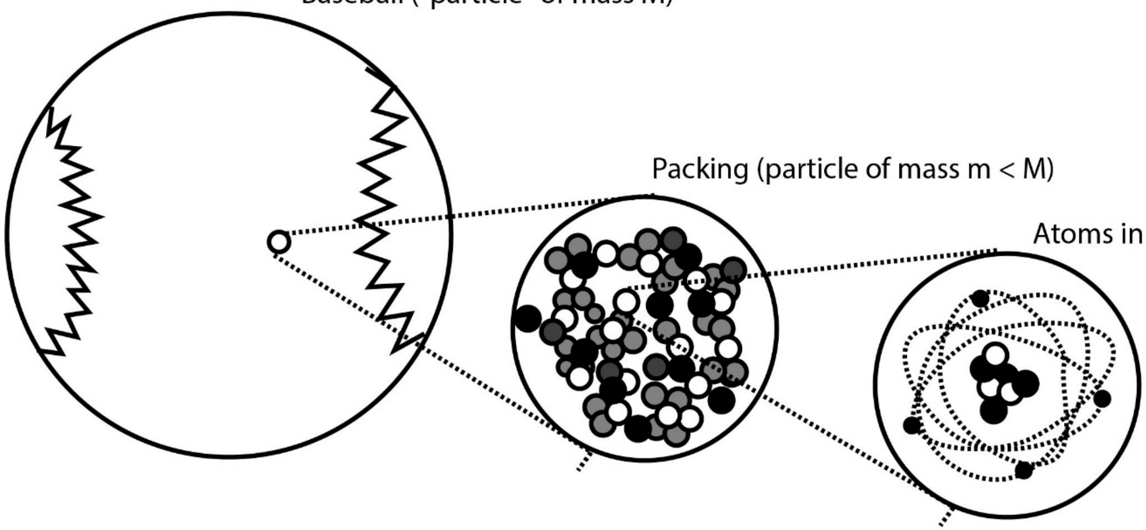
Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Systems of Particles



An object such as a baseball is not really a particle. It is made of *many*, *many* particles – even the atoms it is made of are made of many particles *each*. Yet it *behaves* like a particle as far as Newton's Laws are concerned.

We will obtain this collective behavior by **averaging**, or *summing* over at successively larger scales, the physics that we know applies at the smallest scale to things that *really* are particles.

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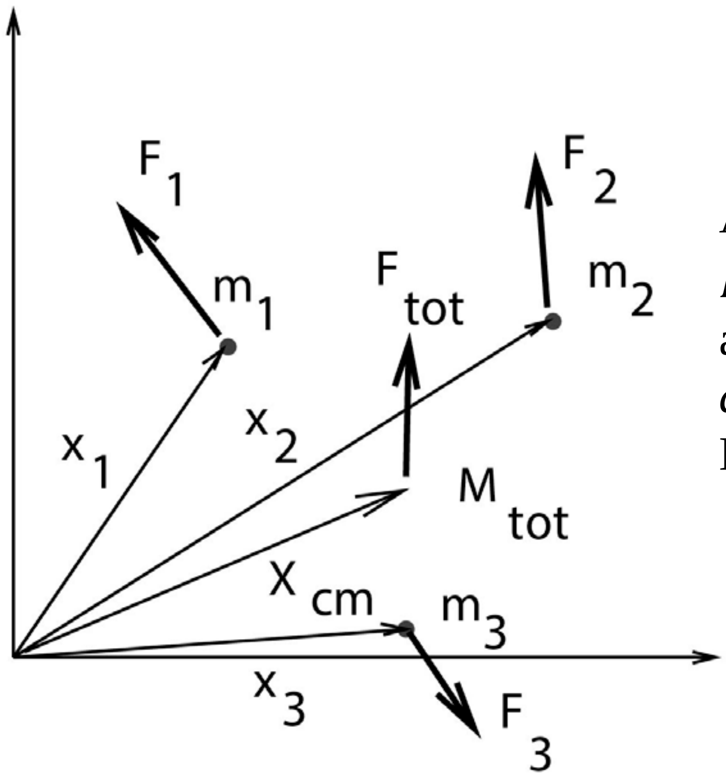
Newton's Laws for a System of Particles – Center of Mass



A system of N = 3 particles is shown, with various forces $\vec{F}_i$ acting on the masses (which therefore each their own accelerations $\vec{a}_i$). From this, we construct a *weighted average* acceleration of the system, in such a way that Newton's Second Law is satisfied for the *total* mass.

Suppose we have a system of N particles, each of which is experiencing a force. Some (part) of those forces are "external" – they come from outside of the system. Some (part) of them may be "internal" – equal and opposite force pairs between particles that help hold the system together (solid) or allow its component parts to interact (liquid or gas).

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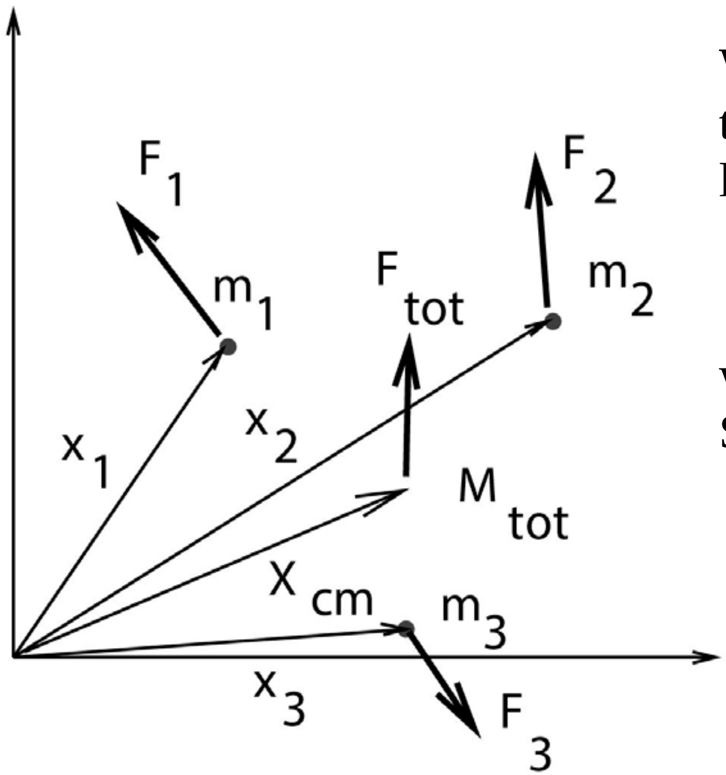
Newton's Laws for a System of Particles – Center of Mass



We would like the total force to act on the total mass of this system as if it were a "particle". That is, we would like for:

$$\vec{F}_{tot} = M_{tot}\vec{A}$$

where $\vec{A}$ is the "acceleration of the system". Newton's Second Law for a system of particles is written as:

$$\vec{F}_{tot} = \sum_i \vec{F}_i = \sum_i m_i \frac{d^2\vec{x}_i}{dt^2} =$$

$$= \left(\sum_i m_i\right)\frac{d^2\vec{X}}{dt^2} = M_{tot}\frac{d^2\vec{X}}{dt^2} = M_{tot}\vec{A}$$

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Newton's Laws for a System of Particles – Center of Mass

$$\sum_i m_i \frac{d^2\vec{x}_i}{dt^2} = M_{tot}\frac{d^2\vec{X}}{dt^2}$$

Basically, if we define an $\vec{X}$ such that this relation is true then Newton's second law is recovered for the entire system of particles "located at $\vec{X}$" as if that location were indeed a particle of mass M_{tot} itself. We can rearrange this a bit as:

$$\frac{d\vec{V}}{dt} = \frac{d^2\vec{X}}{dt^2} = \frac{1}{M_{tot}}\sum_i m_i \frac{d^2\vec{x}_i}{dt^2} = \frac{1}{M_{tot}}\sum_i m_i \frac{d\vec{v}_i}{dt}$$

and can integrate twice on both sides. The first integral is:

$$\frac{d\vec{X}}{dt} = \vec{V} = \frac{1}{M_{tot}}\sum_i m_i\vec{v}_i + \vec{V}_0 = \frac{1}{M_{tot}}\sum_i m_i\frac{d\vec{x}_i}{dt} + \vec{V}_0$$

and the second is: $\vec{X} = \frac{1}{M_{tot}}\sum_i m_i\vec{x}_i + \vec{V}_0 t + \vec{X}_0$

Newton’s Laws for a System of Particles – Center of Mass

We define the position of **the center of mass** to be:

$$M\vec{X}_{\text{cm}} = \sum_i m_i \vec{x}_i \qquad \text{or } \vec{X}_{\text{cm}} = \frac{1}{M}\sum_i m_i \vec{x}_i$$

Not all systems we treat will appear to be made up of point particles. Most solid objects or fluids appear to be made up of a *continuum* of mass, a *mass distribution*. In this case we need to do the sum by means of *integration*, and our definition becomes:

$$M\vec{X}_{\text{cm}} = \int \vec{x}\,dm \qquad \text{or} \qquad \vec{X}_{\text{cm}} = \frac{1}{M}\int \vec{x}\,dm$$

Momentum

Momentum is a useful idea that follows naturally from our decision to treat collections as objects. It is a way of combining the mass (which is a characteristic of the object) with the velocity of the object. We define **the momentum** to be:

$$\vec{p} = m\vec{v}$$

Thus (since the mass of an object is generally constant):

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d}{dt}(m\vec{v}) = \frac{d\vec{p}}{dt}$$

is another way of writing Newton's second law.

Note that there exist systems (like rocket ships, cars, etc.) where the mass is **not** constant. As the rocket rises, its thrust (the force exerted by its exhaust) can be constant, but it continually gets lighter as it burns fuel. Newton's second law (expressed as $\vec{F} = m\vec{a}$) **does** tell us what to do in this case – but only if we treat each little bit of burned and exhausted gas as a "particle", which is a pain. On the other hand, Newton's second law expressed as $\vec{F} = \frac{d\vec{p}}{dt}$ still works fine and makes perfect sense – it simultaneously describes the loss of mass and the increase of velocity as a function of the mass correctly.

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Momentum

Clearly we can repeat our previous argument for the sum of the momenta of a collection of particles:

$$\vec{P}_{tot} = \sum_i \vec{p}_i = \sum_i m\vec{v}_i$$

so that

$$\frac{d\vec{P}_{tot}}{dt} = \sum_i \frac{\vec{p}_i}{dt} = \sum_i \vec{F}_i = \vec{F}_{tot}$$

Differentiating our expression for the position of the center of mass above, we also get:

$$\frac{d\sum_i m\vec{x}_i}{dt} = \sum_i m\frac{d\vec{x}_i}{dt} = \sum_i \vec{p}_i = \vec{P}_{tot} = M_{tot}\vec{v}_{cm}$$

The Law of Conservation of Momentum

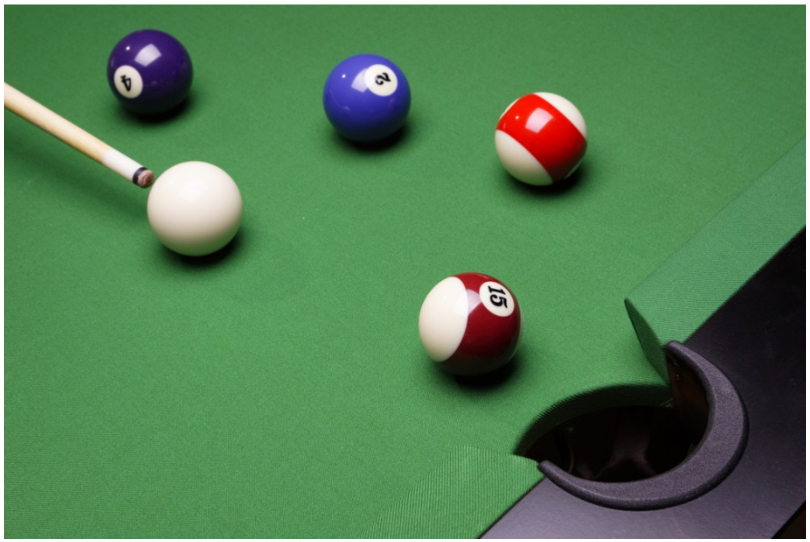
We are now in a position to state and trivially prove the Law of Conservation of Momentum.

> ***If and only if*** the total external force acting on a system is zero, ***then*** the total momentum of a system (of particles) is a constant vector.

You are welcome to learn this in its more succinct algebraic form:

> If and only if $\vec{F}_{tot} = 0$ then $\vec{P}_{tot} = \vec{P}_{initial} = \vec{P}_{final}$ = a constant vector.

Impulse



Let us imagine a typical collision: one pool ball approaches and strikes another, causing both balls to recoil from the collision in some (probably different) directions and at different speeds. Before they collide, they are widely separated and exert no force on one another.

As the surfaces of the two (hard) balls come into contact, they “suddenly” exert relatively large, relatively violent, equal and opposite forces on each other over a relatively short time, and then the force between the objects once again drops to zero as they either bounce apart or stick together and move with a common velocity.

“Relatively” here in all cases means ***compared to all other forces acting on the system during the collision*** in the event that those forces are not actually zero.

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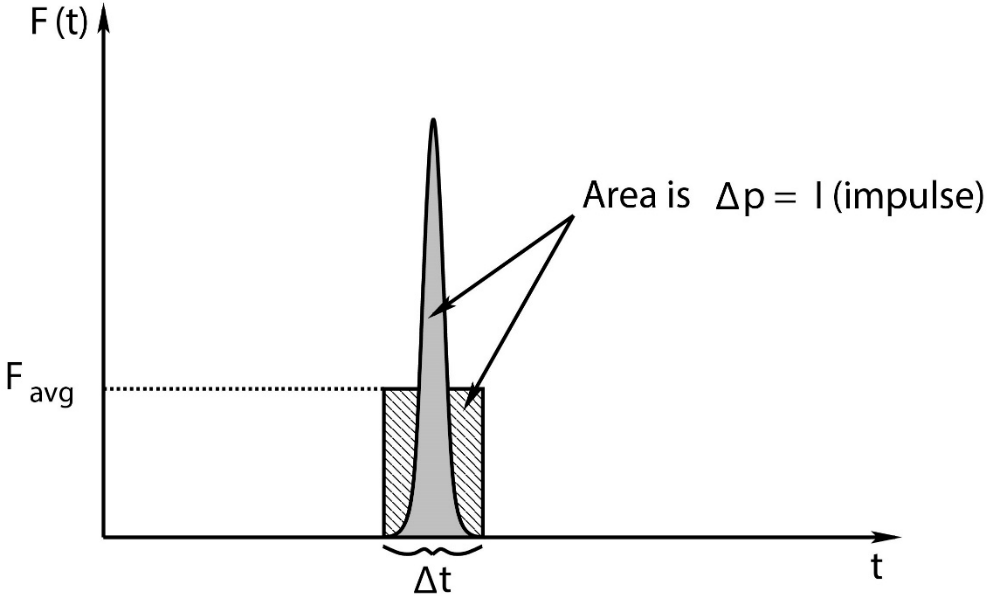
Impulse

Let us begin, then, by defining the average force over the (short) time *Δt* of any given collision, assuming that we *did* know $\vec{F} = \vec{F}_{21}(t)$, the force one object (say m_1) exerts on the other object (m_2).

The magnitude of such a force (one perhaps appropriate to the collision of pool balls) is sketched below in figure where for simplicity we assume that the force acts only along the line of contact and is hence effectively one dimensional in this direction.



The time average of this force is computed the same way the time average of any other timedependent quantity might be:

Impulse

The time average of this force is computed the same way the time average of any other time-dependent quantity might be:

$$\vec{F}_{avg} = \frac{1}{\Delta t}\int_0^{\Delta t} \vec{F}(t)dt$$

We can evaluate the integral using Newton's Second Law expressed in terms of momentum:

$$\vec{F}(t) = \frac{d\vec{p}}{dt}$$

so that (multiplying out by *dt* and integrating):

$$\vec{p}_{2f} - \vec{p}_{2i} = \Delta\vec{p}_2 = \int_0^{\Delta t} \vec{F}(t)dt$$

Note that the momentum change of the first ball is equal and opposite. From Newton's Third Law, $\vec{F}_{12}(t) = -\vec{F}_{21}(t) = \vec{F}$ and:

$$\vec{p}_{1f} - \vec{p}_{1i} = \Delta\vec{p}_1 = -\int_0^{\Delta t} \vec{F}(t)dt = -\Delta\vec{p}_2$$

Impulse

The integral of a force $\vec{F}$ over an interval of time is called the ***impulse*** imparted by the force

$$\vec{I} = \int_{t_1}^{t_2} \vec{F}(t)dt = \int_{t_1}^{t_2} \frac{d\vec{p}}{dt}(t)dt = \int_{p_1}^{p_2} d\vec{p} = \vec{p}_2 - \vec{p}_1 = \Delta\vec{p}$$

This proves that the (vector) impulse is equal to the (vector) change in momentum over the same time interval, a result known as the ***impulse-momentum theorem***. From our point of view, the impulse is just the momentum transferred between two objects in a collision in such a way that the *total* momentum of the two is unchanged.

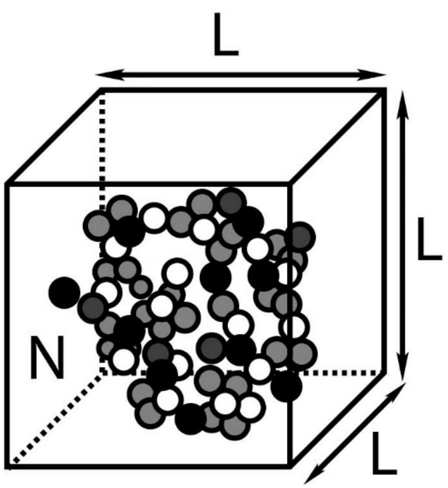
Returning to the average force, we see that the average force in terms of the impulse is just:

$$\vec{F}_{avg} = \frac{\vec{I}}{\Delta t} = \frac{\Delta p}{\Delta t} = \frac{\vec{p}_f - \vec{p}_i}{\Delta t}$$

Impulse, Fluids, and Pressure

Another valuable use of impulse is when we have many objects colliding with something – so many that even though each collision takes only a short time Δt, there are so many collisions that they exert a nearly continuous force on the object.

This is critical to understanding the notion of pressure exerted by a fluid, because microscopically the fluid is just a lot of very small particles that are constantly colliding with a surface and thereby transferring momentum to it, so many that they exert a nearly continuous and smooth force on it that is the average force exerted per particle times the number of particles that collide.



Suppose you have a cube with sides of length L containing N molecules of a gas.

Impulse, Fluids, and Pressure

Let's suppose that all of the molecules have a mass *m* and an average speed in the *x* direction of v_x, with (on average) one half going left and one half going right at any given time.

In order to be in equilibrium (so v_x doesn't change) the change in momentum of any molecule that hits, say, the right hand wall perpendicular to *x* is $\Delta p_x = 2mv_x$. This is the *impulse* transmitted to the wall per molecular collision. To find the total impulse in the time Δt, one must multiply this by one half the number of molecules in in a volume $L^2 v_x \Delta t$. That is,

$$\Delta p_{tot} = \frac{1}{2}\left(\frac{N}{L^3}\right) L^2 v_x \Delta t (2mv_x)$$

Let's call the volume of the box $L^3 = V$ and the area of the wall receiving the impulse $L^2 = A$.

$$P = \frac{F_{avg}}{A} = \frac{\Delta p_{tot}}{A\Delta t} = \left(\frac{N}{V}\right)\left(\frac{1}{2}mv_x^2\right) = \left(\frac{N}{V}\right) K_{x,avg}$$

where the average force per unit area applied to the wall is the ***pressure***, which has SI units of Newtons/meter^2 or ***Pascals***.

Impulse, Fluids, and Pressure

If we add a result called the ***equipartition theorem***:

$$K_{x,avg} = \frac{1}{2} m v_x^2 = \frac{1}{2} k_b T^2$$

$$\Delta p_{tot} = \frac{1}{2}\left(\frac{N}{L^3}\right) L^2 v_x \Delta t (2 m v_x)$$

where k_b is Boltzmann’s constant and T is the temperature in degrees absolute, one gets:

$$PV = NkT$$

which is the **Ideal Gas Law**.

Collisions

A “collision” in physics occurs when two bodies that are more or less not interacting (because they are too far apart to interact) come “in range” of their mutual interaction force, strongly interact for a short time, and then separate so that they are once again too far apart to interact.

There are three general “types” of collision:

- Elastic
- Fully Inelastic
- Partially Inelastic

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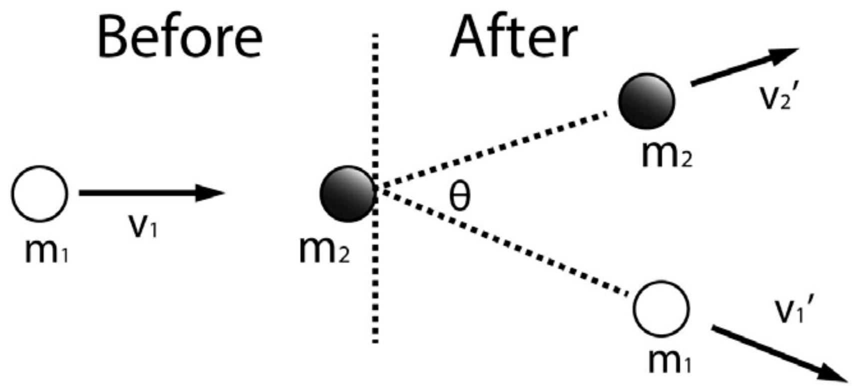
Elastic collision

By definition, an ***elastic collision*** is one that ***also*** conserves ***total kinetic energy*** so that the total scalar kinetic energy of the colliding particles before the collision must equal the total kinetic energy after the collision. This is an additional independent equation that the solution must satisfy.



General relationships:

1. Conservation of momentum $\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$
2. Conservation of kinetic energy: $\frac{1}{2}m_1\vec{v}_{1i}^2 + \frac{1}{2}m_2\vec{v}_{2i}^2 = \frac{1}{2}m_1\vec{v}_f^{2\prime} + \frac{1}{2}m_2\vec{v}_{2f}^2$
3. For head-on collisions: $v_1' = \frac{(m_1-m_2)}{(m_1-m_2)}v_1$; $v_2' = \frac{2m_1}{(m_1+m_2)}v_1$
4. For head-on collisions the velocity of approach is equal to the velocity of separation

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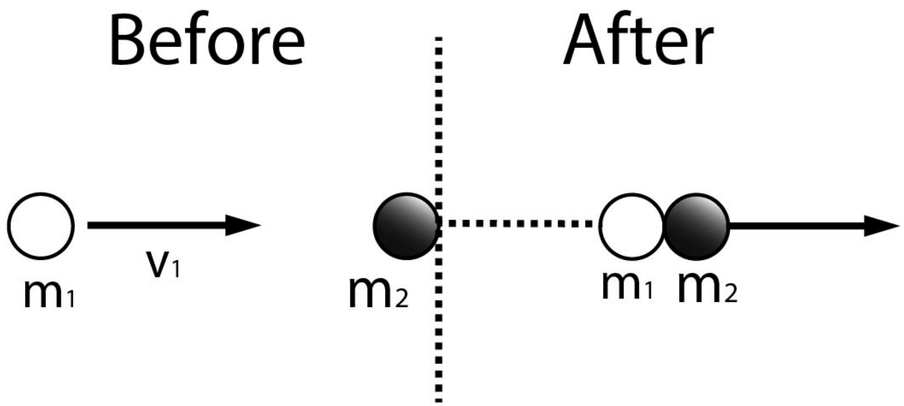
Inelastic collision

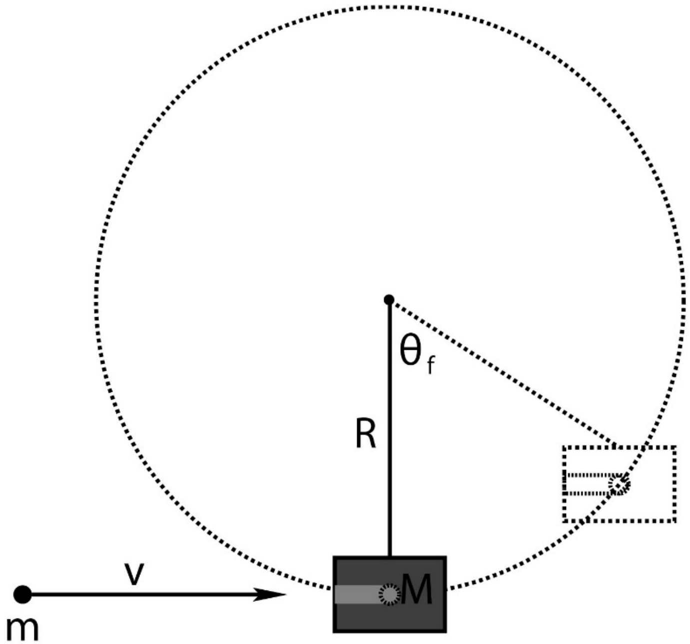
A fully inelastic collision is where two particles collide and *stick* together. As always, momentum is conserved in the impact approximation, but now kinetic energy is not!



$$\vec{p}_{i,n\ tot} = m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = (m_1 + m_2)\vec{v}_f = (m_1 + m_2)\vec{v}_{cm} = \vec{p}_{f,tot}$$

In other words, in a fully inelastic collision, the velocity of the outgoing combined particle is the velocity of the center of mass of the system, which we can easily compute from a knowledge of the initial momenta or velocities and masses.

Example: Ballistic Pendulum



The "ballistic pendulum", where a bullet strikes and sticks to/in a block, which then swings up to a maximum angle θ_f before stopping and swinging back down.

The classic ballistic pendulum question gives you the mass of the block M, the mass of the bullet m, the length of a string or rod suspending the "target" block from a free pivot, and the initial velocity of the bullet v_0. It then asks for the maximum angle θ_f through which the pendulum swings after the bullet hits and sticks to the block (or alternatively, the maximum height H through which it swings).

Solution:

During the collision momentum is conserved in the impact approximation, which in this case basically implies that the block has no time to swing up appreciably "during" the actual collision.

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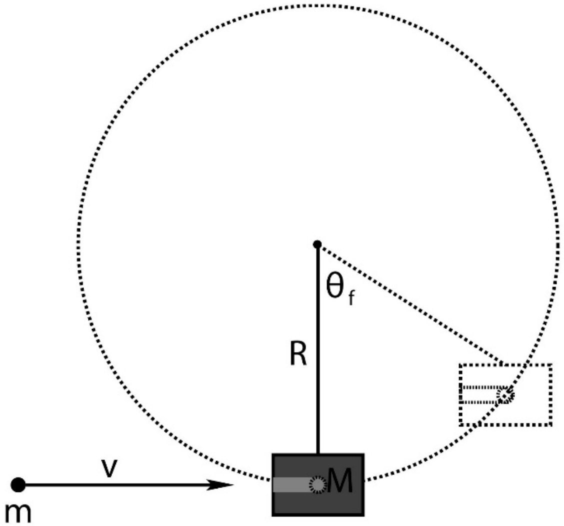
Example: Ballistic Pendulum



Solution:

- **During the collision momentum is conserved** in the impact approximation, which in this case basically implies that the block has no time to swing up appreciably "during" the actual collision.
- **After the collision mechanical energy is conserved**. Mechanical energy is *not* conserved during the collision (see solution above of straight up inelastic collision).

Momentum conservation: $p_{m,0} = mv_0 = p_{M+m,f}$

kinetic part of mechanical energy conservation in terms of momentum:

$$E_0 = \frac{p_{B+b,f}^2}{2(M+m)} = \frac{p_{b,0}^2}{2(M+m)} = E_f = (M+m)gH = (M+m)gR(1-\cos\theta_f)$$

Thus: $\theta_f = \cos^{-1}(1 - \frac{(mv_0)^2}{2(M+m)^2 gR})$ which only has a solution if mv_0 is less than some maximum value.

Torque and Rotation

Rotations in One Dimension are rotations of a solid object about a *single* axis. Since we are free to choose any arbitrary coordinate system we wish in a problem, we can without loss of generality select a coordinate system where the *z*-axis represents the (positive or negative) direction or rotation, so that the rotating object rotates "in" the *xy* plane. Rotations of a rigid body in the *xy* plane can then be described by a single angle θ, measured by convention in the counterclockwise direction from the positive *x*-axis.

Time-dependent Rotations can thus be described by:

a) The ***angular position*** as a function of time, $\theta(t)$.

b) The ***angular velocity*** as a function of time,

$$w(t) = \frac{d\theta}{dt}$$

c) The ***angular acceleration*** as a function of time,

$$\alpha(t) = \frac{dw}{dt} = \frac{d^2\theta}{dt^2}$$

Torque and Rotation

- Forces applied to a rigid object perpendicular to a line drawn from an ***axis of rotation*** exert a ***torque*** on the object. The torque is given by:

$$\tau = rF\sin(\varphi) = rF_{\perp} = r_{\perp}F$$

- The torque (as we shall see) is a *vector* quantity and by convention its direction is *perpendicular* to the plane containing $\vec{r}$ and $\vec{F}$ in the direction given by ***the right hand rule***. Although we won't really work with this until next week, the "proper" definition of the torque is:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

- ***Newton's Second Law for Rotation*** in one dimension is:

$$\tau = I\alpha$$

where I is the moment of inertia of the rigid body being rotated by the torque about a given/specified axis of rotation. The direction of this (one dimensional) rotation is the righthanded direction of the axis – the direction your right handed thumb points if you grasp the axis with your fingers curling around the axis in the direction of the rotation or torque.

Torque and Rotation

- The ***moment of inertia of a point particle*** of mass m located a (fixed) distance r from some axis of rotation is:

$$I = mr^2$$

- The moment of inertia of a rigid collection of point particles is:

$$I = \sum_i m_i r_i^2$$

- The moment of inertia of a continuous solid rigid object is:

$$I = \int r^2 dm$$

- The rotational kinetic energy of a rigid body (total kinetic energy of all of the chunks of mass that make it up) is:

$$K_{rot} = \frac{1}{2} I w^2$$

Conditions for Static Equilibrium

An object at rest remains at rest unless acted on by a net external force.

Previously we showed that *Newton's Second Law* also applies to *systems of particles*, with the replacement of the position of the particle by the position of the *center of mass* of the system and the force with the total external force acting on the entire system.

We also learned that the *force equilibrium* of particles acted on by conservative force occurred at the points where the potential energy was maximum or minimum or neutral (flat), where we named maxima "unstable equilibrium points", minima "stable equilibrium points" and flat regions "neutral equilibria".

However, we learned enough to now be able to see that force equilibrium *alone* is ***not sufficient*** to cause an extended object or collection of particles to be in equilibrium. We can easily arrange situations where *two* forces act on an object in opposite directions (so there is no net force) but along lines such that together they exert a nonzero *torque* on the object and hence cause it to angularly accelerate and gain kinetic energy without bound, hardly a condition one would call "equilibrium".

Conditions for Static Equilibrium

The Newton's Second Law for Rotation is sufficient to imply Newton's First Law for Rotation:

> ***If, in an inertial reference frame, a rigid object is initially at rotational rest (not rotating), it will remain at rotational rest unless acted upon by a net external torque.***

That is, $\vec{\tau} = I\vec{\alpha} = 0$ implies $\vec{w} = 0$ and constant. We will call the condition where $\vec{\tau} = 0$ and a rigid object is not rotating ***torque equilibrium***.

Therefore we now *define* the conditions for the ***static equilibrium of a rigid body*** to be:

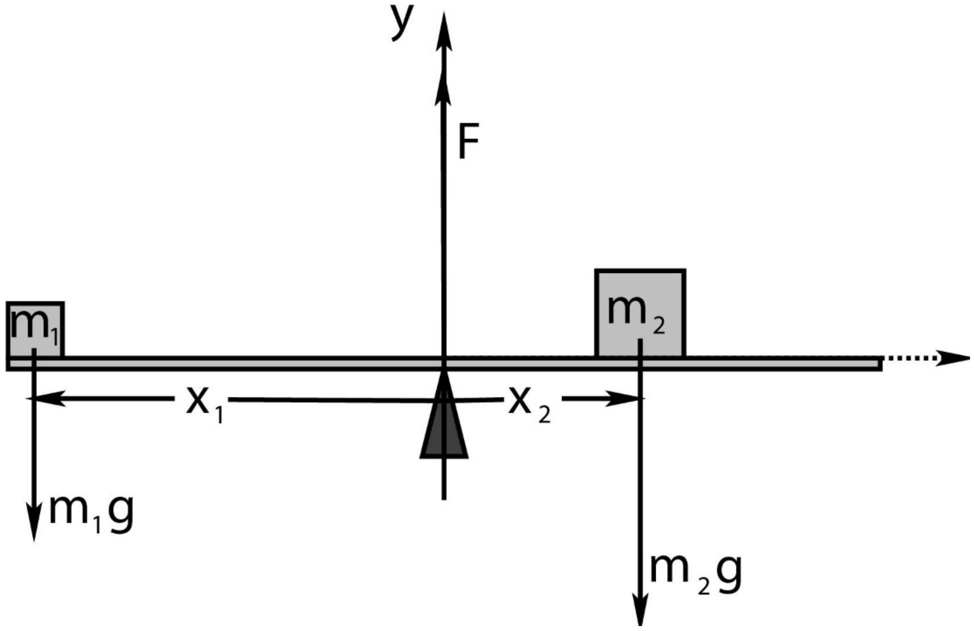
> ***A rigid object is in static equilibrium when both the vector torque and the vector force acting on it are zero.***

That is:

> ***If $\vec{F}_{tot} = 0$ and $\vec{\tau}_{tot} = 0$, then an object initially at translational and rotational rest will remain at rest and neither accelerate nor rotate.***

Balancing a See-Saw

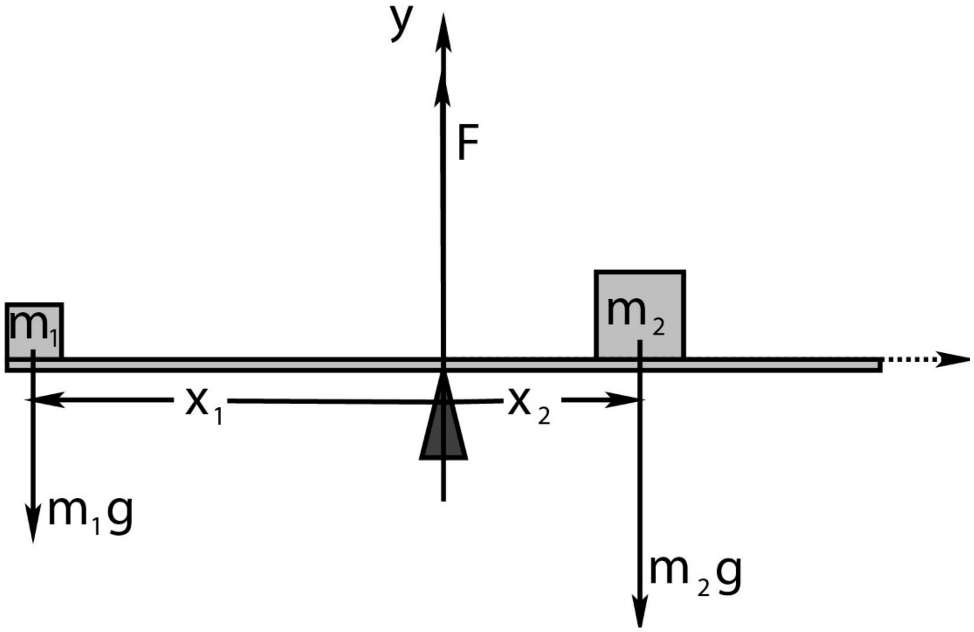
You are given m_1, x_1, and x_2 and are asked to find m_2 and F such that the see-saw is in *static equilibrium*.



One typical problem in statics is balancing weights on a see-saw type arrangement – a uniform plank supported by a fulcrum in the middle. This particular problem is really only one dimensional as far as force is concerned, as there is no force acting in the *x*-direction or *z*-direction.

Balancing a See-Saw

Let's imagine that in this particular problem, the mass m_1 and the distances x_1 and x_2 are given, and we need to find m_2 and F.



We have two choices to make – where we select the pivot and which direction (in or out of the page) we are going to define to be "positive". A perfectly reasonable choice is to select the pivot at the fulcrum of the see-saw where the unknown force F is exerted, and to select the +z-axis as positive rotation.

$$\sum F_y = F - m_1 g - m_2 g = 0$$

$$\sum \tau_z = x_1 m_1 g - x_2 m_2 g = 0$$

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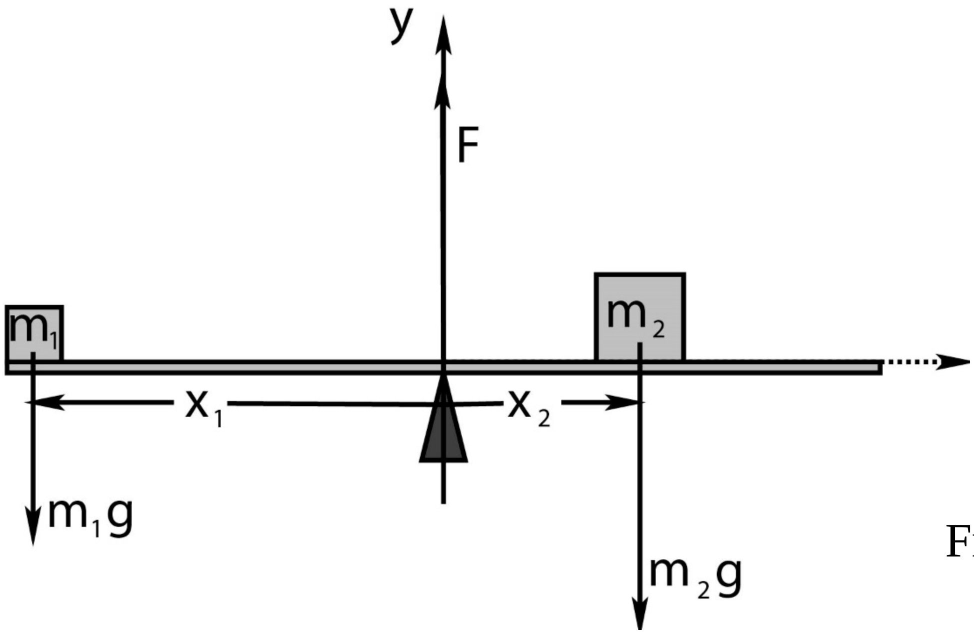
Balancing a See-Saw



$$\sum F_y = F - m_1 g - m_2 g = 0$$

$$\sum \tau_z = x_1 m_1 g - x_2 m_2 g = 0$$

$$m_2 = \frac{m_1 g x_1}{g x_2} = \left(\frac{x_1}{x_2}\right) m_1$$

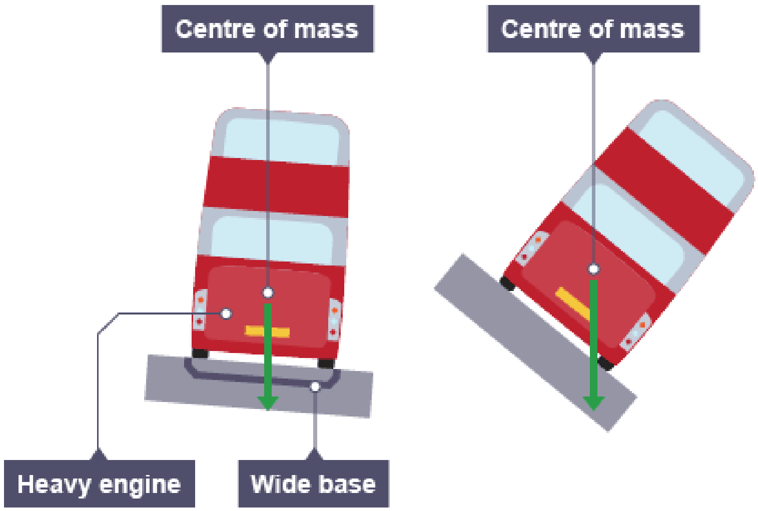
From the first equation and the solution for m_2:

$$F = m_1 g + m_2 g = m_1 g \left(1 + \left(\frac{x_1}{x_2}\right)\right) = m_1 g \left(\frac{x_1 + x_2}{x_2}\right)$$

Tipping

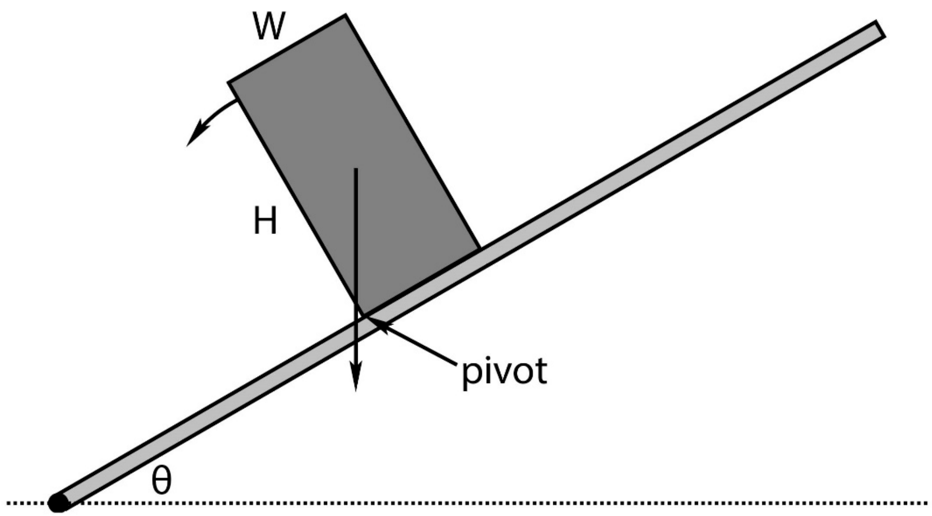
Another important application of the ideas of static equilibrium is to ***tipping problems***. A tippingproblem is one where one uses the ideas of static equilibrium to identify the particular angle or force combination that will *marginally* cause some object to tip over.

The idea of tipping is simple enough. An object placed on a flat surface is typically stable as long as the center of gravity is vertically ***inside the edges*** that are in contact with the surface, so that the torque created by the gravitational force around this limiting pivot is opposed by the torque exerted by the (variable) normal force.



Tipping Versus Slipping

A rectangular block either tips first or slips (slides down the incline) first as the incline is gradually increased. Which one happens first? The figure is show with the block just past the ***tipping angle***.



At some angle we know that the block will start to slide. This will occur because the normal force is decreasing with the angle (and hence, so is the maximum force static friction can exert) and at the same time, the component of the weight of the object that points *down* the incline is increasing. Eventually the latter will exceed the former and the block will slide. However, at some angle the block will *also* tip *over*. We know that this will happen because the normal force can only prevent the block from rotating *clockwise* (as drawn) around the pivot consisting of the lower left corner of the block.

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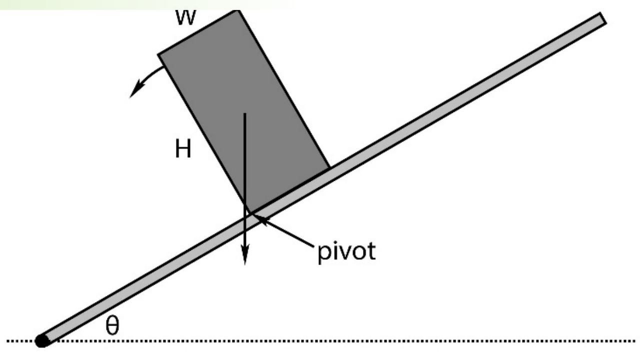
Tipping Versus Slipping



The ***tipping point***, or ***tipping angle*** is thus the angle where the ***center of gravity*** is directly ***over the pivot*** that the object will "tip" around as it falls over.

Let's find the slipping angle θs. Let "down" mean "down the incline". Then:

$$\sum F_{down} = mg\sin(\theta) - F_s = 0$$

$$\sum F_{\perp} = N - mg\cos(\theta) = 0$$

From the latter, as usual: $N = mg\cos(\theta)$ and $F_s \leq F_s^{max} = \mu_s N$

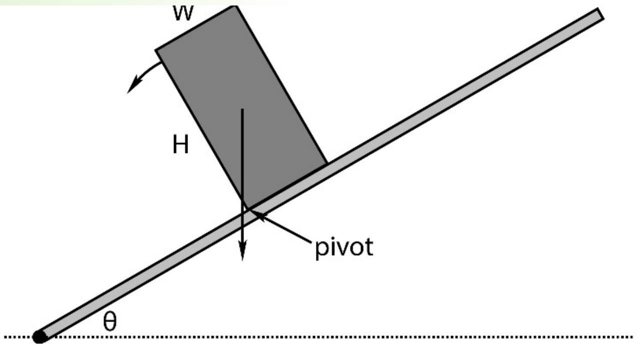
When $mg\sin(\theta_s) = F_s^{max} = \mu_s N\cos(\theta_s)$

The force of gravity down the incline precisely balances the force of static friction. We can solve for the angle where this occurs:$\theta_s = \tan^{-1}(\mu_s)$

This happens when the center of mass passes ***directly over the pivot***.

Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Tipping Versus Slipping



From inspection of the figure (which is drawn *very close* to the tipping point) it should be clear that the tipping angle θ_t is given by:

$$\theta_t = \tan^{-1}\left(\frac{W}{H}\right)$$

So, which one wins? The smaller of the two, θ_s or θ_t, of course – that's the one that happens first as the plank is raised. Indeed, since both are inverse tangents, the smaller of: μ_s, W/H

determines whether the system slips first or tips first, no need to actually *evaluate* any tangents or inverse tangents!