A counterexample of size 20 for the problem of finding a 3-dimensional stable matching with cyclic preferences

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Abstract. As is known, the problem of finding a three-dimensional stable matching with cyclic preferences (3DSM-CYC) always has a solution, if the number of objects of each type (i.e., the problem size n) does not exceed 5. According to the conjecture proposed by K. Eriksson, J. Söstrand, and P. Strimling (2006), this is true for any n. However, C.-K. Lam and C.G. Plaxton (2019) have proposed an algorithm for constructing preference lists in 3DSM-CYC which has allowed them to disprove the mentioned conjecture. The size of the initially constructed counterexample equals 90; however, according to the results obtained by us recently for the problem with incomplete preference lists, one can use the same construction for getting a counterexample of size 45. The main contribution of this paper consists of reducing the size of the counterexample to $n = 20$. We summarize results of the application of the technique developed by us for constructing counterexamples for 3DSM-CYC. In the final section of the paper we discuss a new variant of 3DSM, whose solution always exists and can be found within the same time as that required for solving 2DSM.

Keywords: 3-dimensional stable matching · cyclic preferences · preference graph · Gale-Shapley algorithm

1 Introduction

The monograph [11] by Donald Knuth devoted to the analysis of the Gale–Shapley algorithm for finding a stable matching has inspired many mathematicians. Let us briefly recall the essence of the problem solved by this algorithm in “matrimonial” terms of the mentioned book by D. Knuth.

Given n representatives of each of two genders (men and women), we assume that each person has a complete list of preferences of representatives of the opposite gender. We consider sets consisting of n disjoint pairs of men and women, in other words, perfect matchings of a complete bipartite graph. We say that a man and a woman who prefer each other to their partners, represent a blocking pair of matching. If no blocking pair exists for matching, then the latter is said to be stable. D. Gale and L.S. Shapley [7] have stated the problem of finding a stable matching (2DSM) and proved that such matching always exists. Moreover, they have proposed a sufficiently simple algorithm for finding a stable matching.

In the case of two genders, the problem with incomplete preference lists (2DSMI) is also easily solvable. It is convenient to symmetrize preference lists in 2DSMI so as to make a man X and a woman x either concurrently enter in preference lists of each other or concurrently be not included in them. If initially this condition is not fulfilled, we can just truncate preference lists by deleting persons, whose sympathies are unrequited. Stable matchings in 2DSMI are not necessarily complete, it is only necessary that single X and x do not enter in preference lists of each other and cannot "steal" someone from his/her "spouse". In other words, a blocking pair in 2DSMI consists of a man and a woman, who would become "happier" when forming a couple (note that though the loneliness is the most unpleasant state for them, it is possible, as distinct from marrying a person, who does not enter in their preference lists).

The technique for solving 2DSMI with the help of the algorithm proposed for solving 2DSM is rather simple. To this end, we supplement arbitrarily preference lists in 2DSMI, solve the resulted 2DSM, and then make persons, who did not enter in initial preference lists of each other, remain single. One can easily make sure that as a result we get a stable matching for 2DSMI.

The main content of the monograph of D. Knuth ([11]) is the analysis of the behavior of the Gale–Shapley algorithm with random preference lists. However, the end part of the mentioned book contains some problems that are beyond this issue. In particular, Problem 11 is stated as follows [11, p. 64]: "Can the stable-matching problem be generalized to three sets of objects (for example men, women, and dogs)?". Since the publication of the monograph by D. Knuth, mathematicians have been studied various three-dimensional generalizations of the theory of stable matchings. A well-known problem is 3DSM-CYC [4]. In a problem instance, we are given n men, n women, and n dogs such that each man (respectively woman, dog) has a strictly ordered preference list over a subset of women (respectively dogs, men). Recall that a 3-dimensional matching (3DM) μ is a partition of the set of all men, women, and dogs into disjoint heterogeneous triples. If (m, w, d) is a triple in μ, we let $\mu(m)$ denote w, $\mu(w)$ denote d and $\mu(d)$ denote m. μ is (weakly) stable if it admits no blocking triple, which is a triple (m, w, d) such that m prefers w to $\mu(m)$, w prefers d to $\mu(w)$, and d prefers m to $\mu(d)$. The 3DSM-CYC problem then is to find a stable matching or report that none exists for a given problem instance.

E. Boros, V. Gurvich, S. Jaslar, and D. Krasner prove that if $n \leq 3$, then 3DSM-CYC is solvable with any preference lists (see [4] for the proof of this fact in the case of k genders, $k > 2$). In [6], K. Eriksson, J. Söstrand, and P. Strimling generalize this result for the case when $n = 4$. Ibid, they state the conjecture that any 3DSM-CYC (or just 3DSM) has a solution with any n. Using the statement of the satisfiability problem and performing an extensive computer-assisted search, K. Pashkovich, L. Poirrier (see [16]) prove the validity of the conjecture stated by K. Eriksson et al. for $n = 5$. In [17], B. Pittel proves that with random preference lists the mean value of stable matchings in 3DSM grows as $\Omega(n^2 \ln^2(n))$. Note that heuristic solution algorithms for 3DSM with random preference lists find a solution rather quickly, if $n \sim 10-30$. In Chapter "Further stable matching problems" of the review book ([15, p. 299]) devoted to stable matchings, D. Manlove writes: "Perhaps the most intriguing open problem in this list at least in view of the number of authors that have mentioned it, concerns 3DSM-CYC, and in particular the question of whether every instance I of this problem admits a weakly stable matching."

However, hopes for the existence of both a solution in 3DSM and a polynomial algorithm for finding this solution were somewhat naive. The paper [2] by P. Biró and E. McDermid is devoted to finding a stable matching with incomplete preference lists in the 3D-case (3DSMI). Note that in this problem similarly to the two-gender case some agents, possibly, remain single. A blocking triple is also defined in a similar way, namely, as a triple consisting of a man, a woman, and a dog, such that all of them would become "happier", when forming a family. However, in contrast to the two-gender case, in the mentioned paper published back in 2010, P. Biró and E. McDermid give a sufficiently simple example of 3DSMI of size $n = 6$, where no stable matching exists. Moreover, they prove that the problem of establishing the solvability of 3DSMI is NP-complete.

The conjecture stated by K. Eriksson et al. has been recently disproved by C.-K. Lam and C.G. Plaxton [12]. They associate 3DSMI with a certain 3DSM problem, where n is 15 times greater than the initial size; this problem is solvable if and only if so is the initial 3DSMI. Therefore, the problem of establishing the solvability of 3DSM is NP-complete.

The example proposed in the paper [2] allows one to construct an instance of 3DSM with no stable matching for $n = 90 = 6 \times 15$. The question on the existence of counterexamples of a lesser size remained open.

An evident way to reduce the size of counterexamples of 3DSM is to solve the P. Biró and E. McDermid problem that implies the search of instances of 3DSMI with no stable matching for $n < 6$. We solve the mentioned problem in the paper [14]. We prove the absence of such instances for $n < 3$ and construct several counterexamples for 3DSMI with $n = 3$. Therefore, the result obtained in [12] allows one to construct an instance of 3DSM with no stable matching for $n = 45$.

Another approach to reduce the size of counterexamples of 3DSM is to reduce the value of the multiplier (its current value equals 15) when constructing an unsolvable instance of 3DSM from an unsolvable 3DSMI problem. We apply this approach in [13]. We prove that one can associate each instance of 3DSMI of size n with no stable matching with an instance of 3DSM of size $8n$ with the same property. This allows us to reduce the size of the counterexample of 3DSM to $3 \times 8 = 24$.

This paper completes the series of works devoted to decreasing the size of counterexamples for 3DSM. We tried to make this text understandable even for an unprepared reader. Developing constructions proposed in the paper [13] and making use of specific features of a certain concrete instance of 3DSMI with no stable matching, we have succeeded in constructing a counterexample for 3DSM of size $n = 20$.

The remainder of the paper has the following structure. In Sect. 2 (similarly to previous papers of this series) we present formal statements of 3DSM-CYC and 3DSMI-CYC in terms of the graph theory. In Sect. 3, we state the Key Lemma given in [13] (in somewhat different terms, which seem to be more clear) and auxiliary lemmas for it, which are used by us in this paper. Ibid, we also give the proof of the main Theorem from [13]. The technique used for proving the main Theorem of this paper is based on the approach applied in the mentioned paper. The main results of this paper are presented in Sect. 4 and 5. In Sect. 4, we improve the Key Lemma for further decreasing the size of counterexamples. In Sect. 5, we propose a counterexample of 3DSM of size 20. In Sect. 6, we summarize the obtained results and discuss in detail some potential future work. In particular, we propose a new variant of 3DSM, whose solution always exists and one can find it within the same time as that consumed for solving 2DSM. In the Appendix, we give a short proof of Theorem 2 from [14]; we use it in this paper.

2 The statement of 3DSM (3DSMI) in terms of graph theory

Let G be some directed graph. Denote the set of its edges by E (or $E(G)$); assume that no edge is multiple. Assume that the vertex set V of the graph G is divided into three subsets, namely, the set of men M, women F, and dogs D. Assume that edges (v, v'), $v, v' \in V$, of this graph are such that either $v \in M, v' \in F$, or $v \in F, v' \in D$, or $v \in D, v' \in M$. Assume that $|M| = |F| = |D|$ (otherwise we supplement the corresponding subgraph with vertices that are not connected with the rest part of the graph). The number $n = |M| = |F| = |D|$ is called the problem size. Evidently, the length of all cycles in the graph G is a multiple of 3. Note also that this condition ensures the possibility of dividing the vertex set of any digraph G into three subsets M, F, and D so that all its edges are directed as is indicated above.

Each edge (v, v'), $v, v' \in V$, corresponds to some positive integer $r(v, v')$; it is called the rank of this edge. For fixed $v \in V$ all possible ranks $r(v, v_1), \ldots, r(v, v_k)$ coincide with $\{1, \ldots, k\}$, where k is the outgoing vertex degree v (if $r(v, v') = 1$, then v' is the best preference for v, and so on).

We understand a *three-sided matching* as a subgraph H of the graph G, $V(H) = V(G) = V$, where each vertex $v \in V$ has at most one outgoing edge and the following condition is fulfilled: if a vertex v has an outgoing edge, then this edge belongs to a cycle of length 3 in the graph H. Cycles of length 3 in the graph H are called *families*. Evidently, each family, accurate to a cyclic shift, takes the form (m, f, d), where $m \in M$, $f \in F$, and $d \in D$. Note that in what follows, for convenience of denotations of families, we do not fix the order of genders in a family, i.e., we treat denotations of families as triples derived from an initial one by a cyclic shift as equivalent.

In what follows, we sometimes use the notion of a family in a wider sense, namely, as any cycle of length 3 in the graph G. However, if some three-sided matching H is fixed, then we describe other cycles of length 3 explicitly, applying the term "a family" only to cycles that enter in a three-sided matching.

A *matching* μ is a collection of all families of a three-sided matching H. For a vertex v, $v \in V$, in the matching μ, the rank $R_\mu(v)$ is defined as the rank of the edge that goes out of this vertex in the subgraph H. If some vertex v in the subgraph H has no outgoing edge, then $R_\mu(v)$ is set to $+\infty$.

A triple (v, v', v'') is said to be *blocking* for some matching μ, if it represents a cycle in the graph G, and

$$r(v, v') < R_\mu(v), \quad r(v', v'') < R_\mu(v'), \quad r(v'', v) < R_\mu(v''). \tag{1}$$

A matching μ is said to be *stable* if no blocking triple exists for it.

Recall that *3DSMI* consists of finding a stable matching for a given graph G. As is well known, such a matching does not necessarily exist. Moreover, the problem of establishing its existence for a given graph G is NP-complete. As was mentioned in the Introduction, the proof of this fact belongs to P. Biró and E. McDermid. They have constructed an explicit example of the graph G of size 6, for which no stable matching exists.

Evidently, 3DSM represents a particular case of the 3DSMI, where the outgoing (and incoming) degree of each vertex of the corresponding graph equals the problem size n.

Let us construct on the set of agents (graph vertices) the same map as that used in [15] and in other papers. Denote it by the symbol μ; the same symbol stands for the matching from which we construct this map. If an agent x remains single, then we put $\mu(x) = x$. Otherwise $\mu(x) = y$, where y is the vertex, which represents the endpoint of an edge in the subraph H that generates the matching μ. Informally speaking, in this case $\mu(x)$ is the partner of x in the family, whom x is "partial" to. Evidently, in 3DSM, as distinct from 3DSMI, the equality $\mu(x) = x$ is impossible for a stable matching. Note that $\mu(\mu(x))$ is also a preimage of the vertex x under the map μ; for brevity, we use the denotation $\mu^{-1}(x)$ for it.

Let H' be some subgraph of the graph of 3DSM. In what follows, we consider certain (specific for this paper) denotations and terms which contain this subgraph. For any vertex $v \in V(H')$, we define the following values:

$$\overline{\rho}_{H'}(v) = \max_{(v,w)\in E(H')} r(v, w), \qquad \underline{\rho}_{H'}(v) = \min_{(v,w)\in E(H')} r(v, w).$$

Let $(x, y) \in E(H')$. We call the subgraph H' an (x, y)-*attractor*, if for any stable matching μ, the equality $\mu(x) = y$ implies the inclusion $\mu(y) \in V(H')$. Informally speaking, an (x, y)-attractor "covers" any family in μ, which contains its edge (x, y).

Let us introduce one more definition. Assume, as above, that $x \in V(H')$. We call the subgraph H' an x-*superattractor*, if for any stable matching μ, the inequality $R_\mu(x) \geq \underline{\rho}_{H'}(x)$ implies the inclusion $\{\mu(x), \mu^{-1}(x)\} \subseteq V(H')$. In other words, $V(H')$ contains a family (x, y, z) from μ if the rank of the edge (x, y) is not less than the rank of some edge (x, v), $(x, v) \in E(H')$.

Evidently, an x-superattractor is an (x, y)-attractor for any $(x, y) \in E(H')$. The above definition also implies that any subgraph H'' of the graph of 3DSM such that $H'' \subseteq H'$ has the following properties:

1. If H'' is an (x, y)-attractor, then H' also is an (x, y)-attractor.
2. Let the set of edges that are incident to the vertex x be one and the same both in H' and in H''. Then if H'' is an x-superattractor, then H' also is an x-superattractor.

We call properties 1 and 2 inheritance properties of the attractor (superattractor) obtained with the extension of the graph.

In what follows, we use symbols G, H and H' for graphs of 3DSM and 3DSMI, as well as their subgraphs; we also use, when appropriate, the symbol H' with various subscripts. In certain graphs, we concurrently consider a subgraph H and subgraphs denoted by the symbol H' with various subscripts when considering them concurrently. In such cases, the symbol H denotes the "central", in a sense, subgraph, while the symbol H' with some subscript denotes a "peripheral" subgraph characterized by its subscript. We use the same subscript in denotations for vertices of the latter subgraph.

3 The correspondence between unsolvable 3DSMI and 3DSM

In this section, we prove that each 3DSMI of size n with no stable matching corresponds to 3DSM of size $8n$ with the same property. As was mentioned in the Introduction, an analogous result with the multiplier 15 belongs to C.-K. Lam and C.G. Plaxton [12]. As a corollary, making use of results obtained by us in the paper [14] (see Lemma 10 in the Appendix), we obtain concrete instances of 3DSM of size 24 with no stable matching.

Let us consider subgraphs of the weighted graph of 3DSM, which include some vertices and edges of the initial graph. Let us first prove the Key Lemma 1. (We divide the proof into several separate parts).

Lemma 1 (The Key Lemma for Theorem 1). *Let some subgraph H' of the graph of 3DSM take the form shown in Fig. 1, in particular,*

$$r(b, s) = r(d, t) = 3, \quad r(x, c) = r'_x, \; r(x, d) = r'_x + 1, \tag{2}$$

where $r'_x \in \{1, \ldots, n - 1\}$. Then H' is an x-superattractor.

Lemma 2. *Assume that some subgraph H'' of the graph of 3DSM takes the form shown in Fig. 2. Then H'' is an (x, c)-attractor.*

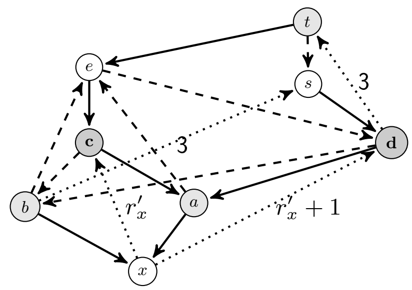

Fig. 1. The subgraph H' of the preference graph considered in Lemma 1. Vertices of various colors correspond to various genders. Bold lines represent edges of rank 1, while dashed ones do those of rank 2. Ranks of edges represented by dotted lines equal r'_x, $r'_x + 1$, or 3 (ranks are indicated near edges).

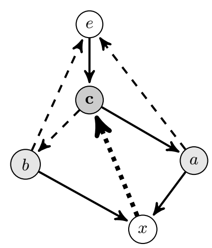

Fig. 2. The subgraph H'' of the preference graph considered in Lemma 2.

Proof. Let μ be a stable matching in this problem, while $\mu(x) = c$ (note that in this case, $R_\mu(e) > 1$). Assume that Lemma 2 is false, i.e., $\mu(c) \notin \{a, b\}$. Then $R_\mu(c) > 2$, $R_\mu(a) > 1$, and $R_\mu(b) > 1$. Consider the triple (c, a, e). We get inequalities $r(c, a) < R_\mu(c)$, $r(e, c) < R_\mu(e)$, and $r(a, e) \leq R_\mu(a)$. Therefore, the triple (c, a, e) is not blocking only if $\mu(a) = e$. But then $R_\mu(b) > 2$ and, consequently, the triple (c, b, e) is blocking. $\square$

In what follows, we repeatedly apply the technique that is used in the proof of Lemma 2. Namely, when considering the potentially blocking triple (c, a, e), with the help of the mentioned technique we conclude that $\mu(a) = e$.

Note also that by the definition of a 3D-matching, for any distinct vertices h and g, equalities $\mu(h) = g$ and $\mu(\mu(g)) = h$ are equivalent. In other words, if $\mu(u_2) = u_1$, then the triple (u_1, v, u_2) does not form families that enter in the matching, only if $\mu(u_1) \neq v$.

Lemma 3. *The subgraph H' shown in Fig. 1 is an (x, d)-attractor.*

Proof. Let μ be a stable matching in 3DSM, while $\mu(x) = d$ (note that this means that $R_\mu(x) = r'_x + 1$, $\mu(e) \neq d$, and $\mu(s) \neq d$). Assume that the assumption of Lemma 3 is violated, i.e., $\mu(d) \notin \{a, b, t\}$.

Then $R_\mu(d) > 3$. Moreover, $\mu(a) \neq x$, otherwise $\mu(d) = \mu^{-1}(x) = a$. Therefore, $R_\mu(a) > 1$. Analogously, $R_\mu(b) > 1$.

Consequently, $\mu(c) = a$, otherwise the triple (c, a, x) is blocking. Analogously, $\mu(b) = s$ (otherwise the triple (d, b, s) is blocking)

Let $\mu(a) \neq e$. Since $\mu(c) = a$, this assumption is equivalent to the condition $\mu(e) \neq c$. Then $R_\mu(a) > 2$, $R_\mu(e) > 2$, and the triple (d, a, e) is blocking. Consequently, the assumption stated at the beginning of this paragraph is violated, and the matching μ contains the family (c, a, e) (see Fig. 3).

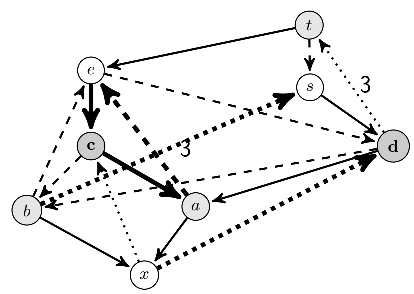

Fig. 3. The part of the preference graph considered in Lemma 3. Bold edges correspond to pairs that enter in families of the matching μ, provided that $\mu(d) \notin \{a, b, t\}$.

Since $\mu(t) \notin \{e, s\}$, we get the inequality $R_\mu(t) > 2$. But then the triple (d, t, s) is blocking. □

Note that the first inheritance property and lemmas 2 and 3 imply that the subgraph H' is concurrently an (x, c)-attractor and an (x, d)-attractor.

Lemma 4. *Assume that some subgraph of the graph of 3DSM takes the form shown in Fig. 4. Let μ be a stable matching in this problem and $R_\mu(x) \geq r'_x$. Then $\mu(x) \in \{c, d\}$.*

Proof. Let us prove this lemma ab contrario. In this case, $R_\mu(x) > r'_x + 1$.

Let us first make sure that $\mu(c) = a$. Assume the contrary. Then $R_\mu(c) > 1$. Consider the triple (c, a, x). It is not blocking, only if $\mu(a) = x$. But then $R_\mu(b) > 1$. Moreover, according to assumptions $\mu(x) \notin \{c, d\}$, i.e., $\mu^{-1}(a) \notin \{c, d\}$. Therefore $R_\mu(d) > 1$ and $R_\mu(c) > 1$. Consequently, the triple (c, b, x) is not blocking, only if $\mu(c) = b$ (see Fig. 4). Since $R_\mu(d) > 2$, the triple (d, b, x) is blocking.

Thus, we have proved that $\mu(c) = a$, consequently, $R_\mu(d) > 1$. Moreover, by assumption, $\mu(a) \neq x$, whence we conclude that $R_\mu(a) > 1$. Then the triple (d, a, x) is blocking. □

The assertion of Lemma 1 evidently follows from proved Lemmas 2, 3, and 4.

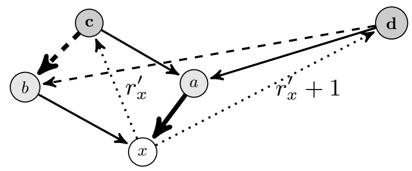

Fig. 4. The part of the preference graph considered in Lemma 4. Solid lines represent edges of rank 1, while dashed ones do those of rank 2. Ranks of edges represented by dotted lines equal r'_x and r'_x+1 (ranks are indicated near edges). Bold edges correspond to pairs that enter in families of the matching μ in the case, when $R_\mu(x) \geq r'_x$, $\mu(c) \neq a$.

Remark 1. In all figures, vertices that characterize genders are colored so as to make graph edges be directed only from white vertices to black ones, from black vertices to gray ones, and from the latter to white ones. However, it is evident that one can "shift these colors modulo 3". For example, in the graph shown in Fig. 1 we can recolor all gray vertices to black. Then certainly we should recolor all vertices that were originally white to gray and color vertices c and d to white; this does not affect the statement of Lemma 1.

Theorem 1. *Let H be the graph of 3DSMI of size n with no stable matching. Let us use it for constructing the graph G of 3DSM in the following way. The graph H is a subgraph of the graph G (with the same ranks of edges). To each vertex x of the graph H we "attach" the corresponding copy of the graph H'_x shown in Fig. 1 with vertices $a_x, b_x, c_x, d_x, e_x, s_x, t_x \notin V(H)$ (all subgraphs H'_x are pairwise disjoint). Moreover, let the value r'_x in formulas (2) equal $\overline{\rho}_H(x)+1$. Let us define ranks of the rest edges of the graph G of 3DSM arbitrarily. Then 3DSM with the graph G has no stable matching. Here the size of 3DSM equals $8n$.*

Proof. Assume the contrary, i.e., assume that for 3DSM with the graph G there exists a stable matching μ_G. We intend to construct the matching μ_H for 3DSMI defined by the graph H from the matching μ_G. To this end, we will make use of Lemma 1. Since the matching μ_H is not stable, we can find for it a blocking triple (v, v', v'') composed of vertices of the subgraph H. Let us prove that the same triple (v, v', v'') is blocking for μ_G.

Let us perform the proof adhering to the above scheme. Let $x \in V(H)$. Denote $y = \mu_G(x)$. The following alternatives are possible:

A) $y \notin V(H)$. Then
$$\overline{\rho}_H(x) < R_{\mu_G}(x). \tag{3}$$
According to Lemma 1, we get the inclusion $\{y, \mu_G^{-1}(x)\} \subseteq V(H'_x)$.

B) $y \in V(H)$. Assume that $\mu_G(y) \notin V(H)$. Then by Lemma 1 we get the inclusion $\{\mu_G(y), \mu_G^{-1}(y)\} \subseteq V(H'_y)$. But since $\mu_G^{-1}(y) = x$, we get a contradiction. Consequently, in this case, $\mu_G(y) \in V(H)$.

Note that the latter property is very important. It means also that if $R_{\mu_G}(x) < \underline{\rho}_{H'_x}(x)$, then the family $(x, \mu(x), \mu^{-1}(x))$ entirely lies in $V(H)$.

Let us associate the matching μ_G with the matching μ_H of 3DSMI with the graph H. Assume that in the case of alternative A, $\mu_H(x) = x$ (i.e., the agent x remains single). In the case of alternative B, we put $\mu_H(x) = \mu_G(x)$.

By condition, the matching μ_H is not stable. Therefore, there exists a blocking triple (v, v', v''), where $v, v', v'' \in V(H)$. The blocking property means that each vertex in this triple satisfies the inequality that connects ranks, for example, for the vertex v this inequality takes the form $r(v, v') < R_{\mu_H}(v)$. If $\mu_H(v) \neq v$, then $\mu_G(v) = \mu_H(v)$ and, consequently, $r(v, v') < R_{\mu_G}(v)$ (because in the case under consideration, $R_{\mu_G}(v) = R_{\mu_H}(v)$).

But if $\mu_H(v) = v$, then $R_{\mu_H}(v) = \infty$. However, now we can make use of inequality (3), which implies that $r(v, v') < R_{\mu_G}(v)$.

Analogous alternatives take place for vertices v' and v'', namely, $r(v', v'') < R_{\mu_G}(v')$ and $r(v'', v) < R_{\mu_G}(v'')$. Therefore, the same triple (v, v', v'') is blocking for the matching μ_G in 3DSM.

Let us calculate the size of 3DSM. Each triple of vertices of the graph H, which are associated with different genders, corresponds to three various subgraphs in the form shown in Fig. 1; see Remark 1 for the principle of their coloring. Note that due to the cyclic shift of colors, any such a triple of vertices in the graph H is supplemented with seven new vertices of each gender of the graph G; in other words, the number of vertices of the graph G becomes 8 times greater. □

Theorem 1, along with the result obtained in the paper [14], allows one to construct instances of 3DSM of size 24 with no stable matching.

4 The Key Lemma for the further reduction of the counterexample size

As was mentioned above, Theorem 1, along with the result obtained in the paper [14], allows one to construct instances of 3DSM of size 24 with no stable matching. In the next section, we reduce the size of such counterexample to 20. As a base we use Lemma 5, which is a certain modification of Lemma 1. Recall that in Lemma 1 we consider an x-superattractor, whose copy is "x-attached" to each vertex of the graph H that defines 3DSMI with no stable matching. In Lemma 5, we consider the subgraph, which in certain cases can have "two attachments" (vertices x and z) to two vertices of such a graph H. The "cost" of this effect is the supplement of the subgraph with the vertex f (apart from the "attached" vertex z). See Fig. 5 for the graph under consideration.

Remark 2. The graph shown in Fig. 1 is a part of the graph shown in Fig. 5, only ranks of three edges in it are different. Namely, now the rank of edges directed from vertices a and b to the vertex z equals 2. Correspondingly, ranks of all edges that go from these vertices, which originally were not less than 2, now are larger by one, i.e., in the new graph, $r(a, e) = r(b, e) = 3$ and $r(b, s) = 4$. Ranks of all the rest edges in the subgraph of the graph shown in Fig. 5, which contains the same vertices x, a, b, c, d, e, s, and t, are the same as in Fig. 1 (and no new edges appear in this subgraph).

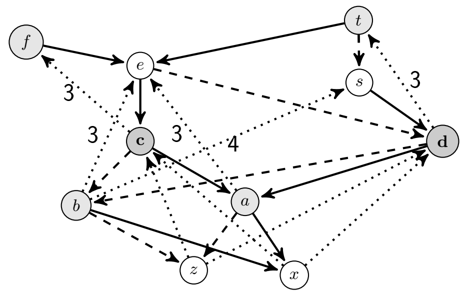

Fig. 5. The subgraph H' of the preference graph considered in Lemma 5. Solid lines represent edges of rank 1, while dashed ones do those of rank 2. Ranks of edges (x, c) and (x, d) equal r'_x and $r'_x + 1$, those of edges (z, c) and (z, d) equal r'_z and $r'_z + 1$, correspondingly, the rank of edges (c, f), $(b, e), (a, e)$, (d, t) equals 3, while $r(b, s) = 4$.

For brevity of the further reasoning, we introduce one more definition (in fact, we have already used it implicitly when considering alternative B in the proof of Theorem 1). Let H' be some subgraph of the graph of 3DSM, $w \in V(H')$. We say that H' is w-*detachable* if for any stable matching μ of 3DSM the inequality $R_\mu(w) < \underline{\rho}_{H'}(w)$ implies that $\mu^{-1}(w) \notin V(H')$. Informally speaking, the latter assertion means that if the current partner of w in a stable matching is more preferable to w than other possible partners in the stratum H', then the family under consideration, except w, entirely lies outside the stratum H'.

Lemma 5 (The Key Lemma for Theorem 2). *Assume that some subgraph H' of the graph of 3DSM takes the form shown in Fig. 5, where ranks are indicated near the corresponding edges.*
A) If H' is z-detachable, then H' is an x-superattractor.
B) If any stable matching μ of 3DSM satisfies the inequality $R_\mu(x) < r'_x$ and H' is x-detachable (i.e., $\mu^{-1}(x) \notin V(H')$), then H' is a z-superattractor.

At the end part of Section 2, we mention inheritance properties possessed by an attractor and a superattractor with the extension of the subgraph. In this section, we need one more technique for constructing attractors and superattractors.

Proposition 1. *Assume that H' is a subgraph of the graph of 3SDM, which contains a certain edge (v, w). Assume also that $\mu(v) \neq w$ for any stable matching μ in the considered problem. Let H'' be obtained from H' by deleting the edge (v, w) and subtracting one from ranks of all edges outgoing from the vertex v, which initially exceeded $r(v, w)$. If the resulting graph H'' is some (x, y)-attractor or x-superattractor, then so is the initial graph H'.*

The validity of Proposition 1 follows from the definition of an attractor and a superattractor, because the order of ranks of edges in the graph H' is the same as that in the graph H''.

Let us prove Lemma 5 with the help of Lemma 4. But first let us consecutively prove analogs of Lemmas 2 and 3.

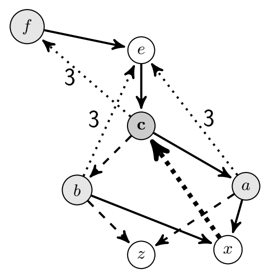

Fig. 6. The subgraph H'' of the preference graph considered in Lemma 6.

Lemma 6. *Assume that some subgraph H'' of the graph of 3DSM takes the form shown in Fig. 6, where ranks are indicated near the corresponding edges. Then H'' is an (x, c)-attractor.*

Proof. Let μ be a stable matching, and $\mu(x) = c$ (note that this implies the inequality $R_\mu(e) > 1$). Assume that Lemma 6 is false, i.e., $\mu(c) \notin \{a, b, f\}$. Then $R_\mu(c) > 3$, $R_\mu(a) > 1$, and $R_\mu(b) > 1$. The triple (a, e, c) (the triple (b, e, c)) is not blocking, only if either $\mu(a) = z$, or $\mu(a) = e$ (either $\mu(b) = z$, or $\mu(b) = e$). Since $\{\mu^{-1}(e), \mu^{-1}(z)\} = \{a, b\}$, we conclude that $\mu(f) \neq e$ and $R_\mu(f) > 1$. But then the triple (f, e, c) is blocking. $\square$

Lemma 7. *Assume that some subgraph H' of the graph of 3DSM takes the form shown in Fig. 5. If H' is z-detachable, then H' is an (x, d)-attractor.*

Proof. Let μ be a stable matching in 3DSM. In the case when $R_\mu(z) < r'_z$, we can make use of Proposition 1 and the fact that H' is z-detachable. Really, by removing edges, which are incident to the vertex z, from the graph H' and reducing ranks of edges (a, e), (b, e), and (b, s) by one, we get a graph, which by Lemma 3 is an (x, d)-attractor.

It remains to consider the case, when $R_\mu(z) \geq r'_z$. By condition, $R_\mu(x) = r'_x + 1$ and $R_\mu(s) > 1$. As above, we perform the proof ab contrario, i.e., we assume that $\mu(d) \notin \{a, b, t\}$.

We get inequalities $R_\mu(d) > 3$, $R_\mu(b) > 1$, and $R_\mu(a) > 1$. The triple (c, a, x) is not blocking, only if $\mu(c) = a$.

Note that if $R_\mu(z) > r'_z + 1$ and $R_\mu(a) > 2$, then the triple (d, a, z) is blocking for the matching μ. Therefore μ contains the family (c, a, z).

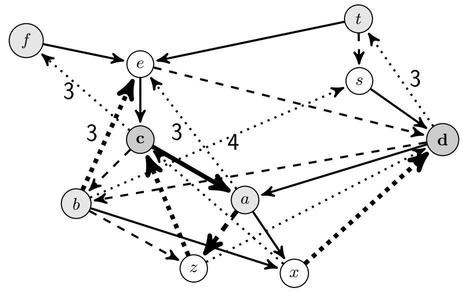

Fig. 7. The subgraph H' of the preference graph considered in Lemma 7. Bold lines represent edges that correspond to pairs that enter in families of a stable matching μ in the case, when $R_\mu(z) \geq r'_z$ and $\mu(d) \notin \{a, b, t\}$.

As a corollary we get inequalities $R_\mu(e) > 2$ and $R_\mu(b) > 2$. Consequently, $\mu(b) = e$, otherwise the triple (d, b, e) is blocking (see Fig. 7). Hence we get the inequality $R_\mu(t) > 1$. Recall that by assumption, $R_\mu(d) > 3$. But then the triple (d, t, e) is blocking. □

Now we can prove Lemma 5.

Proof. The validity of item A of the lemma follows from lemmas 6, 7, and 4 (cf. the proof of Lemma 1). It remains to prove item B.

Let us use Proposition 1. Let us remove the vertex x together with all edges that are incident to it from the graph shown in Fig. 5 and reduce ranks of all edges that are incident to vertices a and b by one. If now we use the symbol x instead of z, then the modified in such a way graph in Fig. 5 will contain the subgraph shown in Fig. 1. It satisfies Lemma 1. Therefore, in this case, according to the inheritance properties of superattractors (see property 2 at the end part of Section 2), H' is a z-superattractor. □

5 An example of unsolvable 3DSM of size 20

Let us make use of the concrete example of 3DSMI of size 3 with no stable matching, which is given in [14, Theorem 2]. See Fig. 8 for the graph H of the considered problem. See the Appendix for the proof of the fact that 3DSMI that corresponds to this graph has no stable matching.

Theorem 2. *Let the graph G of 3DSM contain the subgraph H shown in Fig. 8. Assume that for all considered below subgraphs of the graph G, which are copies of graphs in Fig. 1 and Fig. 5, the value r'_y in the corresponding copies coincides with $\rho_H(y) + 1$; here y is a certain vertex (we specify its number later). Thus, the graph G contains five disjoint subgraphs H'_0, H'_1, H'_2, H'_4, and H'_7, which are copies of the graph in Fig. 1; the role of the vertex x is played there, correspondingly, by vertices 0, 1, 2, 4, and 7 of the graph H. Subgraphs H'_0, H'_1, H'_2, H'_4,*

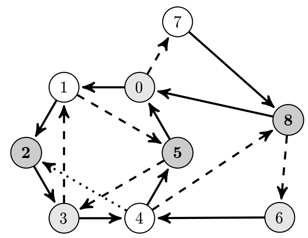

Fig. 8. The graph H of 3DSMI of size 3 with no stable matching. For convenience, we numerate vertices H with numbers v, $v = 0, 1, \ldots, 8$. The value $v \bmod 3$ specifies the gender that corresponds to the vertex v. The rank of each edge, which is represented by a solid line, equals 1. Dashed lines represent edges whose rank equals 2. The rank of the edge $(4, 2)$ equals 3.

and H'_7 have no other common vertices with the graph H. Moreover, the graph G has two disjoint subgraphs $H'_{3,6}$ and $H'_{5,8}$; they are copies of subgraphs mentioned in Fig. 5, the role of the vertex x is played there by vertices 3 and 5, while the role of the vertex z is played by vertices 6 and 8, correspondingly. Subgraphs $H'_{3,6}$ and $H'_{5,8}$ have no more common points with the graph H. Assume that the graph G has no vertices except those considered above, i.e.,

$$V(G) = V(H'_{3,6}) \cup V(H'_{5,8}) \bigcup_{v \in \{0,1,2,4,7\}} V(H'_v).$$

We treat ranks of edges of the graph G, which were not considered above, as arbitrary values. Then 3DSM defined by the graph G has no stable matching.

Lemma 8. *Let μ be a stable matching in 3DSM mentioned in assumptions of Theorem 2. Then $\{\mu(3), \mu^{-1}(3)\} \in V(H)$, and the subgraph $H'_{3,6}$ is a 6-superattractor.*

Proof. Let us first note that the edge $(6, 4)$ is the only edge of the graph H in Fig. 8, which originates from vertex 6. Consequently, if $R_\mu(6) < r'_6$, then $\mu(6) = 4$. Since the subgraph H'_4 is a 4-superattractor, in this case, $\mu^{-1}(6) = \mu(4) \in V(H'_4) \cup \{5, 8\}$. In other words, the subgraph $H'_{3,6}$ is 6-detachable. The subgraph $H'_{3,6}$ satisfies conditions stated in item A of Lemma 5, it is a 3-superattractor.

Assume that $R_\mu(3) > 2$, then $\{\mu(3), \mu^{-1}(3)\} \in V(H'_{3,6})$, $R_\mu(2) > 1$. The triple $(1, 2, 3)$ is not blocking, only if $\mu(1) = 2$. But since H'_2 is a 2-superattractor and $\mu^{-1}(2) \notin V(H'_2)$, this implies that $R_\mu(2) < r'_2 = 2$, i.e., $\mu(2) = 3$. We get a contradiction with the assumption made at the beginning of this paragraph.

Consequently, $\mu(3) \in \{1, 4\}$. The subgraph H'_1 is an 1-superattractor, therefore, if $\mu(3) = 1$, then $R_\mu(1) < r'_1$, i.e., $\mu(1) = \mu^{-1}(3) \in V(H)$. Analogously, in the case, when $\mu(3) = 4$, we get the inclusion $\mu^{-1}(3) \in V(H)$. Thus, $\mu^{-1}(3) \notin V(H'_{3,6})$. Consequently, the subgraph $H'_{3,6}$ satisfies conditions stated in item B of Lemma 5, and the subgraph $H'_{3,6}$ is a 6-superattractor. $\square$

Lemma 9. *Let μ be a stable matching in 3DSM mentioned in assumptions of Theorem 2. Then $\{\mu(5), \mu^{-1}(5)\} \in V(H)$, and the subgraph $H'_{5,8}$ is an 8-superattractor.*

Proof. Assume that $R_\mu(1) > 3$. Since the subgraph H'_1 is an 1-superattractor, we get the embedding $\{\mu(1), \mu^{-1}(1)\} \subseteq V(H'_1)$. Then $R_\mu(0) > 1$. The triple $(0, 1, 5)$ is not blocking, only if $\mu(5) = 0$. Let us now make use of the fact that the subgraph H'_0 is a 0-superattractor such that $\mu^{-1}(0) \notin V(H'_0)$. Then $R_\mu(0) < 3$ and $\mu(0) \neq 1$, therefore, $\mu(0) = 7$, i.e., the matching μ contains the family $(7, 5, 0)$. Thus, $R_\mu(7) > 1$, $\mu(7) \notin V(H'_7)$, which contradicts the fact that the subgraph H'_7 is a 7-superattractor. Therefore, the assumption made at the beginning of this paragraph is false, $\mu(1) \in \{2, 5\}$.

Note that if $R_\mu(8) < r'_8$, then $\mu(8) \in \{6, 0\}$. In view of Lemma 8 the subgraph $H'_{3,6}$ is a 6-superattractor and by condition of the lemma the subgraph H'_0 is a 0-superattractor. If $\mu(8) = 6$, then $R_\mu(6) < r'_6$, i.e., $\mu(6) = \mu^{-1}(8) \in V(H)$. Analogously, in the case, when $\mu(8) = 0$, we get the inclusion $\mu^{-1}(8) \in V(H)$. Consequently, $\mu^{-1}(8) \notin V(H'_{5,8})$, i.e., the subgraph $H'_{5,8}$ is 8-detachable. The subgraph $H'_{5,8}$ satisfies assumptions made in item A of Lemma 5, it is a 5-superattractor.

Assume that $R_\mu(5) > 2$, then $\mu^{-1}(5) \in V(H'_{5,8})$, consequently, $R_\mu(4) > 1$. The triple $(3, 4, 5)$ is not blocking, only if $\mu(3) = 4$ (see Fig. 9). But for vertex 1

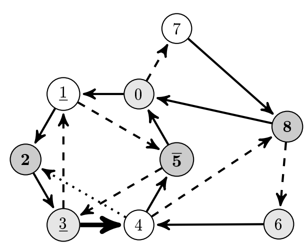

Fig. 9. The subgraph H considered in the proof of Lemma 9 in the case when $R_\mu(5) > 2$. We underline vertices x such that $(x, \mu(x)) \in E(H)$ and overline the rest ones. The solid line that represents the edge $(3, 4)$ illustrates the fact that $\mu(3) = 4$.

the equality $\mu(1) = 5$ is impossible, because $\mu^{-1}(5) \in V(H'_{5,8})$. Therefore, $\mu(1) = 2$, and the 2-superattractor of H'_2 is such that $\mu^{-1}(2) \notin V(H'_2)$, consequently, $R_\mu(2) < 2$. Then the matching μ contains the family $(1, 2, 3)$, which contradicts the equality $\mu(3) = 4$. Therefore, the assumption made at the beginning of this paragraph is false, $R_\mu(5) \leq 2$.

In this case, $\mu(5) \in \{3, 0\}$. If $\mu(5) = 3$, then in view of Lemma 8 we get the equality $\mu^{-1}(5) = \mu(3) \in V(H)$. If $\mu(5) = 0$, then $\mu^{-1}(0) \notin V(H'_0)$. Since the subgraph H'_0 is a 0-superattractor, we conclude that $R_\mu(0) < r'_0$, i.e., $\mu^{-1}(5) \in V(H)$.

Therefore, the subgraph $H'_{5,8}$ satisfies assumptions of item B of Lemma 5, it is a 8-superattractor. $\square$

Now we can prove Theorem 2.

Proof. Assume that in considered 3DSM there exists a stable matching μ_G. According to Lemmas 8, 9 each vertex $x \in V(H)$ satisfies one of the following two alternatives: either $\mu_G(x) \notin V(H)$ and $\mu_G^{-1}(x) \notin V(H)$, or $(x, \mu_G(x)) \in E(H)$ (in particular, the latter case takes place for vertices $x = 3, 5$). Evidently, in the case of the second alternative, $(y, \mu_G(y)) \in E(H)$ for $y = \mu_G(x)$ (see the same correlation in the proof of Theorem 1).

Let us associate the stable matching μ_G with the matching μ_H of 3DSMI with the graph H. Let us do it similarly to the proof of Theorem 1, namely, in the case, when the first alternative takes place for $x \in V(H)$, we put $\mu_H(x) = x$, while in the case, when the second alternative takes place, we do $\mu_H(x) = \mu_G(x)$.

There exists no stable matching for the graph H, therefore for the matching μ_H one can find a blocking triple (v, v', v''), where $v, v', v'' \in V(H)$. Similarly to the proof of Theorem 1, we conclude that the same triple (v, v', v'') is blocking for the matching μ_G of 3DSM. $\square$

Let us calculate the size of 3DSM mentioned in Theorem 2. The simplest way to do this is to calculate the difference between the quantity of vertices that correspond to each gender in Theorem 2 and in Theorem 1 (in the case when the subgraph H in both theorems is one and the same, namely, when it coincides with the graph H shown in Fig. 8). The only difference is the fact that in 3DSM mentioned in Theorem 2, instead of subgraphs H'_3, H'_6, H'_5, and H'_8, one considers subgraphs $H'_{3,6}$ and $H'_{5,8}$. We can easily make sure that the number of vertices that correspond to each gender in the union of the latter subgraphs is reduced by 4. Therefore, the graph G mentioned in Theorem 2 defines 3DSM of size 20.

6 Concluding remarks and open problems

In this paper, we decrease the size of an instance of 3DSM with no stable matching. The size of the initial example proposed by C.K. Lam and C.G. Plaxton equals $n = 90$; the size of the example of the same problem given by us here equals $n = 20$. We tried to get as much useful information from the construction of the superattractor (and from its modifications) as possible.

Earlier K. Pashkovich and L. Poirrier ([16]) have proved that in any 3DSM of size $n \leq 5$ there exists a stable matching. Therefore, the minimum size of 3DSM with no stable matching is not less than 6 and not greater than 20. Its exact value is not known yet. Possibly, the construction, which would allow us to build counterexamples of a small size, principally differs from that used in this paper and in [12].

B. Pittel has pointed out that the average number of stable matchings grows faster than $n^2 \ln^2(n)$ as $n \to \infty$ ([17]). Therefore, it seems natural to suppose that the percentage of unsolvable 3DSM tends to zero as $n \to \infty$. The probabilistic technique [1], which is often used for counting the number of solutions to combinatorial problems, seems to be useful in studying the mentioned problem.

Naturally, it is interesting to obtain a generalization of 2DSM to the case of three genders without such exceptions. As is well known, stable matchings always exist in 3GSM with lexicographically acyclic preferences [5]. In this case, genders are not symmetric; a hierarchy of their "importance" is assumed to be given a priori. However, we can state a symmetric variant of the problem, whose solutions also exist (and their quantity is larger than that in the non-symmetric case).

Let us consider this case in detail. Assume that the following six preference matrices are given: r_{MW} is the matrix of preferences of women among men, r_{MD} is the matrix of preferences of dogs among men; the sense of analogous denotations r_{WM}, r_{WD}, r_{DM}, and r_{DM} is evident. (Following the monograph by D. Knuth, we use matrix denotations instead of order symbols accepted in the game theory.) For denoting ranks of agents x in a 3D-matching μ, we need to modify the denotation $R_\mu(x)$ used by us earlier in this paper. Since now the agent x interacts with two partners, we introduce one more superscript that denotes the gender of the partner. Thus, if $(m, w, d) \in \mu$, then, for example, $R_{\mu,W}(m) = r_{MW}(m, w)$, analogously, $R_{\mu,D}(m) = r_{MD}(m, d)$, and so on. Let $(m, w, d) \notin \mu$. We say that a triple (m, w, d) is *pairwise blocking* for a matching μ, if

$$\begin{aligned} r_{MW}(m,w) &\le R_{\mu,W}(m), & r_{WM}(w,m) &\le R_{\mu,M}(w), \\ r_{DM}(d,m) &\le R_{\mu,M}(d), & r_{MD}(m,d) &\le R_{\mu,D}(m), \\ r_{WD}(w,d) &\le R_{\mu,D}(w), & r_{DW}(d,w) &\le R_{\mu,W}(d) \end{aligned}$$

(cf. inequalities (1)). Note that here m, w, d do not necessarily belong to three distinct triples in the matching μ, it suffices that at least one of three elements of the blocking triple "is new". One can easily prove that due to the latter condition, at least four inequalities among six ones given above are strict. We say that a matching μ is *pairwise stable*, if it contains no pairwise blocking triple.

We can easily construct such matchings in the following way. First we solve 2DSM on pairs from $M \times W$ with preference matrices r_{MW} and r_{WM}, and do 2DSM on pairs from $M \times D$ with preference matrices r_{MD} and r_{DM}. Then we "combine" solutions by transforming pairs (m', w') in the first solution and pairs (m', d') in the second one (i.e., pairs consisting of two solutions with the same elements from M) to triples (m', w', d') of the matching μ of the three-dimensional problem. One can easily see that in this case, any triple (m, w, d), which does not enter in μ, cannot be pairwise blocking, because otherwise one of the first four inequalities stated above is violated (in which case we would obtain a contradiction to the stability of the 2D matchings in either the 2DSM instance on $M \times W$ or the 2DSM instance on $M \times D$).

We can analogously define a new variant of 3DSMI as a problem with incomplete preference lists such that one of its solutions can be found in an evident way.

In the case of 2DSM, one can introduce a natural partial order on all its solutions (see [11]). J. Conway and D. Knuth have proved that this order forms a distributive lattice. Moreover, as was proved later [3,9], the correspondence between distribution lattices and stable matchings is biunique. The authors of the mentioned papers have also explicitly described a partially ordered set (POSET), whose ideals define a lattice of stable matchings. Elements of this POSET represent the so-called rotations (i.e., cycles that include alternately men and women) defined by preference matrices of 2DSM [10] (see the monograph [8] for more detail about rotations and their connection with the distributive lattice). The structure of this lattice allows one to find a stable matching with the minimum regret for this problem in quadratic time (recall [15] that the regret is defined as $\max_{(m,w)\in\mu}\max\{r_{MW}(m,w), r_{WM}(w,m)\}$) and to solve other optimization problems in the class of two-dimensional stable matchings in polynomial time.

We are interested in finding an analogous structure for the three-dimensional problem defined above. However, this is not so easy, because in this problem the existence of some two-dimensional projection of a solution to 3DSM, which solves the corresponding 2DSM, is not guaranteed.

Let us give a concrete problem instance. Let $n = 6$. Assume that with $\ell = 1, 2$ any pair (x, y) such that $r(x, y) = \ell$ satisfies the equality $r(y, x) = \ell$ ("love and strong sympathies are always mutual"). Consider a graph with 18 vertices (all agents). Let us construct in this graph edges (x, y) such that $r(x, y) = 1$. Let the resulted graph represent a union of three cycles of length 6. Moreover, if we supplement this graph with edges (x, y) such that $r(x, y) = 2$, then all of them will lie inside the connectivity component formed by cycles. Evidently, such construction is possible and (accurate to isomorphism) uniquely defines elements of ranks 1 and 2 in matrices r_{MW}, r_{MD}, r_{WM}, r_{WD}, r_{DM}, and r_{DW}). We prove that independently of the rest ranks in these matrices, this problem has a pairwise stable matching, which consists, evidently, of 6 triples and has the following property: by deleting representatives of any fixed gender from these triples one cannot get a solution to the corresponding 2DSM.

Let us explain this property in more detail. For convenience, denote elements of three cycles mentioned above as $(m_{i,0}, w_{i,0}, d_{i,0}, m_{i,1}, w_{i,1}, d_{i,1})$, $i = 1, 2, 3$. Then one can easily see that the only solution to 2DSM with matrices r_{MW}, r_{WM} takes the form $(m_{i,k}, w_{i,k})$, where $i = 1, 2, 3$ and $k = 0, 1$. In other words, the unique solution to 2DSM consists of "loving couples". Really, if some pair does not enter in the solution, then it is blocking. Analogously, the unique solution to 2DSM with matrices r_{WD}, r_{DW} takes the form $(w_{i,k}, d_{i,k})$, $i = 1, 2, 3$ and $k = 0, 1$; while the unique solution to 2DSM with matrices r_{DM}, r_{MD} takes the form $(d_{i,k}, m_{i,(k+1) \bmod 2})$, $i = 1, 2, 3$ and $k = 0, 1$.

At the same time, 3DSM under consideration has the following solution:

$$\begin{aligned}\mu = \{&(m_{1,0}, w_{1,0}, d_{1,0}), (m_{1,1}, w_{1,1}, d_{1,1}),\\ &(w_{2,0}, d_{2,0}, m_{2,1}), (w_{2,1}, d_{2,1}, m_{2,0}),\\ &(d_{3,0}, m_{3,1}, w_{3,1}), (d_{3,1}, w_{3,0}, m_{3,0})\}.\end{aligned}$$

Evidently, one cannot obtain all 6 families that enter in this solution by combining solutions of only two fixed 2DSM problems (on pairs $M \times W$ and $W \times D$ or on pairs $W \times D$ and $D \times M$ or on pairs $D \times M$ and $M \times W$).

The absence of the pairwise blocking triple for it is due to the fact that for any x in this solution $R_\mu(x) \le 2$. Consequently, the blocking triple can belong only to one connectivity component of the graph defined above. The graph of this connectivity component defines the problem of finding a pairwise stable matching of size 2 such that the obtained triple is pairwise blocking for the matching in this problem, which is defined by the separate line in the above formula. Evidently, this condition is not fulfilled for the obtained triple, because each pair of families that enters in the separate line represents a combination of solutions of two 2DSM (of size 2) and therefore is pairwise stable.

Therefore, 3DSM considered here always has a solution; however, the structure of all solutions to this problem is rather complicated. We are going to proceed studying this issue, including related optimization problems, on the solution set of considered 3DSM.

Appendix. An instance of 3DSMI of size $n = 3$ with no stable matching

Lemma 10 ([14], Theorem 2). *3DSMI with the graph G shown in Fig. 8 has no stable matching.*

Proof. There exist 7 families that form matchings in this problem, namely, $(0, 1, 5)$, $(0, 7, 8)$, $(1, 2, 3)$, $(1, 5, 3)$, $(2, 3, 4)$, $(3, 4, 5)$, and $(4, 8, 6)$.

Recall that a matching μ in 3DSMI defined by the graph G is said to be *complementable*, if there exists a triple of vertices (v, v', v'') such that $\mu(v) = v$, $\mu(v') = v'$, $\mu(v'') = v''$, and $\{(v, v'), (v', v'')(v'', v)\} \subseteq E(G)$.

Evidently, any complementable matching is not stable, the triple (v, v', v'') mentioned in the above paragraph is blocking for it. Therefore, for proving the absence of a stable matching it suffices to find blocking triples for all noncomplementable matchings. For the graph shown in Fig. 8 there exists 8 noncomplementable matchings. Below we give their complete list together with blocking triples:

1) $\{(0, 1, 5), (2, 3, 4)\}$, the blocking triple is $(4, 8, 6)$;
2) $\{(0, 1, 5), (4, 8, 6)\}$, the blocking triple is $(1, 2, 3)$;
3) $\{(0, 7, 8), (1, 2, 3)\}$, the blocking triple is $(3, 4, 5)$;
4) $\{(0, 7, 8), (1, 5, 3)\}$, the blocking triple is $(2, 3, 4)$;
5) $\{(0, 7, 8), (2, 3, 4)\}$, the blocking triple is $(0, 1, 5)$;
6) $\{(0, 7, 8), (3, 4, 5)\}$, the blocking triple is $(0, 1, 5)$;
7) $\{(1, 2, 3), (4, 8, 6)\}$, the blocking triple is $(0, 7, 8)$ or $(3, 4, 5)$;
8) $\{(1, 5, 3), (4, 8, 6)\}$, the blocking triple is $(0, 7, 8)$. □

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