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Concerning the Theory of τ-measurable Operators Affiliated to a Semifinite von Neumann Algebra, II

A. M. Bikchentaev*

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Lobachevskii Institute of Mathematics and Mechanics, Kazan (Volga Region) Federal University, Kazan, Tatarstan, 420008 Russia
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Abstract—Let a von Neumann algebra $\mathcal{M}$ of operators act on a Hilbert space $\mathcal{H}$, let τ be a faithful normal semifinite trace on $\mathcal{M}$. Let $S(\mathcal{M},\tau)$ be the *-algebra of all τ-measurable operators. Assume that $X, Y \in S(\mathcal{M},\tau)$ and $|X| = \sqrt{X^*X}$. We have (i) if $|Y| \leq |X|$, then $\ker(X) \subset \ker(Y)$; (ii) if X is left invertible with $X_l^{-1} \in \mathcal{M}$, then $\operatorname{ran}(X^*) = \mathcal{H}$. The following generalizes the C. R. Putnam theorem (1951), see also Problem 188 in the book (P. R. Halmos, A Hilbert space problem book. D. van Nostrand company, inc., London, 1967): A positive self-commutator $A^*A - AA^*$ ($A \in S(\mathcal{M},\tau)$) cannot have the inverse in $\mathcal{M}$. Let I be the unit of the algebra $\mathcal{M}$ and $\tau(I) = +\infty$, let $A, B \in S(\mathcal{M},\tau)$ and $A = A^3$. Then, the commutator $[A, B]$ cannot have the form $\lambda I + K$, where $\lambda \in \mathbb{C} \setminus \{0\}$ and an operator $K \in S(\mathcal{M},\tau)$ is τ-compact.

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1. INTRODUCTION

Let a von Neumann algebra $\mathcal{M}$ of operators act on a Hilbert space $\mathcal{H}$, let τ be a faithful normal semifinite trace on $\mathcal{M}$. Let $S(\mathcal{M},\tau)$ be the *-algebra of all τ-measurable operators. This paper continues the investigations of properties of τ-measurable operators, started in [1] and is an English translation of the Russian-language paper [2]. We obtain the following results. Assume that $X, Y \in S(\mathcal{M},\tau)$ and $|X| = \sqrt{X^*X}$. We have

(i) if $|Y| \leq |X|$, then $\ker(X) \subset \ker(Y)$;

(ii) if X is left invertible with $X_l^{-1} \in \mathcal{M}$, then $\operatorname{ran}(X^*) = \mathcal{H}$ (Theorem 1).

In Theorem 2 we present the following generalzation of C.R. Putnam theorem [3] (see also [4, Problem 188]): a positive self-commutator $A^*A - AA^*$ ($A \in S(\mathcal{M},\tau)$) cannot have the inverse in $\mathcal{M}$. The proof of Theorem 2 is new even for a *-algebra $\mathcal{M} = \mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$, endowed with the canonical trace $\tau = \operatorname{tr}$.

Let I be the unit of the algebra $\mathcal{M}$ and $\tau(I) = +\infty$, let $A, B \in S(\mathcal{M},\tau)$ and $A = A^3$. Then, the commutator $[A, B]$ cannot have a form $\lambda I + K$, where $\lambda \in \mathbb{C} \setminus \{0\}$ and an operator $K \in S(\mathcal{M},\tau)$ is τ-compact (Theorem 3). Finally, we present a new and direct proof of Corollary 3.10 from [1].

*E-mail: Airat.Bikchentaev@kpfu.ru

2. NOTATION AND DEFINITIONS

Let $\mathcal{M}^{\text{pr}}$ be the lattice of projections ($P = P^2 = P^*$) in $\mathcal{M}$ and $P^\perp = I - P$ for $P \in \mathcal{M}^{\text{pr}}$, let $\mathcal{M}^+$ be the cone of all positive operators in $\mathcal{M}$ and $||\cdot||$ be the C^*-norm on $\mathcal{M}$.

A mapping $\varphi : \mathcal{M}^+ \to [0, +\infty]$ is called *a trace*, if $\varphi(X+Y) = \varphi(X) + \varphi(Y)$, $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{M}^+$, $\lambda \geq 0$ (moreover, $0 \cdot (+\infty) \equiv 0$); $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{M}$. A trace φ is called (see [5, Chap. V, § 2])

- *faithful*, if $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$, $X \neq 0$;
- *normal*, if $X_i \uparrow X$ $(X_i, X \in \mathcal{M}^+) \Rightarrow \varphi(X) = \sup \varphi(X_i)$;
- *semifinite*, if $\varphi(X) = \sup\{\varphi(Y) : \ Y \in \mathcal{M}^+, \ Y \leq X, \ \varphi(Y) < +\infty\}$ for every $X \in \mathcal{M}^+$.

An operator on $\mathcal{H}$ (not necessarily bounded or densely defined) is said to be *affiliated to the von Neumann algebra* $\mathcal{M}$ if it commutes with any unitary operator from the commutant $\mathcal{M}'$ of the algebra $\mathcal{M}$. Let τ be a faithful normal semifinite trace on $\mathcal{M}$. A closed operator X, affiliated to $\mathcal{M}$ and possesing a domain $\mathfrak{D}(X)$ everywhere dense in $\mathcal{H}$ is said to be τ-*measurable* if, for any $\varepsilon > 0$, there exists a $P \in \mathcal{M}^{\text{pr}}$ such that $P\mathcal{H} \subset \mathfrak{D}(X)$ and $\tau(P^\perp) < \varepsilon$. The set $S(\mathcal{M}, \tau)$ of all τ-measurable operators is a *-algebra under passage to the adjoint operator, multiplication by a scalar, and operations of strong addition and multiplication resulting from the closure of the ordinary operations [6, Chap. IX]. Let $\mathcal{L}^+$ and $\mathcal{L}^{\text{h}}$ denote the positive and Hermitian parts of a family $\mathcal{L} \subset S(\mathcal{M}, \tau)$, respectively. We denote by $\leq$ the partial order in $S(\mathcal{M}, \tau)^{\text{h}}$ generated by its proper cone $S(\mathcal{M}, \tau)^+$. If $X \in S(\mathcal{M}, \tau)$ and $X = U|X|$ is the polar decomposition of X, then $U \in \mathcal{M}$ and $|X| = \sqrt{X^*X} \in S(\mathcal{M}, \tau)^+$.

The generalized singular value function $\mu(X) : t \to \mu(t; X)$ of the operator X is defined by setting

$$\mu(t; X) = \inf\{||XP|| : \ P \in \mathcal{M}^{\text{pr}} \text{ and } \tau(P^\perp) \leq t\}, \quad t > 0.$$

Lemma 1. ([7]). *Let $X, Y \in S(\mathcal{M}, \tau)$. Then,*

(i) $\mu(t; X) = \mu(t; |X|) = \mu(t; X^*)$ *for all* $t > 0$;

(ii) $\mu(t; \lambda X) = |\lambda|\mu(t; X)$ *for all* $\lambda \in \mathbb{C}$ *and* $t > 0$;

(iii) *if* $|X| \leq |Y|$, *then* $\mu(t; X) \leq \mu(t; Y)$ *for all* $t > 0$;

(iv) $\mu(t; |X|^\alpha) = \mu(t; X)^\alpha$ *for all* $\alpha > 0$ *and* $t > 0$;

(v) $X \in \mathcal{M} \Leftrightarrow \sup\limits_{t>0} \mu(t; X) < +\infty$; *moreover*, $\lim\limits_{t\to+0} \mu(t; X) = \sup\limits_{t>0} \mu(t; X) = ||X||$.

Let $S_0(\mathcal{M}, \tau) = \{A \in S(\mathcal{M}, \tau) : \ \lim\limits_{t\to+\infty} \mu(t; A) = 0\}$ be the *-ideal of all τ-compact operators in $S(\mathcal{M}, \tau)$. An operator $A \in S(\mathcal{M}, \tau)$ is called *hyponormal*, if $A^*A \geq AA^*$; *cohyponormal*, if A^* is hyponormal. An operator $X \in S(\mathcal{M}, \tau)$ is called a *commutator*, if $X = [A, B] = AB - BA$ for some $A, B \in S(\mathcal{M}, \tau)$. The *selfcommutator* of an operator $A \in S(\mathcal{M}, \tau)$ is the operator $[A^*, A] = A^*A - AA^*$.

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, the *-algebra of all bounded linear operators on $\mathcal{H}$, and $\tau = \text{tr}$ is the canonical trace, then $S(\mathcal{M}, \tau)$ coincides with $\mathcal{B}(\mathcal{H})$, $S_0(\mathcal{M}, \tau)$ coincides with the *-ideal of all compact operators in $\mathcal{B}(\mathcal{H})$ and

$$\mu(t; X) = \sum_{n=1}^{\infty} s_n(X)\chi_{[n-1,n)}(t), \quad t > 0,$$

where $\{s_n(X)\}_{n=1}^\infty$ is the sequence of s-numbers of the operator X; χ_A is the indicator function of the set $A \subset \mathbb{R}$.

If $\mathcal{M}$ is Abelian (i.e., commutative), then $\mathcal{M} \simeq L^\infty(\Omega, \Sigma, \nu)$ and $\tau(f) = \int_\Omega f \mathrm{d}\nu$, where (Ω, Σ, ν) is a localized measure space, the *-algebra $S(\mathcal{M}, \tau)$ coincides with the algebra of all complex measurable functions f on (Ω, Σ, ν), which are bounded everywhere, but for a set of finite measure. The function $\mu(t; f)$ coincides with the nonincreasing rearrangement of the function $|f|$; see properties of such rearrangements in [8].

3. THE MAIN RESULTS

Theorem 1. *Assume that* $X, Y \in S(\mathcal{M}, \tau)$. *We have*

(i) *if* $|Y| \le |X|$, *then* $\ker(X) \subset \ker(Y)$;

(ii) *if* X *is left invertible with* $X_l^{-1} \in \mathcal{M}$, *then* $\operatorname{ran}(X^*) = \mathcal{H}$.

Proof. (i) Since $\ker(Z) = \ker(|Z|)$ for all $Z \in S(\mathcal{M}, \tau)$, we may assume that $0 \le Y \le X$. Then, there exists an operator $A \in \mathcal{M}$ with $||A|| \le 1$ such that $Y^{1/2} = AX^{1/2}$ [9, Proposition on p. 261]. Therefore,

$$\ker(X^{1/2}) \subset \ker(Y^{1/2}). \qquad (1)$$

Note that $X = X^{1/2} \cdot X^{1/2}$ and

$$\ker(X^{1/2}) \subset \ker(X),$$

$\mathfrak{D}(X) \subset \mathfrak{D}(X^{1/2})$. If $\xi \in \mathfrak{D}(X)$ and $X\xi = 0$, then $\langle X\xi, \xi\rangle = ||X^{1/2}\xi||^2 = 0$ and

$$\ker(X^{1/2}) \subset \ker(X).$$

Hence $\ker(X^{1/2}) = \ker(X)$ and the assertion follows from (1).

(ii) Let us show that for every vector $\xi \in \mathcal{H}$ there exists a vector $\eta \in \mathcal{H}$ such that $X^*\eta = \xi$. For an arbitrary vector $\zeta \in \mathfrak{D}(X)$ there exists a vector $h \in \operatorname{ran}(X)$ such that $X\zeta = h$ and $\zeta = X_l^{-1}h$. A linear functional $\varphi(h) = \langle \xi, \zeta\rangle = \langle \xi, X_l^{-1}h\rangle$ is bounded on $\operatorname{ran}(X)$, since

$$|\varphi(h)| \le ||\xi||\, ||X_l^{-1}h|| \le ||X_l^{-1}||\, ||\xi||\, ||h||.$$

Let us check that a linear set $\operatorname{ran}(X)$ is closed in $\mathcal{H}$. Let a sequence $\{\psi_n\}_{n=1}^{\infty} \subset \mathfrak{D}(X)$ be such that $\{X\psi_n\}_{n=1}^{\infty}$ is a $||\cdot||$-Cauchy sequence in $\operatorname{ran}(X)$. Then, $X\psi_n \to f \in \mathcal{H}$ as $n \to \infty$. Since $X_l^{-1}(X\psi_n) = \psi_n$ for all $n \in \mathbb{N}$, $\{\psi_n\}_{n=1}^{\infty}$ is a $||\cdot||$-Cauchy sequence. There exists a vector $\psi \in \mathcal{H}$ such that $\psi_n \to \psi$ as $n \to \infty$. Therefore, $\psi_n \to \psi$ and $X\psi_n \to f$ as $n \to \infty$. Since the graph $\Gamma(X) = \{(g, Xg) : g \in \mathfrak{D}(X)\}$ is $||\cdot||$-closed in $\mathcal{H} \times \mathcal{H}$, we have $f \in \mathfrak{D}(X)$ and $f = X\psi$. By Riesz theorem on representation of linear functionals, there exists the unique vector $\eta \in \mathcal{H}$ such that $\varphi(h) = \langle \eta, h\rangle$, i. e., $\langle \xi, \zeta\rangle = \langle \eta, X\zeta\rangle$ for all $\zeta \in \mathfrak{D}(X)$. Thus $\eta \in \mathfrak{D}(X^*)$ and $X^*\eta = \xi$. Theorem is proved. □

Corollary 1. *If an operator* $X \in S(\mathcal{M}, \tau)$ *is hyponormal, then* $\ker(X) \subseteq \ker(X^*)$.

Proof. Let $A \in S(\mathcal{M}, \tau)^+$, numbers $0 < \alpha < \beta$ and a vector $\xi \in \mathcal{H}$. By relations

$$A^{\alpha}\xi = 0 \quad \Rightarrow \quad A^{\beta}\xi = A^{\beta-\alpha}(A^{\alpha}\xi) = 0;$$
$$A\xi = 0 \quad \Rightarrow \quad 0 = \langle A\xi, \xi\rangle = \langle A^{1/2}\xi, A^{1/2}\xi\rangle = ||A^{1/2}\xi||^2$$

follows $\ker(A^q) = \ker(A)$ for all $q > 0$. Hence

$$\ker(X) = \ker(|X|) = \ker(|X|^2) \subseteq \ker(|X^*|^2) = \ker(|X^*|) = \ker(X^*).$$

□

If a cohyponormal operator $X \in S(\mathcal{M}, \tau)$ has a left inverse in the *-algebra $S(\mathcal{M}, \tau)$, then X is invertible in $S(\mathcal{M}, \tau)$ [10, Corollary 11]. Moreover, if $X_l^{-1} \in \mathcal{M}$, then X is invertible in $\mathcal{M}$, i.e., there exists the operator $X^{-1} \in \mathcal{M}$, see the proof of Theorem 2 in [10].

The following assertion generalizes the classical Putnam theorem for bounded hyponormal operator [3] (see also [4, Problem 188]) to a case of τ-measurable unbounded hyponormal operator.

Theorem 2. *A positive self-commutator* $A^*A - AA^*$ $(A \in S(\mathcal{M}, \tau))$ *cannot have an inverse in* $\mathcal{M}$.

Proof. If $\tau(I) < +\infty$, then every hyponormal (or cohyponormal) operator $A \in S(\mathcal{M}, \tau)$ is normal, i.e. $A^*A - AA^* = 0$, see [11, Corollary 2.6]. Let as consider the case of $\tau(I) = +\infty$. Let, on the contrary, an operator $A \in S(\mathcal{M}, \tau)$ possess the inverse in $\mathcal{M}$. Then, $A^*A - AA^* \ge \varepsilon I$ for some number $\varepsilon > 0$. For every number $t > 0$ we have

$$\mu(t; AA^*) = \mu(t; |A^*|^2) = \mu(t; |A^*|)^2 = \mu(t; |A|)^2 = \mu(t; |A|^2) = \mu(t; A^*A)$$
$$\ge \mu(t; AA^* + \varepsilon I) \ge \mu(t; AA^*)$$

by items (iv) and (iii) of Lemma 1. Therefore,

$$\mu(t; AA^* + \varepsilon I) = \mu(t; AA^*) \quad \text{for all} \quad t > 0.$$

On the other hand, since $AA^* \neq 0$ and $\varepsilon I \geq \lim\limits_{t\to+\infty} \mu(t; \varepsilon I) \cdot I = \varepsilon I$, by [12, Proposition 2.2] there exists a number $s > 0$ such that

$$\mu(s; AA^*) < \mu(s; AA^* + \varepsilon I).$$

We have a contradiction. □

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, $\tau = \mathrm{tr}$, the space $\mathcal{H}$ is separable and $\dim \mathcal{H} = +\infty$, then an operator $X \in \mathcal{M}$ is a commutator $\Leftrightarrow X$ is non-representable as a sum $\lambda I + K$, where $\lambda \in \mathbb{C} \setminus \{0\}$ and an operator $K \in \mathcal{M}$ is compact [13, Theorem 3] and [4, Corollary from Problem 182].

Theorem 3. *Let $\tau(I) = +\infty$, operators $A, B \in S(\mathcal{M}, \tau)$ and $A = A^3$. Then, the commutator $[A, B]$ cannot have a form $\lambda I + K$, where $\lambda \in \mathbb{C} \setminus \{0\}$ and an operator $K \in S_0(\mathcal{M}, \tau)$.*

Proof. Assume that

$$AB - BA = \lambda I + K \tag{2}$$

with some $\lambda \in \mathbb{C} \setminus \{0\}$ and $K \in S_0(\mathcal{M}, \tau)$. Then, $A, B \notin S_0(\mathcal{M}, \tau)$. Multiply both sides of equality (2) from the right by the operator A and obtain

$$ABA = \lambda A + BA^2 + KA. \tag{3}$$

Multiply both sides of equality (2) from the left by operator A, and achieve

$$ABA = -\lambda A + A^2 B - AK. \tag{4}$$

Subtract term by term (4) from (3), and obtain

$$2\lambda A = A^2 B - BA^2 - KA - AK. \tag{5}$$

Let $A = P - Q$ be the representation of the tripotent A [14, Proposition 1] with $P = P^2$, $Q = Q^2$ from $S(\mathcal{M}, \tau)$ and $PQ = QP = 0$. Then, $A^2 = P + Q$ is an idempotent and (5) can be rewritten in the form

$$2\lambda(P - Q) = (P + Q)B - B(P + Q) - K(P - Q) - (P - Q)K. \tag{6}$$

Multiply both sides of equality (6) from the left and the right by the idempotent P and obtain $2\lambda P = -2PKP \in S_0(\mathcal{M}, \tau)$, i. e., $P \in S_0(\mathcal{M}, \tau)$. Multiply both sides of equality (6) from the left and the right by the idempotent Q and conclude that $-2\lambda Q = 2QKQ \in S_0(\mathcal{M}, \tau)$, i.e., $Q \in S_0(\mathcal{M}, \tau)$. Therefore, $A = P - Q \in S_0(\mathcal{M}, \tau)$. We have a contradiction. □

Recall Corollary 3.10 of [1]:

"*Let an operator $A \in S(\mathcal{M}, \tau)$ and $A = A^2$. If A is hyponormal or cohyponormal, then A is normal, hence $A \in \mathcal{M}^{pr}$*".

Let us present a new and direct proof of this assertion. Note that, if $A \in S(\mathcal{M}, \tau)$, then $A = A^2$ if and only if $A = |A^*|\,|A|$, see [1, Theorem 3.3]. For a hyponormal operator A we have

$$|A|^2 = A^*A = |A|\,|A^*| \cdot |A^*|\,|A| = |A| \cdot AA^* \cdot |A| \leq |A| \cdot A^*A \cdot |A| = |A| \cdot |A|^2 \cdot |A| = |A|^4$$

and $|A|^2 \leq |A|^4$. Hence $|A| \leq |A|^2$ and for all numbers $t > 0$

$$\mu(t; A) = \mu(t; |A|) \leq \mu(t; |A|^2) = \mu(t; |A|)^2 = \mu(t; A)^2$$

by items (i) and (iv) of Lemma 1. Therefore, $\mu(t; A) \in \{0\} \bigcup [1, +\infty)$ for all $t > 0$. On the other hand,

$$|A^*|^2 = AA^* = |A^*|\,|A| \cdot |A|\,|A^*| = |A^*| \cdot A^*A \cdot |A^*| \geq |A^*| \cdot AA^* \cdot |A^*| = |A^*| \cdot |A^*|^2 \cdot |A^*| = |A^*|^4$$

and $|A^*|^2 \geq |A^*|^4$. Hence $|A^*| \geq |A^*|^2$ and for all numbers $t > 0$

$$\mu(t; A) = \mu(t; A^*) = \mu(t; |A^*|) \geq \mu(t; |A^*|^2) = \mu(t; |A^*|)^2 = \mu(t; A^*)^2 = \mu(t; A)^2$$

by items (i) and (iv) of Lemma 1. Therefore, $\mu(t; A) \in [0, 1]$ for all $t > 0$.

Thus, $\mu(t; A) \in \{0, 1\}$ for all $t > 0$ and by item (v) of Lemma 1 we obtain $||A|| \leq 1$. Therefore, either $A = 0$, or $||A|| = 1$. Hence $A \in \mathcal{M}^{\mathrm{pr}}$.

For a cohyponormal operator $A = A^2$ we note that the operator $A^* = (A^*)^2$ is hyponormal.

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