Positivity (2026) 30:24
https://doi.org/10.1007/s11117-026-01180-0

Positivity

RESEARCH

Trace inequalities and characterizations of tracial functionals in operator algebras

Airat M. Bikchentaev[1] · **Mohammad Sal Moslehian**[2]

Received: 31 December 2025 / Accepted: 9 March 2026

Abstract

For a positive normal linear functional φ on a von Neumann algebra $\mathscr{A}$, we prove that the following conditions are equivalent: (i) φ is tracial, (ii) $|\varphi(\mathrm{Re}(A^2)| \le \varphi(|A|^2)$ for all $A \in \mathscr{A}$, and (iii) $|\varphi(A^2)| \le \varphi(|A|^2)$ for all $A \in \mathscr{A}$. Based on this result, we present some criteria for commutativity of a von Neumann algebra. For a trace φ on a C^*-algebra $\mathscr{A}$, we prove that $-\varphi(A^2B^2) \le \varphi((AB)^2) \le \varphi(A^2B^2)$ for certain elements of $\mathscr{A}$, and show that when φ is faithful, the equality in the second inequality is achieved if and only if $AB = BA$. Moreover, we partially generalize the Araki–Lieb–Thirring inequality to arbitrary traces on any C^*-algebras and to self-adjoint elements. Furthermore, we present a simple joint proof for $\mathrm{Tr}(AB) \pm \mathrm{Tr}(X^*X) \le \mathrm{Tr}(A)\,\mathrm{Tr}(B) \pm \mathrm{Tr}(X^*)\,\mathrm{Tr}(X)$ provided that $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ is positive semidefinite, without using the fact that $\Phi(X) = X + (\mathrm{Tr}\,X)I$ is completely copositive, and then present a characterization of the trace on the full matrix algebra $\mathbb{M}_n$.

Keywords C^*-algebra · Von Neumann algebra · Trace · Positive linear functional · Positive semidefinite block matrix

Mathematics Subject Classification 47A63 · 47B15 · 46L10 · 15A15 · 46L05

1 Preliminaries and introduction

Traces and weights on C^*-algebras are fundamental tools in operator theory and its applications. It is important to recall that the main inequalities of Quantum

---

✉ Mohammad Moslehian
moslehian@um.ac.ir; moslehian@yahoo.com

Airat Bikchentaev
Airat.Bikchentaev@kpfu.ru

[1] N.I. Lobachevsky Institute of Mathematics and Mechanics, Kazan (Volga Region) Federal University, Kremlevskaya ul. 18, 420008 Kazan, Tatarstan, Russia

[2] Department of Pure Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, P. O. Box 1159, 91775 Mashhad, Iran

Statistical Mechanics, such as Hölder's inequality, Cauchy–Schwarz inequality, Golden–Thompson inequality, Peierls–Bogoliubov inequality, Young's inequality, and Araki–Lieb–Thirring inequality, are trace inequalities for products of operators.

In this article, we further explore this line of investigation and examine trace inequalities for products of operators.

A left (respectively, right) ideal in an algebra $\mathscr{A}$ is a vector subspace $\mathcal{J}$ of $\mathscr{A}$ such that

$$A \in \mathscr{A} \text{ and } B \in \mathcal{J} \Rightarrow AB \in \mathcal{J} \text{ (respectively, } BA \in \mathcal{J}).$$

A C^*-algebra is considered a closed $*$-subalgebra of the $*$-algebra $\mathbb{B}(\mathscr{H})$ of all bounded linear operators acting on a complex Hilbert space $\mathscr{H}$ equipped with the operator norm $\|\cdot\|$ and with the unit I. For a C^*-algebra $\mathscr{A}$, we denote by $\mathscr{A}^{\mathrm{sa}}$, $\mathscr{A}^{\mathrm{pr}}$, and $\mathscr{A}^+$ its subsets of self-adjoint elements, projections, and positive (semidefinite) elements, respectively. If $A \in \mathscr{A}$, then $|A| = (A^*A)^{1/2} = \sqrt{A^*A} \in \mathscr{A}^+$. The real part of A is denoted by $\mathrm{Re}(A) = (A + A^*)/2$.

A *weight* on a C^*-algebra $\mathscr{A}$ is a mapping $\varphi : \mathscr{A}^+ \to [0, \infty]$ such that $\varphi(X+Y) = \varphi(X)+\varphi(Y)$ and $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathscr{A}^+, \lambda \geq 0$ (in this case, $0\cdot(\infty) \equiv 0$). A weight φ is called:

- *faithful* if $\varphi(X) = 0$ implies $X = 0$, for all $X \in \mathscr{A}^+$;
- *trace* if $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathscr{A}$;
- *finite* if $\varphi(X) < \infty$ for all $X \in \mathscr{A}^+$.

For the weight φ, we define (see [10, Ch. II, II.6.7.3], [20, Ch. 2, §11])

$\mathfrak{M}_\varphi^+ = \{X \in \mathscr{A}^+ \colon \ \varphi(X) < \infty\}$, $\mathfrak{M}_\varphi^{\mathrm{sa}} = \mathrm{lin}_{\mathbb{R}}\mathfrak{M}_\varphi^+$, $\mathfrak{M}_\varphi = \mathrm{lin}_{\mathbb{C}}\mathfrak{M}_\varphi^+$, and $\mathfrak{N}_\varphi = \{A \in \mathscr{A} \colon A^*A \in \mathfrak{M}_\varphi^+\}$ is a left ideal of $\mathscr{A}$ (a $*$-ideal in $\mathscr{A}$ for a trace φ). We also set $\|A\|_{\varphi,2} = \sqrt{\varphi(A^*A)}$ $(A \in \mathfrak{N}_\varphi)$, which is a seminorm (a norm for faithful φ) on $\mathfrak{N}_\varphi$ (see [22, Ch. I, §9, formula (5)]). Recall the Cauchy–Schwarz inequality [22, Ch. I, §9, formula (2)]

$$|\varphi(Y^*X)|^2 \leq \varphi(X^*X)\varphi(Y^*Y) \ \text{ for all } \ X, Y \in \mathfrak{N}_\varphi. \tag{1.1}$$

The restriction $\varphi|_{\mathfrak{M}_\varphi^+}$ can be correctly extended by linearity to a functional on $\mathfrak{M}_\varphi$, which we denote by the same letter φ. This extension allows us to identify the finite weights with positive linear functionals on $\mathscr{A}$.

The commutant of the set $\mathcal{X} \subseteq \mathcal{B}(\mathcal{H})$ is the set $\mathcal{X}' = \{Y \in \mathbb{B}(\mathscr{H}) \colon XY = YX \text{ for all } X \in \mathcal{X}\}$. A von Neumann algebra acting on a Hilbert space $\mathscr{H}$ is a C^*-subalgebra $\mathscr{A}$ of $\mathbb{B}(\mathscr{H})$ for which $\mathscr{A} = \mathscr{A}''$. For $P, Q \in \mathscr{A}^{\mathrm{pr}}$, we write $P \sim Q$ (the Murray–von Neumann equivalence) if $P = U^*U$ and $Q = UU^*$ for some $U \in \mathscr{A}$.

A trace φ on a von Neumann algebra $\mathscr{A}$ is called

- *normal* if $\varphi(\sup X_i) = \sup \varphi(X_i)$ for every bounded increasing net $\{X_i\}$ in $\mathscr{A}^+$;
- *semifinite* if $\varphi(A) = \sup\{\varphi(B) : B \in \mathscr{A}^+, \ B \leq A, \ \varphi(B) < \infty\}$ for all $A \in \mathscr{A}^+$.

A normal weight φ on a von Neumann algebra $\mathscr{A}$ is a trace if and only if it meets the condition: "for $A \in \mathscr{A}$ the inequality $\varphi(|A|) < \infty$ implies that $A \in \mathfrak{M}_\varphi$ and $|\varphi(A)| \leq \varphi(|A|)$", see [21, Corollary 2].

In the case where $\dim \mathscr{H} = n < \infty$, the algebra $\mathbb{B}(\mathscr{H})$ can be identified with the full matrix algebra $\mathbb{M}_n$. The space $\mathbb{M}_n$ forms a Hilbert space under the inner product defined by $\langle X, Y\rangle = \mathrm{Tr}(XY^*)$, where $\mathrm{Tr}(\cdot)$ denotes the standard trace. Applying the Riesz representation theorem, we deduce that the positive linear functional φ on $\mathbb{M}_n$ can be expressed as $\varphi(A) = \mathrm{Tr}(SA)$ for a positive semidefinite matrix S with $\mathrm{Tr}(S) = n$.

By employing the spectral decomposition of S, and using the fact that the trace is invariant under unitary conjugation, we can simplify the problem. Specifically, by replacing φ with the map $\varphi \circ \mathrm{Ad}_U$, where $\mathrm{Ad}_U(X) = UXU^*$ for a suitably chosen unitary matrix U, we may assume that S is diagonal, that is, $S = \mathrm{diag}(s_1, s_2, \ldots, s_n)$. If we want to show that a positive linear functional on $\mathbb{M}_n$ is a multiple of the standard trace, we have to show that all the diagonal entries s_i are equal.

Let φ be a trace on a C^*-algebra $\mathscr{A}$. We prove that if $A \in \mathfrak{N}_\varphi$, then $A^2 \in \mathfrak{M}_\varphi$ and $|\varphi(\mathrm{Re}(A^2))| \le \varphi(|A|^2)$ (Proposition 2.4). For a positive normal linear functional φ on a von Neumann algebra $\mathscr{A}$, the following conditions are equivalent: (i) φ is tracial; (ii) $|\varphi(\mathrm{Re}(A^2))| \le \varphi(|A|^2)$ for all $A \in \mathscr{A}$; (iii) $|\varphi(A^2)| \le \varphi(|A|^2)$ for all $A \in \mathscr{A}$ (Theorem 2.5). For a von Neumann algebra $\mathscr{A}$, the following conditions are equivalent: (i) $\mathscr{A}$ is commutative; (ii) $\mathrm{Re}(A^2) \le |A|^2$ for all $A \in \mathscr{A}$; (iii) $|A^2| \le |A|^2$ for all $A \in \mathscr{A}$ (Corollary 2.6). Let φ be a trace on the C^*-algebra $\mathscr{A}$ and let $A, B \in \mathscr{A}^{\mathrm{sa}}$ with $AB \in \mathfrak{M}_\varphi$. Then, $\varphi(A^2B^2) \ge 0$ and $-\varphi(A^2B^2) \le \varphi((AB)^2) \le \varphi(A^2B^2)$. If the trace φ is faithful, then the equality in the second inequality is achieved if and only if $AB = BA$ (Theorem 2.7). Let $A \in \mathscr{A}^+$ and $B \in \mathscr{A}^{\mathrm{sa}}$ with $AB \in \mathfrak{M}_\varphi$. Then $\varphi((\sqrt{A}B\sqrt{A})^2) \le \varphi(AB^2A)$ (Corollary 2.8), which provides a partial generalization of the Araki–Lieb–Thirring inequality to any traces on arbitrary C^*-algebras and to self-adjoint elements.

In Section 4, we present a simple joint proof for $\mathrm{Tr}(AB) \pm \mathrm{Tr}(X^*X) \le \mathrm{Tr}(A)\,\mathrm{Tr}(B) \pm \mathrm{Tr}(X^*)\,\mathrm{Tr}(X)$, without using the fact that $\Phi(X) = X + (\mathrm{Tr}\,X)I$ is completely copositive. As a consequence, we present similar inequalities for partial traces. Furthermore, we prove that the matrix trace is the unique positive linear functional φ on $\mathbb{M}_n$ such that $\varphi(I) = n$ and the inequality

$$\varphi(\sqrt{A}B\sqrt{A}) \pm \varphi(X^*X) \le \varphi(A)\varphi(B) \pm \varphi(X^*)\varphi(X)$$

holds for all $A, B, X \in \mathbb{M}_n$ provided that $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ is positive semidefinite (Theorem 4.1).

2 Trace inequalities for von Neumann algebras and C^*-algebras

We show that if $A \in \mathbb{M}_n$ is a normal matrix, then

$$\varphi(|\mathrm{Re}(A^2)|) \le \varphi(A^*A)$$

for any tracial positive linear functional φ on $\mathbb{M}_n$:

It is shown in [17, Theorem 2.1] that for such a functional φ, for a matrix $X \in \mathbb{M}_n$ and a positive semidefinite matrix $Y \in \mathbb{M}_n$, if $\begin{bmatrix} Y & X \\ X^* & Y \end{bmatrix} \geq 0$ and $\begin{bmatrix} Y & X^* \\ X & Y \end{bmatrix} \geq 0$, then

$$\begin{bmatrix} \varphi(Y) & \varphi(|\mathrm{Re} X|) \\ \varphi(|\mathrm{Re} X|) & \varphi(Y) \end{bmatrix} \geq 0.$$

The normality of A ensures that $\begin{bmatrix} A^*A & A \\ A^* & I \end{bmatrix} \geq 0$. Therefore,

$$\begin{bmatrix} A^*A & A^2 \\ A^{*2} & A^*A \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & A^* \end{bmatrix} \begin{bmatrix} A^*A & A \\ A^* & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix} \geq 0.$$

By a similar method, we can show that $\begin{bmatrix} A^*A & A^{*2} \\ A^2 & A^*A \end{bmatrix} \geq 0$. Thus,

$$\begin{bmatrix} \varphi(A^*A) & \varphi(|\mathrm{Re}(A^2)|) \\ \varphi(|\mathrm{Re}(A^2)|) & \varphi(A^*A) \end{bmatrix} \geq 0.$$

It follows from [18, Proposition 2.4.19] that $\varphi(|\mathrm{Re}(A^2)|) \leq \varphi(A^*A)$.

We can establish a variation of this inequality as follows.

Let $A = [a_{ij}] \in \mathbb{M}_n$ be an arbitrary matrix with complex eigenvalues $\lambda_1, \ldots, \lambda_n$. The spectral theorem yields that

$$|\,\mathrm{Tr}(\mathrm{Re}\, A^2)| = |\mathrm{Re}\ \mathrm{Tr}(A^2)| = \left|\mathrm{Re} \sum_{k=1}^{n} \lambda_k^2\right| \leq \sum_{k=1}^{n} \left|\mathrm{Re}\,(\lambda_k^2)\right| \leq \sum_{k=1}^{n} |\lambda_k|^2,$$

as $|\mathrm{Re}\,(z)| \leq |z|$ for any $z \in \mathbb{C}$.

Using Schur's inequality [24, Theorem 9.5], we can deduce that

$$\sum_{k=1}^{n} |\lambda_k|^2 \leq \sum_{i,j=1}^{n} |a_{i,j}|^2 = \mathrm{Tr}(A^*A),$$

Therefore,

$$\left|\mathrm{Tr}(\mathrm{Re}\, A^2)\right| \leq \mathrm{Tr}(A^*A).$$

Now, we extend the above inequality to the context of von Neumann algebras by establishing characterizations of positive normal linear functionals on von Neumann algebras. To achieve this, we require some lemmas and facts.

Lemma 2.1 *Let φ be a weight on a C^*-algebra $\mathscr{A}$. If $A \in \mathfrak{N}_\varphi$, then $|A^2| \in \mathfrak{M}_\varphi^+$ and $\varphi(|A^2|) \leq \varphi(|A|^2)$.*

Proof Without loss of generality, we assume that $\varphi(|A^2|) \neq 0$ and $\|A\| \leq 1$. Then

$$A^{2*}AA^*A^2 \leq A^{2*}IA^2 \leq |A^2|.$$

Since $\mathfrak{N}_\varphi$ is a left ideal in $\mathscr{A}$, we have $A^*A^2 \in \mathfrak{N}_\varphi$. By utilizing the Cauchy–Schwarz inequality (1.1) with $Y = A$ and $X = A^*A^2$, the monotonicity of the weight φ on the cone $\mathscr{A}^+$ and the monotonicity of the real function $t \mapsto \sqrt{t}$ on $\mathbb{R}^+$, we obtain

$$\varphi(|A^2|) = |\varphi(A^* \cdot A^*A^2)| \leq \sqrt{\varphi(A^{2*}AA^*A^2)}\sqrt{\varphi(A^*A)} \leq \sqrt{\varphi(|A^2|)}\sqrt{\varphi(|A|^2)}.$$

Therefore, $\sqrt{\varphi(|A^2|)} \leq \sqrt{\varphi(|A|^2)}$. Hence, $|A^2| \in \mathfrak{M}_\varphi^+$ and $\varphi(|A^2|) \leq \varphi(|A|^2)$. □

If φ is a tracial functional on the C^*-algebra $\mathscr{A}$, then $|\varphi(X)| \leq \varphi(|X|)$ for all $X \in \mathscr{A}$, as shown in [12]. Therefore, we have:

$$|\varphi(X^2)| \leq \varphi(|X^2|) \leq \varphi(|X|^2) \quad \text{for all} \quad X \in \mathscr{A}. \tag{2.1}$$

Lemma 2.2 ([19, Exercise 6.3]). *Let φ be a trace on a C^*-algebra $\mathscr{A}$. Then, $\varphi(ST) = \varphi(TS)$ for all $S \in \mathfrak{M}_\varphi$ and $T \in \mathscr{A}$.*

Lemma 2.3 *Let φ be a trace on a C^*-algebra $\mathscr{A}$. If $A, B \in \mathfrak{N}_\varphi$ and $X \in \mathscr{A}$, then $A^*XB, AXB \in \mathfrak{M}_\varphi$.*

Proof Since $\mathfrak{N}_\varphi$ is a $*$-ideal in $\mathscr{A}$, we have $A^*, XB \in \mathfrak{N}_\varphi$. Therefore, it suffices to show that $A^*B \in \mathfrak{M}_\varphi$. From the inequalities $(A \pm B)^*(A \pm B) \geq 0$, we have

$$0 \leq A^*A + B^*B + A^*B + B^*A \leq 2A^*A + 2B^*B \in \mathfrak{M}_\varphi^+.$$

Hence,

$$A^*B + B^*A = A^*A + B^*B + A^*B + B^*A - \frac{1}{2}(2A^*A + 2B^*B) \in \mathfrak{M}_\varphi^{\mathrm{sa}}. \tag{2.2}$$

From the inequalities $(A \pm \mathrm{i}B)^*(A \pm \mathrm{i}B) \geq 0$ we have $0 \leq A^*A + B^*B - \mathrm{i}A^*B + \mathrm{i}B^*A \leq 2A^*A + 2B^*B \in \mathfrak{M}_\varphi$. Hence,

$$\mathrm{i}A^*B - \mathrm{i}B^*A \in \mathfrak{M}_\varphi^{\mathrm{sa}}. \tag{2.3}$$

From (2.2) and (2.3) it follows that

$$A^*B = \frac{1}{2}(A^*B + B^*A - \mathrm{i}(\mathrm{i}A^*B - \mathrm{i}B^*A)) \in \mathfrak{M}_\varphi.$$

Since $AA^* \in \mathfrak{M}_\varphi^+$, analogously we obtain $AXB \in \mathfrak{M}_\varphi$. □

Proposition 2.4 *Let φ be a trace on a C^*-algebra $\mathscr{A}$. If $A \in \mathfrak{N}_\varphi$, then $A^2 \in \mathfrak{M}_\varphi$ and $|\varphi(\mathrm{Re}(A^2))| \leq |\varphi(A^2)| \leq \varphi(|A|^2)$.*

Proof Putting $B = A^*$ in the proof of Lemma 2.3 and obtaining $A^{*2} \in \mathfrak{M}_\varphi$, we can conclude that $A^2 = (A^{*2})^* \in \mathfrak{M}_\varphi$. For all $X \in \mathfrak{M}_\varphi$ we have $X^* \in \mathfrak{M}_\varphi$ and, since $\varphi(X^*) = \overline{\varphi(X)}$ for all $X \in \mathfrak{M}_\varphi$, we have $\varphi(\mathrm{Re}(X)) = \mathrm{Re}(\varphi(X))$ for all $X \in \mathfrak{M}_\varphi$. Thus, $|\varphi(\mathrm{Re}(A^2))| \le |\varphi(A^2)|$. By the Cauchy–Schwarz inequality (1.1) with $Y = A^*$ and $X = A$ we have $|\varphi(A^2)|^2 = |\varphi((A^*)^*A)|^2 \le \varphi(A^*A)\varphi(AA^*) = \varphi(A^*A)^2 = \varphi(|A|^2)^2$ and $|\varphi(A^2)| \le \varphi(|A|^2)$. □

Theorem 2.5 *For a positive normal linear functional φ on a von Neumann algebra $\mathscr{A}$, the following conditions are equivalent:*
(i) *φ is tracial;*
(ii) $|\varphi(\mathrm{Re}(A^2))| \le \varphi(|A|^2)$ *for all $A \in \mathscr{A}$;*
(iii) $|\varphi(A^2)| \le \varphi(|A|^2)$ *for all $A \in \mathscr{A}$.*

Proof The implication (i) ⇒ (ii) was proved in Proposition 2.4, while (i) ⇒ (iii) follows from (2.1).

Here, we demonstrate, in analogy with several comparable situations (see, for example, [12] or [23]), that verifying the converse implications for an arbitrary von Neumann algebra can be reduced to the special case of the matrix algebra $\mathbb{M}_2$.

It is known [12] that a positive normal linear functional φ on a von Neumann algebra $\mathscr{A}$ is a trace precisely when $\varphi(P) = \varphi(Q)$ holds for all $P, Q \in \mathscr{A}^{\mathrm{pr}}$ satisfying $PQ = 0$ and $P \sim Q$ (see also [23, Lemma 2]).

Consider the reduced algebra $(P + Q)\mathscr{A}(P + Q)$ and let the $*$-algebra $\mathcal{N}$ be generated by a partial isometry $V \in \mathscr{A}$ that implements the equivalence between P and Q. Then $\mathcal{N}$ is $*$-isomorphic to $\mathbb{M}_2$, and the inequalities appearing in (ii) and (iii) continue to hold for operators belonging to $\mathcal{N}$ under the restricted functional $\varphi|\mathcal{N}$. We establish that this restriction is itself a trace functional on $\mathcal{N}$,from which it follows that $\varphi(P) = \varphi(Q)$.

As mentioned in the introduction, every linear functional φ on $\mathbb{M}_2$ can be represented as $\varphi(\cdot) = \mathrm{Tr}(S\,\cdot)$. Without loss of generality, we can assume that

$$S = \mathrm{diag}\left(\frac{1}{2} - s, \frac{1}{2} + s\right), \quad 0 \le s \le \frac{1}{2}.$$

Thus, $\varphi(X)$ equals $(1/2 - s)x_{11} + (1/2 + s)x_{22}$ for $X = [x_{ij}]_{i,j=1}^2$ from $\mathbb{M}_2$.

(ii) ⇒ (i). We shall show that $s = 0$.

Set $A := \begin{pmatrix} 1 & a \\ 1 & a \end{pmatrix} \in \mathbb{M}_2$ for $0 \le a < 1$. We choose this one-parameter family of test matrices because it constrains the parameters in a way that makes the induced density matrix S proportional to the identity.

Then $2\mathrm{Re}(A^2) = (1+a)\begin{pmatrix} 2 & 1+a \\ 1+a & 2a \end{pmatrix}$ and $2A^*A = 4\begin{pmatrix} 1 & a \\ a & a^2 \end{pmatrix}$. Therefore

$$2\varphi(\mathrm{Re}(A^2)) = (1 - 2s)(1 + a) + (1 + 2s)(a + a^2) = (1 + a)^2 - 2s(1 - a^2),$$

$$2\varphi(A^*A) = (1 - 2s)2 + (1 + 2s) \cdot 2a^2 = 2(1 + a^2) - 4s(1 - a^2).$$

Now the inequality in (ii) can be rewritten as $2s(1-a^2) \le (1-a)^2$. By dividing both sides of the last inequality by $1-a^2 > 0$, we obtain

$$0 \le 2s \le \frac{1-a}{1+a}.$$

This inequality is met for all $0 \le a < 1$ only when $s = 0$.

Since $\varphi(X^*) = \overline{\varphi(X)}$ for all $X \in \mathscr{A}$, we have $\varphi(\mathrm{Re}(X)) = \mathrm{Re}(\varphi(X))$ for all $X \in \mathscr{A}$. Therefore, we have $|\varphi(\mathrm{Re}(X))| = |\mathrm{Re}(\varphi(X))| \le |\varphi(X)| \le \varphi(|X|)$, and this ensures (iii)⇒(ii)⇒(i). Let's provide a direct and concise proof of the implication (iii)⇒(i). Let us again consider a simple one-parameter matrix such that it forces the density matrix S associated with the functional to be a scalar multiple of the identity. For every nonzero $a \in (0,\infty)$, consider the 2×2 matrix

$$A = \begin{bmatrix} 0 & 1-a \\ 1+a & 0 \end{bmatrix}.$$

Then, $A^2 = \mathrm{diag}(1-a^2, 1-a^2)$, $A^*A = \mathrm{diag}((1+a)^2, (1-a)^2)$,

$$\varphi(A^2) = (1-a^2)\Big(\frac{1}{2} - s\Big) + (1-a^2)\Big(\frac{1}{2} + s\Big) = 1 - a^2,$$

and

$$\varphi(A^*A) = (1+a)^2\Big(\frac{1}{2} - s\Big) + (1-a)^2\Big(\frac{1}{2} + s\Big) = 1 - 4as + a^2.$$

The inequality in (iii) turns into $4as \le 2a^2$, that is, $2s \le a$. This inequality holds for all $a > 0$ only if $s = 0$. □

Corollary 2.6 *For a von Neumann algebra $\mathscr{A}$, the following conditions are equivalent:*
(i) *$\mathscr{A}$ is commutative;*
(ii) $\mathrm{Re}(A^2) \le |A|^2$ *for all $A \in \mathscr{A}$;*
(iii) $|A^2| \le |A|^2$ *for all $A \in \mathscr{A}$.*

Proof We prove the implication (ii)⇒(i). For any positive functional φ on $\mathscr{A}$, the inequality from item (ii) of Theorem 2.5 holds. This implies that any positive functional on $\mathscr{A}$ is tracial. In other words, $\varphi(XY) = \varphi(YX)$ for any elements $X, Y \in \mathscr{A}$. As the positive linear functionals on a C^*-algebra separate points (meaning that if A and B are two distinct elements of the C^*-algebra, then there exists a positive functional φ with $\varphi(A) \ne \varphi(B)$), the assumption that $\varphi(XY - YX) = 0$ for every positive linear functional φ implies $XY - YX = 0$, which shows that the von Neumann algebra $\mathscr{A}$ is commutative. □

Another criterion for the commutativity of C^*-algebras is presented in [13].

Theorem 2.7 *Let φ be a trace on the C^*-algebra $\mathscr{A}$ and $A, B \in \mathscr{A}^{\mathrm{sa}}$ with $AB \in \mathfrak{M}_\varphi$. Then, $\varphi(A^2B^2) \ge 0$ and*

$$-\varphi(A^2B^2) \le \varphi((AB)^2) \le \varphi(A^2B^2). \tag{2.4}$$

If the trace φ is faithful, then the equality in the second inequality from (2.4) is achieved if and only if $AB = BA$.

Proof By Lemma 2.2, we have $\varphi(A^2B^2) = \varphi(A \cdot AB^2) = \varphi(AB^2A) \geq 0$, since $AB^2A \geq 0$. Next, $BA = (AB)^* \in \mathfrak{M}_\varphi$ and

$$\begin{aligned}\|AB - BA\|_{\varphi,2}^2 &= \varphi((AB - BA)^*(AB - BA))\\ &= \varphi(BA^2B + AB^2A - BABA - ABAB)\\ &= 2\varphi(A^2B^2) - 2\varphi((AB)^2)\end{aligned}$$

due to Lemma 2.2 and the linearity of the trace φ extended to $\mathfrak{M}_\varphi$.

If the trace φ is faithful, then $\|\cdot\|_{\varphi,2}$ is a norm on $\mathfrak{N}_\varphi$. Therefore, the equality $\varphi(A^2B^2) = \varphi((AB)^2)$ is equivalent to the equality $AB = BA$.

Since $AB + BA \in \mathfrak{M}_\varphi^{\mathrm{sa}}$, we have

$$0 \leq \varphi((AB+BA)^2) = \varphi((AB)^2 + (BA)^2 + AB^2A + BA^2B) = 2\varphi((AB)^2) + 2\varphi(A^2B^2)$$

and, therefore, $-\varphi(A^2B^2) \leq \varphi((AB)^2)$. □

Corollary 2.8 *Let φ be a trace on the C^*-algebra $\mathscr{A}$, $A \in \mathscr{A}^+$ and $B \in \mathscr{A}^{\mathrm{sa}}$ with $AB \in \mathfrak{M}_\varphi$. Then*

$$\varphi((\sqrt{A}B\sqrt{A})^2) \leq \varphi(AB^2A). \tag{2.5}$$

Proof By Lemma 2.2 we have $\varphi(A^2B^2) = \varphi(AB^2A)$ and

$$\varphi((AB)^2) = \varphi(A \cdot BAB) = \varphi(\sqrt{A} \cdot BAB \cdot \sqrt{A}) = \varphi((\sqrt{A}B\sqrt{A})^2).$$

□

If φ is a normal semifinite trace on a von Neumann algebra $\mathscr{A}$, then the Araki–Lieb–Thirring inequality [15] holds:

$$\varphi((\sqrt{A}B\sqrt{A})^{rp}) \leq \varphi((\sqrt{A^r}B\sqrt{A^r})^p) \tag{2.6}$$

for all numbers $r > 1$, $p > 0$ and operators $A, B \in \mathscr{A}^+$. Our inequality (2.5) is a generalization of the Araki–Lieb–Thirring inequality (2.6) to arbitrary traces on any C^*-algebras and to self-adjoint elements B for $r = 2$ and $p = 1$. In connection with Theorem 2.5, we note that for a positive functional φ on a C^*-algebra $\mathscr{A}$, the fulfillment of inequality (2.6) for some fixed numbers $r > 1$, $p > 0$, and for all $A, B \in \mathscr{A}^+$ is equivalent to the fact that the functional φ is tracial, see [4, Theorem 2]. A similar inequality characterizes traces in the class of all normal semifinite weights on the von Neumann algebra [6, Theorem 2].

3 Trace inequalities and a characterization of the trace on the full matrix algebra

A consequence of the subadditivity inequality for q-entropies is presented in [2, Example 1] as follows:

$$\operatorname{Tr}(AB) - \operatorname{Tr}(X^*X) \leq \operatorname{Tr}(A)\operatorname{Tr}(B) - \operatorname{Tr}(X^*)\operatorname{Tr}(X), \tag{3.1}$$

provided the block matrix $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{M}_{2n}$ with $A = [a_{ij}]$, $B = [b_{ij}]$, $X = [x_{ij}] \in \mathbb{M}_n$ is positive semidefinite. There are some remarkable proofs of the above mentioned inequality in the literature; see [3, 25]. Under the same conditions and by using the complete copositivity of $\Phi(X) = X + (\operatorname{Tr} X)I$, the trace inequality

$$\operatorname{Tr}(AB) + \operatorname{Tr}(X^*X) \leq \operatorname{Tr}(A)\operatorname{Tr}(B) + \operatorname{Tr}(X^*)\operatorname{Tr}(X) \tag{3.2}$$

is elegantly proved in [14, Theorem 2.2].

The left-hand side may be negative. For example, consider the matrices $A = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & \beta \end{bmatrix}$, and $X = \begin{bmatrix} 0 & \gamma \\ 0 & 0 \end{bmatrix}$, where α and β are positive numbers and $\gamma \neq 0$ is a complex number such that $\alpha\beta \geq |\gamma|^2$. Then $\operatorname{Tr}(AB) - \operatorname{Tr}(X^*X) = 0 - |\gamma|^2 < 0$; see also [16, p. 918]. However, if both $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ and $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}$ are positive semidefinite, then $\operatorname{Tr}(AB) - \operatorname{Tr}(X^*X) \geq 0$, as shown in [16, Theorem 2.1].

The right hand side is always positive because the matrix $\begin{bmatrix} \operatorname{Tr}(A) & \operatorname{Tr}(X) \\ \operatorname{Tr}(X^*) & \operatorname{Tr}(B) \end{bmatrix}$ is positive definite. This is a result of the positivity of the principal submatrix $\begin{bmatrix} a_{ii} & x_{ii} \\ \overline{x_{ii}} & b_{ii} \end{bmatrix}$ of $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{M}_{2n}$ as a matrix in $\mathbb{M}_{2n}$ and the fact that the sum of positive semidefinite matrices is positive semidefinite; see [9, 11].

We now provide simple proofs of (3.1) and (3.2) by utilizing induction on $n \geq 2$.

First, we consider the spectral decomposition $A = UDU^*$ of the positive semidefinite matrix $A \in \mathbb{M}_n$ [24, Theorem 3.4] and the tracial property of the trace. This allows us to rewrite (3.1) and (3.2) as follows:

$$\operatorname{Tr}(DU^*BU) \pm \operatorname{Tr}(X^*X) \leq \operatorname{Tr}(D)\operatorname{Tr}(U^*BU) \pm \operatorname{Tr}(X^*)\operatorname{Tr}(X).$$

Therefore, we can assume that the $(1, 1)$-entry in the matrix $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$, which corresponds to A, is diagonal.

For $n = 2$, inequalities (3.1) and (3.2) are clear. Assuming that (3.1) and (3.2) hold for n, let's consider the positive semidefinite matrix $\begin{bmatrix} \widetilde{A} & \widetilde{X} \\ \widetilde{X}^* & \widetilde{B} \end{bmatrix} \in \mathbb{M}_{2(n+1)}$, where

$$\widetilde{A} = \begin{bmatrix} A & \mathbf{a} \\ \mathbf{a}^* & \alpha \end{bmatrix}, \ \widetilde{X} = \begin{bmatrix} X & \mathbf{x} \\ \mathbf{x}' & \gamma \end{bmatrix}, \ \widetilde{B} = \begin{bmatrix} B & \mathbf{b} \\ \mathbf{b}^* & \beta \end{bmatrix},$$

in which A and B are positive semidefinite matrices, $\mathbf{a}, \mathbf{b}, \mathbf{x} = \begin{bmatrix} x_1 \cdots x_n \end{bmatrix}^T$ are column vectors, $\mathbf{x}' = \begin{bmatrix} x_1' \cdots x_n' \end{bmatrix}$ is a row vector, and α and β are nonnegative real numbers.

$$\operatorname{Tr}(\widetilde{A}\widetilde{B}) = \operatorname{Tr}\left(AB + \mathbf{a}\mathbf{b}^*\right) + \operatorname{Tr}\left(\mathbf{a}^*\mathbf{b}\right) + \alpha\beta,$$

$$\operatorname{Tr}(\widetilde{X}^*\widetilde{X}) = \operatorname{Tr}\left(X^*X + \mathbf{x}'^*\mathbf{x}'\right) + \operatorname{Tr}\left(\mathbf{x}^*\mathbf{x}\right) + |\gamma|^2,$$

and

$$\operatorname{Tr}(\widetilde{A}) = \operatorname{Tr}(A) + \alpha \quad \operatorname{Tr}(\widetilde{B}) = \operatorname{Tr}(B) + \beta, \quad \text{and} \quad \operatorname{Tr}(\widetilde{X}) = \operatorname{Tr}(X) + \gamma.$$

As explained previously, we can assume that A is diagonal and $\mathbf{a} = 0$. Therefore,

$$\operatorname{Tr}(\widetilde{A}\widetilde{B}) = \operatorname{Tr}(AB) + \alpha\beta.$$

In addition,

$$\operatorname{Tr}(\widetilde{X}^*\widetilde{X}) = \operatorname{Tr}(X^*X) + \sum_{i=1}^{n} |x_i'|^2 + \sum_{i=1}^{n} |x_i|^2 + |\gamma|^2.$$

Since the principal submatrix $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ is positive semidefinite, the induction hypothesis ensures that $\operatorname{Tr}(AB) \pm \operatorname{Tr}(X^*X) \le \operatorname{Tr}(A)\operatorname{Tr}(B) \pm \operatorname{Tr}(X^*)\operatorname{Tr}(X)$.

Note that inequalities (3.1) and (3.2) for the above $(n+1) \times (n+1)$ matrices hold true if

$$\begin{aligned} \alpha\beta \pm \left(\sum_{i=1}^{n} |x_i'|^2 + \sum_{i=1}^{n} |x_i|^2 + |\gamma|^2 \right) \\ \le (\beta \operatorname{Tr}(A) + \alpha \operatorname{Tr}(B) + \alpha\beta) \pm \left(|\gamma|^2 + 2\operatorname{Re}\left(\overline{\gamma}\operatorname{Tr}(X)\right) \right), \end{aligned}$$

or respectively,

$$-\left(\sum_{i=1}^{n} |x_i'|^2 + \sum_{i=1}^{n} |x_i|^2 \right) \le \beta \operatorname{Tr}(A) + \alpha \operatorname{Tr}(B) - 2\operatorname{Re}\left(\overline{\gamma}\operatorname{Tr}(X)\right) \tag{3.3}$$

and

$$\sum_{i=1}^{n} |x_i'|^2 + \sum_{i=1}^{n} |x_i|^2 \leq \beta \operatorname{Tr}(A) + \alpha \operatorname{Tr}(B) + 2\operatorname{Re}\,(\overline{\gamma} \operatorname{Tr}(X)). \tag{3.4}$$

Proof of inequality (3.3): Since $\sum_{i=1}^{n} |x_i'|^2 + \sum_{i=1}^{n} |x_i|^2 \geq 0$, it is enough to prove that

$$2\operatorname{Re}\,(\overline{\gamma} \operatorname{Tr}(X)) \leq \beta \operatorname{Tr}(A) + \alpha \operatorname{Tr}(B).$$

We have

$$\begin{aligned}
2\operatorname{Re}\,(\overline{\gamma} \operatorname{Tr}(X)) &\leq 2|\gamma|\,|\operatorname{Tr}(X)| \\
&\leq 2\sqrt{\alpha\beta}\,|\operatorname{Tr}(X)| \\
&\text{(since the principal submatrix } \begin{bmatrix} \alpha & \gamma \\ \overline{\gamma} & \beta \end{bmatrix} \text{ is positive semidefinite)} \\
&\leq 2\sqrt{\alpha\beta}\sqrt{\operatorname{Tr}(A)\operatorname{Tr}(B)} \\
&\text{(since } \begin{bmatrix} \operatorname{Tr}(A) & \operatorname{Tr}(X) \\ \operatorname{Tr}(X^*) & \operatorname{Tr}(B) \end{bmatrix} \text{ is positive semidefinite)} \\
&\leq 2\sqrt{(\beta \operatorname{Tr}(A))(\alpha \operatorname{Tr}(B))} \\
&\leq \beta \operatorname{Tr}(A) + \alpha \operatorname{Tr}(B).
\end{aligned}$$

Proof of inequality (3.4): Let $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \geq 0$. For each $i = 1, \ldots, n$, consider the principal submatrix $\begin{bmatrix} U & W \\ W^* & V \end{bmatrix}$ of $\begin{bmatrix} \widetilde{A} & \widetilde{X} \\ \widetilde{X}^* & \widetilde{B} \end{bmatrix}$, where

$$U = \begin{bmatrix} \lambda_i & 0 \\ 0 & \alpha \end{bmatrix} \geq 0, \quad W = \begin{bmatrix} X_{ii} & x_i \\ x_i' & \gamma \end{bmatrix}, \quad V = \begin{bmatrix} B_{ii} & b_i \\ b_i' & \beta \end{bmatrix} \geq 0.$$

By the base case $n = 2$,

$$\operatorname{Tr}(UV) + \operatorname{Tr}(W^*W) \leq \operatorname{Tr}(U)\operatorname{Tr}(V) + \operatorname{Tr}(W^*)\operatorname{Tr}(W).$$

A straitforward computation yields that $\operatorname{Tr}(UV) = \lambda_i B_{ii} + \alpha\beta$, $\operatorname{Tr}(W^*W) = |X_{ii}|^2 + |x_i|^2 + |x'i|^2 + |\gamma|^2$, $\operatorname{Tr}(U) = \lambda_i + \alpha$, $\operatorname{Tr}(V) = B_{ii} + \beta$, and $\operatorname{Tr}(W^*)\operatorname{Tr}(W) = |X_{ii} + \gamma|^2$. By substituting these, we get

$$\lambda_i B_{ii} + \alpha\beta + |X_{ii}|^2 + |x_i|^2 + |x_i'|^2 + |\gamma|^2 \leq (\lambda_i + \alpha)(B_{ii} + \beta) + |X_{ii} + \gamma|^2.$$

Expanding the right-hand side and canceling the common terms $\lambda_i B_{ii} + \alpha\beta + |X_{ii}|^2 + |\gamma|^2$ from both sides, we arrive at

$$|x_i|^2 + |x_i'|^2 \leq \lambda_i \beta + \alpha B_{ii} + 2\operatorname{Re}\,(\overline{X_{ii}}\gamma).$$

Summing over $1 \leq i \leq n$ produces

$$\sum_{i=1}^{n} |x_i|^2 + \sum_{i=1}^{n} |x_i'|^2 \leq \beta \sum_{i=1}^{n} \lambda_i + \alpha \sum_{i=1}^{n} B_{ii} + 2 \sum_{i=1}^{n} \mathrm{Re}\,(\overline{X_{ii}}\gamma)$$
$$= \beta\, \mathrm{Tr}(A) + \alpha\, \mathrm{Tr}(B) + 2\mathrm{Re}\, \left(\overline{\mathrm{Tr}(X)}\gamma\right),$$

which is precisely (3.4).

Let $n, m \geq 1$. Recall that the tensor product of two matrices $C = [c_{ij}] \in \mathbb{M}_n$ and $D = [d_{ij}] \in \mathbb{M}_m$ is defined as a matrix in $\mathbb{M}_{mn}$, represented by $C \otimes D$, where the element in the (i, p)-th row and (j, q)-th column is given by $c_{ij} d_{pq}$. In other words,

$$C \otimes D = \begin{bmatrix} c_{11}D & c_{12}D & \cdots & c_{1n}D \\ c_{21}D & c_{22}D & \cdots & c_{2n}D \\ \vdots & \vdots & \vdots & \vdots \\ c_{n1}D & c_{m2}D & \cdots & c_{nn}D \end{bmatrix}.$$

The partial traces $\mathrm{Tr}_1 : \mathbb{M}_n \otimes \mathbb{M}_m \to \mathbb{M}_m$ and $\mathrm{Tr}_2 : \mathbb{M}_n \otimes \mathbb{M}_m \to \mathbb{M}_n$ are linearly defined by

$$\mathrm{Tr}_1(C \otimes D) = (\mathrm{Tr}_n\, C)\, D, \qquad \text{and} \qquad \mathrm{Tr}_2(C \otimes D) = (\mathrm{Tr}_m\, D)\, C,$$

for $C \in \mathbb{M}_n$ and $D \in \mathbb{M}_m$. In addition, let Tr_{nm} denote the full trace on $\mathbb{M}_n \otimes \mathbb{M}_m$, defined by $\mathrm{Tr}_{nm}(C \otimes D) = \mathrm{Tr}_n(C)\ \mathrm{Tr}_m(D)$.

Corollary 3.1 *Let $A, B, X \in \mathbb{M}_n \otimes \mathbb{M}_m$ such that $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ is positive semidefinite. Then*

$$\mathrm{Tr}_m\big(\mathrm{Tr}_1(A)\ \mathrm{Tr}_1(B)\big) \pm \mathrm{Tr}_m\big(\mathrm{Tr}_1(X)^*\ \mathrm{Tr}_1(X)\big) \leq \mathrm{Tr}_{nm}(A)\ \mathrm{Tr}_{nm}(B) \pm \mathrm{Tr}_{nm}(X^*)\, \mathrm{Tr}_{nm}(X).$$

and

$$\mathrm{Tr}_n\big(\mathrm{Tr}_2(A)\ \mathrm{Tr}_2(B)\big) \pm \mathrm{Tr}_n\big(\mathrm{Tr}_2(X)^*\ \mathrm{Tr}_2(X)\big) \leq \mathrm{Tr}_{nm}(A)\ \mathrm{Tr}_{nm}(B) \pm \mathrm{Tr}_{nm}(X^*)\, \mathrm{Tr}_{nm}(X).$$

Proof Let's define the map Φ as $I_2 \otimes \mathrm{Tr}_1$, where

$$\Phi : \mathbb{M}_2 \otimes (\mathbb{M}_n \otimes \mathbb{M}_m) \longrightarrow \mathbb{M}_2 \otimes \mathbb{M}_m,$$

given by

$$\Phi(M) = \begin{bmatrix} \mathrm{Tr}_1(A) & \mathrm{Tr}_1(X) \\ \mathrm{Tr}_1(X)^* & \mathrm{Tr}_1(B) \end{bmatrix}.$$

Since the partial trace Tr_1 is completely positive, Φ is a positive map. Therefore, $\Phi(M) \geq 0$ in $\mathbb{M}_2 \otimes \mathbb{M}_m$.

Employing the trace inequalities (3.1) and (3.2) with the trace Tr_m on this 2×2 block matrix, we arrive at

$$\begin{aligned}\mathrm{Tr}_m(\mathrm{Tr}_1(A)\,\mathrm{Tr}_1(B)) &\pm \mathrm{Tr}_m(\mathrm{Tr}_1(X)^*\,\mathrm{Tr}_1(X)) \\ &\le \mathrm{Tr}_m(\mathrm{Tr}_1(A))\,\mathrm{Tr}_m(\mathrm{Tr}_1(B)) \pm \mathrm{Tr}_m(\mathrm{Tr}_1(X)^*)\,\mathrm{Tr}_m(\mathrm{Tr}_1(X)).\end{aligned} \tag{3.5}$$

By definition, $\mathrm{Tr}_m\big(\mathrm{Tr}_1(A)\big) = \mathrm{Tr}_{nm}(A)$, and similarly $\mathrm{Tr}_m\big(\mathrm{Tr}_1(B)\big) = \mathrm{Tr}_{nm}(B)$, and $\mathrm{Tr}_m(\mathrm{Tr}_1(X)) = \mathrm{Tr}_{nm}(X)$. Moreover, $\mathrm{Tr}_m\big(\mathrm{Tr}_1(X)^*\big) = \overline{\mathrm{Tr}_{nm}(X)}$

Substituting these identities into (3.5) yields the first desired inequality.

Utilizing the same reasoning with the roles of the two factors interchanged we can obtain the other required inequality. □

Now, we establish a characterization of the matrix trace. Over the last few decades, multiple mathematicians have explored characterizations of tracial positive linear functionals on matrices and operator algebras. Interested readers are referred to [8, 17] and the references mentioned therein. If φ is a positive functional on $\mathbb{M}_n$ and $A, B \in \mathbb{M}_n^+$, then the value of $\varphi(AB)$ may lies in $\mathbb{C}\setminus\mathbb{R}$. Furthermore, $\varphi(AB) \in \mathbb{R}$ for all projections $A, B \in \mathbb{M}_n$ if and only if φ is tracial [5, Theorem 4.1].

Theorem 3.2 *The matrix trace is the unique positive linear functional φ on $\mathbb{M}_n$ such that $\varphi(I) = n$ and the inequality*

$$\varphi(\sqrt{A}B\sqrt{A}) \pm \varphi(X^*X) \le \varphi(A)\varphi(B) \pm \varphi(X^*)\varphi(X). \tag{3.6}$$

holds for all $A, B, X \in \mathbb{M}_n$ provided that $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ is positive semidefinite (which in particular implies $A, B \ge 0$).

Proof It suffices to show that if a linear functional φ satisfies equation (3.6), then φ must be the trace functional. As mentioned in the introduction, we can assume that $\varphi(A) = \mathrm{Tr}(RA)$ for a positive semidefinite matrix $R = \mathrm{diag}(r_1, r_2, \ldots, r_n)$ with $\mathrm{Tr}(R) = n$. The proof is then reduced to showing that all the diagonal entries r_i are equal. To establish this, it is sufficient to prove that $r_1 = r_2$. Consequently, without loss of generality, we can assume that $n = 2$. Thus, it is enough to prove that if $R = \mathrm{diag}(r, 2 - r)$ for some $0 \le r \le 2$, and

$$\mathrm{Tr}(R\sqrt{A}B\sqrt{A}) \pm \mathrm{Tr}(RX^*X) \le \mathrm{Tr}(RA)\,\mathrm{Tr}(RB) \pm \mathrm{Tr}(RX^*)\,\mathrm{Tr}(RX), \tag{3.7}$$

for all matrices $A, B, X \in M_2$ when $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ is positive semidefinite, then $r = 1$.

Let

$$A = B = \begin{bmatrix} 1 & p \\ p & p^2 \end{bmatrix} \quad \text{and} \quad X = 0,$$

where $p > 1$. Then, A and B are positive semidefinite, and so is $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{M}_{2n}$. Moreover, $A = B = (1+p^2)P$ with the one-dimensional projection

$$P = \begin{bmatrix} \frac{1}{1+p^2} & \frac{p}{1+p^2} \\ \frac{p}{1+p^2} & \frac{p^2}{1+p^2} \end{bmatrix}.$$

We have $\sqrt{A} = \sqrt{1+p^2}\,P$ and $\sqrt{A}B\sqrt{A} = (1+p^2)A$. It follows from (3.7) that

$$(1+p^2)(r+(2-r)p^2) \le (r+(2-r)p^2)^2,$$

whence, $(p^2-1)r \le p^2-1$. Therefore, $r \le 1$. By the same reasoning as above but with

$$A = B = \begin{bmatrix} p^2 & p \\ p & 1 \end{bmatrix} \quad \text{and} \quad X = 0,$$

we infer that $r \ge 1$. Thus, $r = 1$. □

Remark 3.3 It is interesting to extend the considered inequalities to other settings including tracial positive linear functionals on C^*-algebras and von Neumann algebras; see [1, 7].

Acknowledgements The authors are grateful to the referee for several useful comments and suggestions.

Author Contributions AB and MSM wrote the main manuscript text. All authors reviewed the manuscript.

Funding The work of the first author was performed under the development program of Volga Region Mathematical Center (agreement no. 075-02-2025-1725/1).

Data Availability No datasets were generated or analysed during the current study.

Declarations

Conflicts of Interest On behalf of the authors, the corresponding author states that there is no conflict of interest.

References

1. Argerami, M., Moslehian, M.S.: The trace and its extensions in operator algebras, Irish Math. Soc. Bull. **96**, 69–79 (2025)
2. Audenaert, K.M.R.: Subadditivity of q-entropies for $q>1$. J. Math. Phys. **48**(8), 083507 (2007). (**3 pp**)
3. Besenyei, A.: A note on a trace inequality for positive block matrices, Technical Report, (2013)
4. Bikchentaev, A.M.: The Peierls-Bogoliubov inequality in C^*-algebras and characterization of tracial functionals. Lobachevskii J. Math. **32**(3), 175–179 (2011)
5. Bikchentaev, A.M.: Commutation of projections and characterization of traces on von Neumann algebras. III. Internat. J. Theoret. Phys. **54**(12), 4482–4493 (2015)
6. Bikchentaev, A.M.: Characterization of certain traces on von Neumann algebras, Infinite dimensional analysis, quantum probability and applications, 279–289, Springer Proc. Math. Stat., 390, Springer, Cham., (2022)
7. Bikchentaev, A.M., Moslehian, M.S.: Characterizations of tracial functionals on matrix and operator algebras. Bull. Belg. Math. Soc. Simon Stevin **32**(3), 343–361 (2025a)

8. Bikchentaev, A.M., Moslehian, M.S.: Characterizations of tracial functionals on C*-algebras, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **28**, 2550003 (2025). (**14 pp**)
9. Bikchentaev, A.M., Kittaneh, F., Moslehian, M.S., Seo, Y.: Trace inequalities for Matrices and Hilbert Space Operators. Forum for Interdisciplinary Mathematics, Singapore, Springer (2024)
10. Blackadar, B.: Operator algebras, theory of C^*-algebras and von Neumann algebras, in: Encyclopaedia of Mathematical Sciences, in: Operator Algebras and Non-commutative Geometry, III, vol. 122. Berlin, Springer-Verlag, (2006)
11. Fu, X., Gumus, M.: Trace inequalities involving positive semi-definite block matrices. Linear Multilinear Algebra **70**(20), 5987–5994 (2022)
12. Gardner, L.T.: An inequality characterizes the trace. Canad. J. Math. **31**(6), 1322–1328 (1979)
13. Ji, G., Tomiyama, J.: On characterizations of commutativity of C^*-algebras. Proc. Amer. Math. Soc. **131**(12), 3845–3849 (2003)
14. Kittaneh, F., Lin, M.: Trace inequalities for positive semidefinite block matrices. Linear Algebra Appl. **524**, 153–158 (2017)
15. Kosaki, H.: On an inequality of Araki-Lieb-Thirring (von Neumann algebra case). Proc. Amer. Math. Soc. **114**(2), 477–481 (1992)
16. Lin, M.: Inequalities related to 2 × 2 block PPT matrices. Oper. Matrices **9**(4), 917–924 (2015)
17. Moslehian, M.S.: More characterizations of the matrix trace. Rend. Circ. Mat. Palermo (2) **74**(4), 130 (2025). (**8 pp**)
18. Moslehian, M.S., Osaka, H.: Advanced Techniques with Block Matrices of Operators. Frontiers in Mathematics, Birkhäuser, Cham (2024)978-3-031-64545-7
19. Murphy, G.J.: C^*-algebras and operator theory, Boston. Academic Press, Inc, MA etc. (1990)
20. Sherstnev, A.N.: Methods of bilinear forms in non-commutative measure and integral theory, Moscow, Fizmatlit, (2008). (ISBN 978-5-9221-0911-6/hbk) (Russian)
21. Stolyarov, A.I., Tikhonov, O.E., Sherstnev, A.N.: Characterization of normal traces on von Neumann algebras by inequalities for the modulus. Math. Notes **72**(3–4), 411–416 (2002)
22. Takesaki, M.: Theory of operator algebras. I. Encyclopaedia of Mathematical Sciences, 124. Operator Algebras and Non-commutative Geometry, 5. Springer-Verlag, Berlin, (2002)
23. Tikhonov, O.E.: Subadditivity inequalities in von Neumann algebras and characterization of tracial functionals. Positivity **9**(2), 259–264 (2005)
24. Zhang, F.: Matrix Theory. Second edition, Universitext, Springer, New York, Basic Results and Techniques (2011)
25. Zhang, F.: Trace inequality for positive block matrices, IMAGE Solution 50-3.1, Bull. Internat. Linear Algebra Soc. (Fall 2013)