invariance of the norm $\|\cdot\|_p$, we obtain $\max\{\|\operatorname{Re} X^2\|_p, \|\Im X^2\|_p\} \le \|X^*X\|_p$. If $X \in \mathscr{M}^{\mathrm{pr}}$ then this inequality turns into equality.

(ii) If $\tau(I) = 1$ then, by (8), (10), the triangle inequality for $\|\cdot\|_p$, and the unitary equivalence of the norm $\|\cdot\|_p$, we come to the relation

$$\max\{\|\operatorname{Re} X\|_p, \|\Im X\|_p\} \le \frac{1}{2}\left(\frac{1}{\lambda}\|X^*X\|_p + \lambda\right).$$

Note that $\min_{\lambda>0}\left(\frac{1}{\lambda}\|X^*X\|_p + \lambda\right)$ is attained at the point $\lambda_0 = \sqrt{\|X^*X\|_p}$. The corollary is proved. $\square$

Lemma 5 and Theorem 5 give

Corollary 7. *Suppose that $\mathscr{E}$ is an ideal space on $(\mathscr{M}, \tau)$ and $X \in \widetilde{\mathscr{M}}$. If $X \in \mathscr{E}_2$ then $\operatorname{Re} X^2 \in \mathscr{E}$. In particular, if $0 < p < +\infty$ and $X \in L_{2p}(\mathscr{M}, \tau)$ then $\operatorname{Re} X^2 \in L_p(\mathscr{M}, \tau)$ and $\|\operatorname{Re} X^2\|_p \le 2^{\max\{0, 2-2p\}}\|X\|_{2p}$.*

The author is grateful to the referee for sound advice.

References

1. Segal I. E., "A non-commutative extension of abstract integration," Ann. Math., vol. 57, no. 3, 401–457 (1953).
2. Muratov M. A., "Ideal subspaces in the ring of measurable operators," in: *Functional Analysis* [Russian], Tashkent Univ., Tashkent, 1978, 51–58.
3. Muratov M. A., "A condition of being fundamental for ideal subspaces of measurable operators," in: Sb. Nauch. Tr. Tashkent Univ., Tashkent Univ., Tashkent, 1982, no. 689, 37–40.
4. Muratov M. A., "Nonsymmetric ideal spaces of measurable operators," in: *Mathematical Analysis and Differential Equations* [Russian], Tashkent Univ., Tashkent, 1984, 56–60.
5. Chilin V. I., "The triangle inequality in the algebras of locally measurable operators," in: *Mathematical Analysis and Algebra* [Russian], Tashkent Univ., Tashkent, 1986, 77–81.
6. Takesaki M., *Theory of Operator Algebras. I*, Springer-Verlag, Berlin (2002) (Encyclopaedia Math. Sci.; vol. 124. Operator Algebras and Non-Commutative Geometry, 5).
7. Nelson E., "Notes on non-commutative integration," J. Funct. Anal., vol. 15, no. 2, 103–116 (1974).
8. Yeadon F. J., "Non-commutative L^p-spaces," Math. Proc. Cambridge Philos. Soc., vol. 77, no. 1, 91–102 (1975).
9. Bikchentaev A. M., "On a property of L_p spaces on semifinite von Neumann algebras," Math. Notes, vol. 64, no. 2, 159–163 (1998).
10. Ber A. F., Levitina G. B., and Chilin V. I., "Derivations with values in quasi-normed bimodules of locally measurable operators," Siberian Adv. Math., vol. 25, no. 3, 169–178 (2015).
11. Gokhberg I. Ts. and Krein M. G., *An Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space*, Amer. Math. Soc., Providence (1969).
12. Antonevich A. B., Linear Functional Equations. Operator Approach [Russian], Izdat. "Universitetskoe," Minsk (1988).
13. Dodds P. G., Dodds T. K.-Y., and de Pagter B., "Noncommutative Köthe duality," Trans. Amer. Math. Soc., vol. 339, no. 2, 717–750 (1993).
14. Yeadon F. J., "Convergence of measurable operators," Math. Proc. Cambridge Philos. Soc., vol. 74, no. 2, 257–268 (1973).
15. Brown L. G. and Kosaki H., "Jensen's inequality in semifinite von Neumann algebras," J. Operator Theory, vol. 23, no. 1, 3–19 (1990).
16. Bikchentaev A. M., "Convergence of integrable operators affiliated to a finite von Neumann algebra," Proc. Steklov Inst. Math., vol. 293, 67–76 (2016).
17. Bikchentaev A. M., "On idempotent τ-measurable operators affiliated to a von Neumann algebra," Math. Notes, vol. 100, no. 4, 515–525 (2016).
18. Bikchentaev A. M., "Trace and integrable operators affiliated with a semifinite von Neumann algebra," Dokl. Math., vol. 93, no. 1, 16–19 (2016).
19. Hansen F., "An operator inequality," Math. Ann., vol. 246, no. 3, 249–250 (1980).
20. Bikchentaev A. M., "On normal τ-measurable operators affiliated with semifinite von Neumann algebras," Math. Notes, vol. 96, no. 3, 332–341 (2014).
21. Akemann C. A., Anderson J., and Pedersen G. K., "Triangle inequalities in operator algebras," Linear Multilinear Algebra, vol. 11, no. 2, 167–178 (1982).
22. Stroh A. and West Graeme P., "τ-Compact operators affiliated to a semifinite von Neumann algebra," Proc. Roy. Irish Acad., Sect. A., vol. 93, no. 1, 73–86 (1993).
23. Bikchentaev A. M., "Integrable products of measurable operators," Lobachevskii J. Math., vol. 37, no. 4, 397–403 (2016).
24. Bikchentaev A. M., "Block projection operators in normed solid spaces of measurable operators," Russian Math. (Iz. VUZ), vol. 56, no. 2, 75–79 (2012).
25. Bikchentaev A. M., "Concerning the theory of τ-measurable operators affiliated to a semifinite von Neumann algebra," Math. Notes, vol. 98, no. 3, 382–391 (2015).

A. M. Bikchentaev
Kazan Federal University, Kazan, Russia
E-mail address: Airat.Bikchentaev@kpfu.ru