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On Extreme Points of Sets in Operator Spaces and State Spaces

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Dedicated to the 80th birthday of Academician A. S. Holevo

Abstract—We obtain a representation of the set of quantum states in terms of barycenters of nonnegative normalized finitely additive measures on the unit sphere $S_1(\mathcal{H})$ of a Hilbert space $\mathcal{H}$. For a measure on $S_1(\mathcal{H})$, we find conditions in terms of its properties under which the barycenter of this measure belongs to the set of extreme points of the family of quantum states and to the set of normal states. The unitary elements of a unital C*-algebra are characterized in terms of extreme points. We also study extreme points $\operatorname{extr}(\mathcal{E}^1)$ of the unit ball $\mathcal{E}^1$ of a normed ideal operator space $\langle \mathcal{E}, \|\cdot\|_{\mathcal{E}}\rangle$ on $\mathcal{H}$. If $U \in \operatorname{extr}(\mathcal{E}^1)$ for some unitary operator $U \in \mathcal{B}(\mathcal{H})$, then $V \in \operatorname{extr}(\mathcal{E}^1)$ for all unitary operators $V \in \mathcal{B}(\mathcal{H})$. In addition, we construct quantum correlations corresponding to singular states on the algebra of all bounded operators in a Hilbert space.

Keywords—Hilbert space, linear operator, C*-algebra, von Neumann algebra, normed ideal operator space, quantum state, finitely additive measure, barycenter, extreme point, quantum correlations generated by a state.

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INTRODUCTION

The study of the extreme points of normed spaces is an important direction in functional analysis. In the present paper, we deal with three different problems in this direction. First, we investigate the extreme points of the unit ball of the space of positive continuous linear functionals on the algebra of all bounded operators $\mathcal{B}(\mathcal{H})$ in a Hilbert space $\mathcal{H}$. Of course, we primarily focus on singular extreme points, i.e., points that are not vector states. The second problem we address is to describe the extreme points of the unit ball of a normed ideal space, which is a special subspace of a C*-algebra. Finally, the third problem is to study the quantum correlations corresponding to singular states on the algebra $\mathcal{B}(\mathcal{H})$.

A quantum state is a nonnegative normed continuous linear functional on the Banach space $\mathcal{B}(\mathcal{H})$ of all bounded linear operators (see [8, Sect. 2.3.2]). By the Alaoglu theorem, the set of quantum states $\Sigma(\mathcal{H})$ considered as the intersection of the unit sphere with the positive cone in the space of continuous linear functionals on the Banach space $\mathcal{B}(\mathcal{H})$ is convex and compact in the *-weak topology (see [9, Ch. V, Sect. 4]). Therefore, by the Krein–Milman theorem, the set of quantum states coincides with the closure in the *-weak topology of the set of its extreme points.

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To describe the set of extreme points of the set of quantum states, one can use barycentric decompositions of states in the Pettis integral with respect to a nonnegative normalized (to unity) measure on the set of pure vector states (see [8, Sect. 4.1] as well as [4, 5]). The approach we implement in this paper expands the possibilities of the methods presented in [8], since we use not only countably additive but also finitely additive measures. Every state on the algebra $\mathcal{B}(\mathcal{H})$ can be represented (nonuniquely) as a Pettis integral with respect to a measure on the unit sphere of the Hilbert space $\mathcal{H}$, whose points define pure vector states.

On the set of measures on the unit sphere, we introduce an equivalence relation: every equivalence class includes all measures with a common barycenter. We establish a bijection between the set of states and the set of equivalence classes of finitely additive nonnegative normalized measures on the unit sphere of the Hilbert space. The extreme points of the intersection of the unit sphere with the positive cone in the space of finitely additive measures are given by two-valued measures with only two values, 0 and 1 (see [18]). We establish that if a state is an extreme point of the set of quantum states, then the class of measures with barycenter at this state contains a two-valued measure. However, the barycenter of measures from an equivalence class containing a two-valued measure may not be an extreme point of the set of quantum states. We conjecture that for the barycenter of an equivalence class of measures to be an extreme point of the set of states, it suffices that the set of extreme points of the intersection of the equivalence class with the cone of nonnegative measures consists of two-valued measures.

We solve the problem of whether the barycenter of a finitely additive measure on the unit sphere of $\mathcal{H}$ belongs to the set of normal states. A criterion for this is given by the following condition on a finitely additive measure: up to an arbitrary number $\varepsilon > 0$, the measure is concentrated on a compact subset K_ε of $\mathcal{H}$ that belongs to the unit sphere. This condition of the normality criterion is similar to the condition for a countably additive measure to be pseudoconcentrated on a compact set (see [8, Sect. 4.1.2]).

Thus, we obtain a description of quantum states in terms of nonnegative normalized measures on the unit sphere of the Hilbert space. Note that we thus solve the problem, discussed in [3], of describing the dynamics of normal quantum states by describing the evolution of probability distributions.

We establish a unitarity criterion for an arbitrary element of a unital C*-algebra and analyze the properties of the set of extreme points of the unit ball of a normed ideal space (NIS) on $\mathcal{H}$. Let $\langle \mathcal{E}, \|\cdot\|_{\mathcal{E}} \rangle$ be a NIS on $\mathcal{H}$. If $U \in \mathrm{extr}(\mathcal{E}^1)$ for some unitary operator $U \in \mathcal{B}(\mathcal{H})$, then $V \in \mathrm{extr}(\mathcal{E}^1)$ for all unitary operators $V \in \mathcal{B}(\mathcal{H})$.

The study of the properties of the set of quantum correlations has recently allowed to solve a number of problems in the theory of operator algebras that remained open since the second half of the 1970s [13, 2]. An important role in this theory is played by the construction of von Neumann factors $\mathcal{M}$ of type II_1. To construct the set of correlations, one should consider a pair consisting of the factor $\mathcal{M}$ itself and its commutant $\mathcal{M}'$. By means of the Gelfand–Naimark–Segal representation, using an extreme point in the set of singular quantum states, we construct both the factors $\mathcal{M}$ and $\mathcal{M}'$ themselves and the corresponding quantum correlations.

1. NOTATION AND DEFINITIONS

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the Banach space of all bounded linear operators on $\mathcal{H}$ equipped with the operator norm. Let $(\mathcal{B}(\mathcal{H}))^*$ be the dual Banach space of $\mathcal{B}(\mathcal{H})$.

For an arbitrary set M, denote by 2^M the σ-algebra of all subsets of M, by $B(M)$ the Banach space of bounded complex-valued functions on M equipped with the sup-norm, and by $\mathrm{ba}(M)$ the Banach space of complex-valued finitely additive measures of bounded variation on the measurable space $(M, 2^M)$, with the norm of every measure equal to the variation of the measure on the set M. By $\mathrm{ba}^+(M)$ we denote the cone of nonnegative measures in the space $\mathrm{ba}(M)$.

For an arbitrary normed space $\mathcal{X}$, we denote the unit sphere in it by $S_1(\mathcal{X})$, and let $\mathcal{X}^1$ be the unit ball in $\mathcal{X}$. Denote the set of extreme points of a set K in a linear space $\mathcal{L}$ by $\operatorname{extr}(K)$. A C*-algebra is a complex Banach *-algebra $\mathcal{A}$ such that $\|A^*A\| = \|A\|^2$ for all $A \in \mathcal{A}$. Let $\mathcal{A}^{-1}$, $\mathcal{A}^+$, and $\mathcal{A}^{\mathrm{u}}$ be the sets of all invertible, positive, and unitary elements of $\mathcal{A}$, respectively. If $X \in \mathcal{A}$, then $|X| = \sqrt{X^*X} \in \mathcal{A}^+$. Any C*-algebra can be realized as a C*-subalgebra in $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (see [14, Theorem 3.4.1]).

A *-lineal $\mathcal{E} \subset \mathcal{B}(\mathcal{H})$ equipped with a norm $\|\cdot\|_{\mathcal{E}}$ is called a *normed ideal space* (NIS) on $\mathcal{H}$ if

(1) $\|A^*\|_{\mathcal{E}} = \|A\|_{\mathcal{E}}$ for all $A \in \mathcal{E}$;

(2) for all $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{E}$ such that $|A| \le |B|$, we have $A \in \mathcal{E}$ and $\|A\|_{\mathcal{E}} \le \|B\|_{\mathcal{E}}$.

The concept of NIS on $\mathcal{H}$ generalizes symmetrically normed ideals of operators on $\mathcal{H}$, which were studied, for example, in [11, 16]. If the Hilbert space $\mathcal{H}$ is separable, then any NIS on $\mathcal{H}$ is symmetric with respect to its constituent elements (see [11, Ch III, § 2]). If $\mathcal{E}$ is a hereditary C*-subalgebra in $\mathcal{B}(\mathcal{H})$, then $\langle \mathcal{E}, \|\cdot\| \rangle$ is a NIS on $\mathcal{H}$. If $\mathcal{A}$ is a C*-algebra or a NIS on $\mathcal{H}$, then

$$X \in \operatorname{extr}(S_1(\mathcal{A})) \;\Leftrightarrow\; X^* \in \operatorname{extr}(S_1(\mathcal{A})) \qquad \text{and} \qquad X \in \operatorname{extr}(\mathcal{A}^1) \;\Leftrightarrow\; X^* \in \operatorname{extr}(\mathcal{A}^1).$$

Any positive linear functional ρ on a C*-algebra $\mathcal{A}$ defines a sesquilinear form

$$(A,B)_\rho = \rho(B^*A), \qquad A, B \in \mathcal{A}.$$

Factorizing and completing the algebra $\mathcal{A}$ with respect to this form, we obtain a Hilbert space $\mathfrak{H}$ and a $*$-representation π of the algebra $\mathcal{A}$ in $\mathfrak{H}$, called the Gelfand–Naimark–Segal (GNS) representation (see [7]). An important source for constructing von Neumann algebras $\mathcal{M}$ of different types is the closure of the image of a GNS representation of the algebra $\mathcal{A} = \mathcal{B}(\mathcal{H})$. On the other hand, the algebra $\mathcal{B}(\mathcal{H})$ itself can be generated by a pair of factors $\mathcal{M}$ and $\mathcal{M}'$ (a factor is an algebra $\mathcal{M}$ such that $\mathcal{M} \cap \mathcal{M}' = \{\mathbb{C}\mathrm{I}\}$). In this case, the state ρ defines correlations between observables that belong to $\mathcal{M}$ and $\mathcal{M}'$, respectively.

2. EXTREME POINTS OF THE SET OF QUANTUM STATES

We consider the problem of describing the set of extreme points of the set $\Sigma(\mathcal{H})$ of states of a quantum system on a separable Hilbert space $\mathcal{H}$. The set $\Sigma(\mathcal{H})$ is the intersection of the unit sphere $S_1((\mathcal{B}(\mathcal{H}))^*)$ with the cone $(\mathcal{B}(\mathcal{H}))^*_+$ of nonnegative elements of the space dual to the Banach space of bounded linear operators $\mathcal{B}(\mathcal{H})$ equipped with the operator norm. Hence, the set $\Sigma(\mathcal{H})$ is convex and (according to the Alaoglu theorem, see [9]) compact in the space $(\mathcal{B}(\mathcal{H}))^*$ equipped with the *-weak topology. Denote by $\Sigma_{\mathrm{p}}(\mathcal{H})$ and $\Sigma_{\mathrm{n}}(\mathcal{H})$ the sets of pure vector states and normal states, respectively.

A partial description of the set of extreme points of the family of quantum states was given in [4], where the authors showed that for an arbitrary state $\rho \in \Sigma(\mathcal{H})$ there exists a measure $\mu_\rho \in S_1(\mathrm{ba}(S_1(\mathcal{H}))) \cap \mathrm{ba}^+(S_1(\mathcal{H})) \equiv S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$ such that

$$\rho(\mathbf{A}) = \int\limits_{S_1(\mathcal{H})} \rho_u(\mathbf{A})\, d\mu_\rho(u) = \int\limits_{\Sigma_{\mathrm{p}}(\mathcal{H})} \rho(\mu \circ f^{-1})\,(d\rho) \qquad \forall \mathbf{A} \in \mathcal{B}(\mathcal{H}),$$

where $f\colon S_1(\mathcal{H}) \to (\mathcal{B}(\mathcal{H}))^*$ is a function that maps u to ρ_u. When the above condition is satisfied, we say that the state ρ is the barycenter of the measure $\mu \circ f^{-1}$ (measure μ_ρ). In this case the state ρ is also said to be equal to the Pettis integral of the function $f\colon S_1(\mathcal{H}) \to (\mathcal{B}(\mathcal{H}))^*$ with respect to the measure μ_ρ. As shown in [4], any $\rho \in \operatorname{extr}(\Sigma(\mathcal{H}))$ can be represented as the barycenter of some two-valued measure $\mu\colon 2^{S_1(\mathcal{H})} \to \{0,1\}$.

The following observations were also made in [4]: different measures may represent the same state; countably additive measures represent only normal states; and if a measure is concentrated on the set of vectors of an orthonormal basis and is two-valued, then the corresponding state is an extreme point of the set of states $\Sigma(\mathcal{H})$.

Let us give a description of the set of extreme points of the set of quantum states. To this end, on the space of measures $\mathrm{ba}(S_1(\mathcal{H}))$ we introduce a barycentric equivalence relation $\sim$:

$$\mu \sim \nu \qquad \Leftrightarrow \qquad \int\limits_{S_1(\mathcal{H})} \rho_e(\mathbf{A})\,d\mu(e) = \int\limits_{S_1(\mathcal{H})} \rho_e(\mathbf{A})\,d\nu(e) \quad \forall \mathbf{A} \in \mathcal{B}(\mathcal{H}). \tag{2.1}$$

The integrals in (2.1) are the Pettis integrals of a vector-valued function with values in the space $(\mathcal{B}(\mathcal{H}))^*$. The relation $\sim$ is obviously an equivalence relation on the space $\mathrm{ba}(S_1(\mathcal{H}))$.

Here is an example of two equivalent measures. Consider two sequences $\{e_k\}$ and $\{f_k\}$ of vectors in the unit sphere $S_1(\mathcal{H})$ such that the sequence $\{\|e_k - f_k\|_{\mathcal{H}}\}$ is infinitesimal and strictly monotone. If $\mathcal{F}$ is a nonprincipal ultrafilter on the set of positive integers $\mathbb{N}$ and $\nu_{\mathcal{F}}$ is a two-valued measure generated by $\mathcal{F}$ (see [4]), then the measures μ_e and μ_f defined by $\mu_{e,f}(A) = \nu_{\mathcal{F}}(\{n \in \mathbb{N} \colon e_n, f_n \in A\})$, $A \subset S_1(\mathcal{H})$, are different as elements of $\mathrm{ba}(S_1(\mathcal{H}))$ and are equivalent.

The set

$$\mathcal{V}_0 = \left\{ \mu \in \mathrm{ba}(S_1(\mathcal{H})) \colon \int\limits_{S_1(\mathcal{H})} \rho_e(\mathbf{A})\,d\mu(e) = 0 \ \ \forall \mathbf{A} \in \mathcal{B}(\mathcal{H}) \right\}$$

is a linear subspace in the space of measures $\mathrm{ba}(S_1(\mathcal{H}))$.

The quotient space $\widehat{\mathrm{ba}}(S_1(\mathcal{H})) = \mathrm{ba}(S_1(\mathcal{H}))/{\sim}$ is a linear space. For every $\widehat{\mu} \in \widehat{\mathrm{ba}}(S_1(\mathcal{H}))$, we set

$$\rho_{\widehat{\mu}}(\mathbf{A}) = \int\limits_{S_1(\mathcal{H})} \rho_e(\mathbf{A})\,d\widehat{\mu}(e) = \int\limits_{S_1(\mathcal{H})} \rho_e(\mathbf{A})\,d\mu(e), \qquad \mathbf{A} \in \mathcal{B}(\mathcal{H}), \tag{2.2}$$

where $\mu \in \widehat{\mu}$. By the definition of the equivalence relation, the right-hand side of (2.2) does not depend on the choice of a representative $\mu \in \widehat{\mu}$.

On the space $\widehat{\mathrm{ba}}(S_1(\mathcal{H}))$ we introduce a partial order relation $\geq$:

$$\widehat{\mu} \geq \widehat{\nu} \qquad \Leftrightarrow \qquad \int\limits_{S_1(\mathcal{H})} \rho_e(\mathbf{A})\,d\widehat{\mu}(e) \geq \int\limits_{S_1(\mathcal{H})} \rho_e(\mathbf{A})\,d\widehat{\nu}(e) \quad \forall \mathbf{A} \in (\mathcal{B}(\mathcal{H}))^+.$$

Denote by M_1^+ the set

$$\left\{ \widehat{\mu} \in \widehat{\mathrm{ba}}(S_1(\mathcal{H})) \colon \ \widehat{\mu} \cap S_1^+(\widehat{\mathrm{ba}}(S_1(\mathcal{H}))) \neq \varnothing \right\} \equiv \left\{ \widehat{\mu} \in \widehat{\mathrm{ba}}(S_1(\mathcal{H})) \colon \ \widehat{\mu} \geq 0, \int\limits_{S_1(\mathcal{H})} \rho_e(\mathbf{I})\,d\widehat{\mu}(e) = 1 \right\}.$$

Theorem 2.1. *The map*

$$\int \colon \ \widehat{\mu} \to \int\limits_{S_1(\mathcal{H})} \rho_e\,d\widehat{\mu}(e) \equiv \rho_{\widehat{\mu}} \tag{2.3}$$

is a bijection of the set M_1^+ onto $\Sigma(\mathcal{H})$ that preserves convex combinations.

Proof. Every element $\widehat{\mu}$ of $\widehat{\mathrm{ba}}(S_1(\mathcal{H}))$ defines a linear functional $\rho_{\widehat{\mu}}$ on $\mathcal{B}(\mathcal{H})$ according to (2.2). Moreover, if $\widehat{\mu} \in M_1^+$, then the functional $\rho_{\widehat{\mu}}$ is nonnegative and $|\rho_{\widehat{\mu}}(\mathbf{A})| = \left|\int_{S_1(\mathcal{H})} \rho_e(\mathbf{A})\,d\mu_1(e)\right|$ with $\mu_1 \in S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$; in particular, $\rho_{\widehat{\mu}}(\mathbf{I}) = 1$. Hence, $|\rho_{\widehat{\mu}}(\mathbf{A})| \leq \|\mathbf{A}\|_{\mathcal{B}(\mathcal{H})}$, and the functional $\rho_{\widehat{\mu}}$ is a continuous nonnegative normed linear functional on $\mathcal{B}(\mathcal{H})$; i.e., $\int(M_1^+) \subset \Sigma(\mathcal{H})$, and the map (2.3) preserves convex combinations.

If $\widehat{\mu} \neq \widehat{\nu}$, then by (2.1) there exists an element $\mathbf{A} \in \mathcal{B}(\mathcal{H})$ such that $\rho_{\widehat{\mu}}(\mathbf{A}) \neq \rho_{\widehat{\nu}}(\mathbf{A})$; hence, the map (2.3) is injective.

According to [4], for any $\rho \in \Sigma(\mathcal{H})$ there exists a measure $\mu_\rho \in S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$ such that $\rho = \int_{S_1(\mathcal{H})} \rho_e \, d\mu_\rho(e)$. Hence, $\rho = \int_{S_1(\mathcal{H})} \rho_e \, d\widehat{\mu}(e)$ for a class $\widehat{\mu}$ containing the measure μ_ρ. Hence, the map (2.3) is surjective. $\square$

Corollary 2.1. *The map* (2.3) *is a bijection of the set* $\mathrm{extr}(M_1^+)$ *onto the set* $\mathrm{extr}(\Sigma(\mathcal{H}))$.

If a state ρ is the image of a class of measures $\widehat{\mu}$ under the bijective map (2.3), then we will write $\rho = \rho_\mu$ and $\widehat{\mu} = \rho_{\widehat{\mu}}$. Moreover, we will write $\rho_{\widehat{\mu}} = \rho_\mu$ for an arbitrary choice of the measure μ in the class $\widehat{\mu}$.

Let us now give a description of the extreme points of the set M_1^+ of classes of nonnegative normalized measures. It is well known (see [18]) that the set of extreme points of the simplex of measures $S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$ consists of two-valued measures (i.e., measures generated by ultrafilters).

Theorem 2.2. *If* $m \in M_1^+$ *is an extreme point of* M_1^+, *then the equivalence class of* m *contains a two-valued measure.*

Lemma 2.1. *Let* $\widehat{\mu} \in \mathrm{extr}(M_1^+)$. *Let* $\mu \in \widehat{\mu}$ *and suppose that there exists a set* $A_1 \subset S_1(\mathcal{H})$ *such that* $\mu(A_1) = a_1 \in (0,1)$. *If* $\mu_1(A) = \frac{1}{a_1}\mu(A \cap A_1)$, $A \in 2^{S_1(\mathcal{H})}$, *then* $\mu_1 \sim \mu$.

Proof. We put $\nu_1(A) = \mu(A \cap A_1)$ and $\nu_2(A) = \mu(A \setminus A_1)$, $A \in 2^{S_1(\mathcal{H})}$. Then

$$\mu = a_1\mu_1 + (1 - a_1)\mu_2, \qquad \text{where} \quad \mu_1 = \frac{1}{a_1}\nu_1, \quad \mu_2 = \frac{1}{1 - a_1}\nu_2.$$

Since μ is an extreme point of M_1^+, it follows that $\mu_1 \sim \mu_2 \sim \mu$. $\square$

Hence, the class $\widehat{\mu}$ can be represented by any element μ_1 concentrated on any set $A_1 \in 2^{S_1(\mathcal{H})}$ such that $\mu(A_1) > 0$.

Let $\mathcal{P}(\mathcal{H})$ be the set of orthogonal projections acting in $\mathcal{H}$.

Lemma 2.2. *Let* $\widehat{\mu}$ *be an extreme point of* M_1^+. *Then the class* $\widehat{\mu}$ *contains a two-valued measure generated by an ultrafilter of subsets of the set* $S_1(\mathcal{H})$.

Proof. Let $\mu \in \widehat{\mu}$. Consider the set

$$\mathcal{A}_1 = \left\{A_1 \in 2^{S_1(\mathcal{H})} \colon \mu(A_1) > 0\right\}$$

of subsets of $S_1(\mathcal{H})$ partially ordered by inclusion. Then $\mathcal{A}_1$ has the following of the properties characterizing an ultrafilter:

- $\varnothing \notin \mathcal{A}_1$;
- $B \in S_1(\mathcal{H}),\ B \notin \mathcal{A}_1 \ \Rightarrow\ S_1(\mathcal{H}) \setminus B \in \mathcal{A}_1$;
- $A \in \mathcal{A}_1,\ B \supset A \ \Rightarrow\ B \in \mathcal{A}_1$.

However, the following property fails:

- $A \in \mathcal{A}_1,\ B \in \mathcal{A}_1 \ \Rightarrow\ A \cap B \in \mathcal{A}_1$,

since it is unknown whether the measure μ can take positive values on two disjoint sets. To resolve this question, we will show that the family of sets $\mathcal{A}_1$ contains maximal chains that are linearly ordered by inclusion.

Consider the set $\mathcal{E}$ of chains of elements of the partially ordered (by inclusion) set $\mathcal{A}_1$. The set $\mathcal{E}$ is partially ordered by the inclusion relation between the elements of $\mathcal{E}$. By the Hausdorff theorem, $\mathcal{E}$ has a maximal element $\mathcal{C}_1$.

If $A_1, A_2 \in \mathcal{C}_1$, then at least one of the conditions $A_1 \subset A_2$ or $A_2 \subset A_1$ is satisfied, since the elements of the chain $\mathcal{C}_1$ are linearly ordered by inclusion.

By the maximality of the chain $\mathcal{C}_1$, for an arbitrary element $A \in \mathcal{C}_1$ (such that $\mu(A) > 0$) and an arbitrary partition of the set A into two disjoint subsets A' and A'', one and only one of the subsets A' and A'' belongs to the chain $\mathcal{C}_1$. Since $\mu(A') + \mu(A'') = \mu(A) > 0$, at least one of the sets A' and A'' belongs to the family $\mathcal{A}_1$. Since the chain $\mathcal{C}_1$ is linearly ordered by inclusion, it cannot contain both sets A' and A'' simultaneously. At least one of the sets A' and A'' should be an element of the chain $\mathcal{C}_1$, for otherwise the chain $\mathcal{C}_1$ would not be a maximal chain in $\mathcal{A}_1$.

A maximal chain $\mathcal{C}_1$ of subsets of $\mathcal{A}_1$ linearly ordered by inclusion forms an ultrafilter $\mathcal{F}$ of subsets of $S_1(\mathcal{H})$ so that $B \in \mathcal{F}$ if and only if B contains a set from $\mathcal{C}_1$. Then

- $\varnothing \notin \mathcal{C}_1$;
- $A \in \mathcal{C}_1,\ B \supset A \ \Rightarrow\ B \in \mathcal{C}_1$;
- $B \subset S_1(\mathcal{H}),\ B \notin \mathcal{C}_1 \ \Rightarrow\ S_1(\mathcal{H}) \setminus B \in \mathcal{C}_1$ (because $S_1(\mathcal{H}) \in \mathcal{C}_1$, and so one and only one of the sets B and $S_1(\mathcal{H}) \setminus B$ is contained in the chain $\mathcal{C}_1$);
- $A \in \mathcal{C}_1,\ B \in \mathcal{C}_1 \ \Rightarrow\ A \cap B \in \mathcal{C}_1$ (because if $A, B \in \mathcal{C}_1$, then either $A \subset B$ or $A \supset B$).

Thus, any measure $\mu \in S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$ defines an ultrafilter $\mathcal{F}$ of subsets on which μ takes positive values.

For every $A \in \mathcal{F}$, we define a measure $\mu_A = (\mu(A))^{-1}\nu_A$ with $\nu_A(B) = \mu(A \cap B)$, $B \in 2^{S_1}$. By the hypothesis of Lemma 2.2, the class $\widehat{\mu}$ is an extreme point of M_1^+, and $\mu \in \widehat{\mu}$ by the assumption. Therefore, according to Lemma 2.1, we have $\mu_A \sim \mu$ for any $A \in \mathcal{F}$.

Hence, for any $\mathbf{P} \in \mathcal{P}(\mathcal{H})$, there exists a limit $\lim_{\mathcal{F}} \rho_{\mu_A}(\mathbf{P}) = \rho_\mu(\mathbf{P})$. Then $\rho_{\mu_{\mathcal{F}}}(\mathbf{P}) = \rho_\mu(\mathbf{P})$ for any $\mathbf{P} \in \mathcal{P}(\mathcal{H})$, where $\mu_{\mathcal{F}}(A) = 1$ for all $A \in \mathcal{F}$. Hence, $\mu_{\mathcal{F}} \in \widehat{\mu}$, and the two-valued measure $\mu_{\mathcal{F}}$ is equivalent to μ. Thus, any extreme point of M_1^+ is an equivalence class of measures that contains a two-valued measure. $\square$

Remark 2.1. The converse statement does not hold: even if a class $\widehat{\mu}$ contains a two-valued measure $\mu \in S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$, the class $\widehat{\mu}$ may not be an extreme point of M_1^+.

Example. Let $\mathcal{E} = \{e_k\}$ be an orthonormal basis in a Hilbert space E, and let $\mathcal{F} = \{f_k\}$ be a system of unit vectors in E defined as $f_k = \frac{1}{\sqrt{2}}(e_1 + e_{k+1})$, $k \in \mathbb{N}$. Let F be an ultrafilter on the set of positive integers and ν_F be the corresponding two-valued measure on $\mathbb{N}$. Then the measures $\mu_F = \nu_F \circ f^{-1}$, $\mu_E = \nu_F \circ e^{-1}$, and δ_{e_1} are two-valued and hence are extreme points of the set $S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$. However, it can be easily verified that $\rho_{\mu_F} = \frac{1}{2}(\rho_{\mu_E} + \rho_{e_1})$; hence, the equivalence class $\widehat{\mu}$ contains both the two-valued measure μ_F and the measure $\frac{1}{2}(\mu_E + \delta_{e_1})$; therefore, it is not an extreme point of M_1^+.

Every equivalence class $\widehat{\mu} \in M_1^+$, as a set in the space $\mathrm{ba}(S_1(\mathcal{H}))$, is convex and (pre)compact in the *-weak topology (since it is a subset of $S_1(\mathrm{ba}(S_1(\mathcal{H})))$). If $\rho \in \mathrm{extr}(\Sigma(\mathcal{H}))$, then the class of the measure μ_ρ contains a two-valued measure; however, not all measures in the class $\widehat{\mu}_\rho$ are two-valued. For example, if $e \in S_1(\mathcal{H})$ and F is an ultrafilter converging to e, then the class $\widehat{\mu}_\rho$ contains the line segment $\{a\delta_e + (1-a)\mu_F : a \in [0,1]\}$.

For any $\widehat{\mu} \in M_1^+$, the set $\widehat{\mu}$ is not compact in the space $\mathrm{ba}(S_1(\mathcal{H}))$ equipped with the *-weak topology induced by functionals from the Banach space $B(S_1(\mathcal{H}))$ of bounded functions with the sup-norm. Indeed, $\widehat{\mu}$ can be represented as $\mu + \mathcal{N}$, where

$$\mathcal{N} = \bigl\{\nu \in \mathrm{ba}(S_1(\mathcal{H})) \colon\ \nu(\mathbf{A}) = 0\ \ \forall \mathbf{A} \in \mathcal{B}(\mathcal{H})\bigr\}$$

is a linear subspace. However, the unit sphere of the Banach space $\mathrm{ba}(S_1(\mathcal{H}))$ is a compact set in the *-weak topology; therefore, $\widehat{\mu} \cap S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$ is compact in the *-weak topology.

Lemma 2.3. *If $\nu \in \mathrm{extr}\bigl(\widehat{\mu} \cap S_1^+(\mathrm{ba}(S_1(\mathcal{H})))\bigr)$ for an extreme point $\widehat{\mu}$ of M_1^+, then the measure ν is two-valued.*

Proof. Since $\nu \in \mathrm{extr}(\widehat{\mu})$, it follows from the relations $\nu = a\nu_1 + (1-a)\nu_2$, $\nu_1, \nu_2 \in \widehat{\mu}$, $a \in (0,1)$, that $\nu_1 = \nu_2$.

Suppose that there exist measures $\lambda_1, \lambda_2 \in S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$ and a number $b \in (0,1)$ such that $\nu = b\lambda_1 + (1-b)\lambda_2$. Let us prove that then the condition $\lambda_1 = \lambda_2$ holds, that is, the measure ν is an extreme point of $S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$ and hence is two-valued.

Suppose by contradiction that $\lambda_1 \neq \lambda_2$. If λ_1 is not equivalent to λ_2, then $\widehat{\mu} = b\widehat{\lambda}_1 + (1-b)\widehat{\lambda}_2$, $\widehat{\lambda}_1 \neq \widehat{\lambda}_2$, but this contradicts the condition $\widehat{\mu} \in \mathrm{extr}(M_1^+)$.

If the measures λ_1 and λ_2 are equivalent, then they are also equivalent to the measure ν, that is, $\nu, \lambda_1, \lambda_2 \in \widehat{\mu} \cap S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$. Then it follows from the condition $\lambda_1 \neq \lambda_2$ that μ is not an extreme point of $\widehat{\mu} \cap S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$, which contradicts the hypothesis of the lemma.

Thus, it follows from the condition $\nu = b\lambda_1 + (1-b)\lambda_2$, where $\lambda_1, \lambda_2 \in S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$ and $b \in (0,1)$, that $\lambda_1 = \lambda_2$. Consequently, $\nu \in \mathrm{extr}\big(S_1^+(\mathrm{ba}(S_1(\mathcal{H})))\big)$; hence, the measure ν is two-valued. □

Theorem 2.3. *The condition $\widehat{\mu} \in \mathrm{extr}(M_1^+)$ is equivalent to the following condition: the set $\mathrm{extr}\big(\widehat{\mu} \cap S_1^+(\mathrm{ba}(S_1(\mathcal{H})))\big)$ contains only two-valued measures.*

Proof. The necessity is established in Lemma 2.3; let us prove the sufficiency. To this end, we prove that if $\widehat{\mu} \notin \mathrm{extr}(M_1^+)$, then the set $\mathrm{extr}\big(\widehat{\mu} \cap S_1^+(\mathrm{ba}(S_1(\mathcal{H})))\big)$ contains a non-two-valued measure.

Let $\widehat{\mu} \notin \mathrm{extr}(M_1^+)$. Then $\widehat{\mu} = \alpha\widehat{\mu}_1 + (1-\alpha)\widehat{\mu}_2$ with $\widehat{\mu}_{1,2} \in \widehat{\mathrm{ba}}(S_1(\mathcal{H}))$, $\widehat{\mu}_{1,2} \neq \widehat{\mu}$. Therefore, if $\mu_j \in \mathrm{extr}\big[\widehat{\mu}_j \cap S_1^+(\mathrm{ba}(S_1(\mathcal{H})))\big]$, $j = 1,2$, then $\mu_F \sim \alpha\mu_1 + (1-\alpha)\mu_2$, $\mu_{1,2} \nsim \mu_F$. This last relation implies that $\mu_1 \nsim \mu_2$; hence, there exists a set $D \subset S_1(\mathcal{H})$ such that $0 < (\alpha\mu_1 + (1-\alpha)\mu_2)(D) < 1$. Now, if

$$\nu_1 = \frac{1}{a}\big(\alpha\mu_1 + (1-\alpha)\mu_2\big)\big|_D \qquad \text{and} \qquad \nu_2 = \frac{1}{1-a}\big(\alpha\mu_1 + (1-\alpha)\mu_2\big)\big|_{D^\perp}$$

with $a = (\alpha\mu_1 + (1-\alpha)\mu_2)(D) \in (0,1)$ and $D^\perp = S_1(\mathcal{H}) \setminus D$, then $\nu_1, \nu_2 \in S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$; moreover, $\nu_1(S_1(\mathcal{H}) \setminus D) = 0 = \nu_2(D)$ and

$$\widehat{\mu} \ni a\nu_1 + (1-a)\nu_2.$$

Let

$$\mathcal{M}_\mu^a = \big\{a\mu_1 + (1-a)\mu_2 \in \widehat{\mu}\colon\ \mu_{1,2} \in S_1^+(\mathrm{ba}(S_1(\mathcal{H}))),\ \mu_2(D) = 0 = \mu_1(D^\perp)\big\}.$$

The set $\mathcal{M}_\mu^a$ is convex and, by the Banach–Alaoglu theorem, is compact in the topology τ_B generated by all bounded functions on $S_1(\mathcal{H})$. Hence, this set contains its extreme points.

Lemma A. *If $a\nu_1 + (1-a)\nu_2 \in \mathrm{extr}(\mathcal{M}_\mu^a)$, then the measure ν_1 cannot decomposed into a convex combination of two measures from $S_1(\mathrm{ba}(S_1(\mathcal{H})))$.*

Proof. If $\nu_1 = \beta m_1 + (1-\beta)m_2$ for some $m_1, m_2 \in S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$, then we have $m_1(D^\perp) = 0 = m_2(D^\perp)$; moreover,

$$a\nu_1 + (1-a)\nu_2 = \beta(am_1 + (1-a)\nu_2) + (1-\beta)(am_2 + (1-a)\nu_2).$$

However, this contradicts the fact that $a\nu_1 + (1-a)\nu_2 \in \mathrm{extr}(\mathcal{M}_\mu^a)$. □

Let ν^* be an extreme point of the convex set $\mathcal{M}_\mu^a$, which is compact in the topology τ_B. Then ν^* has the form $\nu^* = a\nu_1^* + (1-a)\nu_2^*$ with $\nu_2^*(D) = 0 = \nu_1^*(D^\perp)$, $\nu_1^*, \nu_2^* \in S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$.

Let us prove that ν^* is an extreme point of the set $\widehat{\mu} \cap S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$.

Suppose the contrary. Then the measure $a\nu_1^* + (1-a)\nu_2^* \in \mathrm{extr}(\mathcal{M}_\mu^a)$ satisfies the condition

$$a\nu_1^* + (1-a)\nu_2^* = b\lambda_1 + (1-b)\lambda_2, \qquad \lambda_1, \lambda_2 \in S_1^+(\mathrm{ba}(S_1(\mathcal{H}))),$$

with $\lambda_1, \lambda_2 \in \widehat{\mu}$ and $b \in (0,1)$. Therefore,

$$\frac{1}{a} b\lambda_1\big|_D + \frac{1}{a}(1-b)\lambda_2\big|_D = a\nu_1^*.$$

Since $\lambda_1 \neq \lambda_2$, we can assume that $\lambda_1|_D \neq \lambda_2|_D$ (the case of $\lambda_1|_{D^\perp} \neq \lambda_2|_{D^\perp}$ is treated similarly). Then if $\lambda_1(D)\lambda_2(D) \neq 0$, the measure ν_1 is decomposed into a complex combination of two measures from $S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$, which leads to a contradiction by Lemma A. This contradiction shows that in fact $\lambda_1(D)\lambda_2(D) = 0$. Hence, either $a = b$ or $a = 1-b$. Consequently, $a\nu_1 + (1-a)\nu_2 \in \operatorname{extr}(S_1^+(\mathrm{ba}(S_1(\mathcal{H}))) \cap \widehat{\mu})$ and the set $S_1^+(\mathrm{ba}(S_1(\mathcal{H}))) \cap \widehat{\mu}$ contains an extreme point that is not a two-valued measure. $\square$

Remark 2.2. For every $\mathbf{P} \in \mathcal{P}(\mathcal{H})$, the family of sets $\mathcal{A}_{\mathbf{P}} = \{f_{\mathbf{P}}^{-1}(B),\, B \in \mathcal{B}(\mathbb{R})\}$, where $f_{\mathbf{P}}(e) = (\mathbf{P}e, e)$, $e \in S^1(\mathcal{H})$, is a σ-algebra. Let $\mathcal{A}_{\mathcal{P}}$ be the σ-algebra generated by the family of sets $\bigcup_{\mathbf{P}\in\mathcal{P}} \mathcal{A}_{\mathbf{P}}$. Then measures $\mu, \nu \in \mathrm{ba}(S_1(\mathcal{H}))$ are equivalent if and only if their restrictions to the algebra $\mathcal{A}_{\mathcal{P}}$ coincide. Hence, the sets M_1^+ and $S_1^+(\mathrm{ba}(S_1(\mathcal{H}), \mathcal{A}_{\mathcal{P}}))$ are isomorphic, and the map (2.3) defines a bijection of the set $S_1^+(\mathrm{ba}(S_1(\mathcal{H}), \mathcal{A}_{\mathcal{P}}))$ onto the set of quantum states $\Sigma(\mathcal{H})$.

Let us characterize the classes of measures whose barycenters are normal states. Recall that a measure $\mu \in S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$ is *inner regular* if for any $\varepsilon > 0$ there exists a compact set $K \subset S_1(\mathcal{H})$ such that $\mu(S_1(\mathcal{H}) \setminus K) < \varepsilon$.

Lemma 2.4. *If a measure μ is inner regular, then the state ρ_μ is normal.*

Proof. Take an $\varepsilon > 0$. Let $K \subset S_1(\mathcal{H})$ be a compact set in $\mathcal{H}$ such that $\mu(S_1(\mathcal{H}) \setminus K) < \varepsilon$. Let $\{f_1, \dots, f_m\} \subset S_1(\mathcal{H})$ be an ε-net in K. Then there exists a finite-dimensional orthogonal projection $\mathbf{P}_m$ onto the subspace $\mathrm{span}(f_1, \dots, f_m)$ such that

$$\rho_\mu(\mathbf{P}_m) = \int\limits_{S_1(\mathcal{H})} (\mathbf{P}_m e, e)\, d\mu(e) \geq \int\limits_K (\mathbf{P}_m e, e)\, d\mu(e) \geq \mu(K)\sqrt{1-\varepsilon^2} > (1-\varepsilon)^{3/2}.$$

Since $\varepsilon > 0$ is arbitrary, it follows (see [15, Theorem 9.2]) that the state ρ_μ is normal. $\square$

If ρ is a normal state (and even a pure vector state), not every measure $\mu \in \widehat{\mu}_\rho$ is necessarily inner regular. For example, let $\rho = \rho_u$, $u \in S_1(\mathcal{H})$, and $\{e_k\}$ be a sequence of unit vectors that form a dense subset of the sphere $S_1(\mathcal{H})$. Let $\mathcal{F}$ be the filter generated by the system of sets $\mathbb{N}_\varepsilon = \{k \in \mathbb{N}\colon \|e_k - u\|_{\mathcal{H}} < \varepsilon\}$, $\varepsilon \in (0,1)$. Let F be an ultrafilter containing the filter $\mathcal{F}$. Then $u = \lim_F e_k$, $\mu_F \in \widehat{\mu}_{\rho_u}$, but the ultrafilter F need not be concentrated on compact sets.

Definition 2.1. A measure $\mu \in S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$ is said to be *normal* if for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset S_1(\mathcal{H})$ such that $\mu(O_\varepsilon(K_\varepsilon)) > 1 - \varepsilon$. (Here $O_\varepsilon(K_\varepsilon) = \{y \in S_1(\mathcal{H})\colon \inf_{x\in K_\varepsilon} \|x - y\|_{\mathcal{H}} < \varepsilon\}$.)

Remark 2.3. If $\dim \mathcal{H} < \infty$, then every measure $\mu \in S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$ is normal.

Remark 2.4. The normality of a finitely additive measure $\mu \in S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$ is an analog of the pseudoconcentration of a countably additive measure on $S_1(\mathcal{H})$.

Lemma 2.5. *If ρ_u is a vector state, then every measure $\mu \in \widehat{\mu}_{\rho_u}$ is normal.*

Proof. Let us demonstrate that $\mu(O_\varepsilon(u)) = 1$ for every $\varepsilon > 0$. Suppose by contradiction that $\mu(S_1(\mathcal{H}) \setminus O_\varepsilon(u)) = \delta > 0$. Then

$$1 = \rho_u(\mathbf{P}_u) = \int\limits_{S_1(\mathcal{H})} |(u,e)|^2\, d\mu(e) \leq \mu(O_\varepsilon(u)) + \mu(S_1(\mathcal{H}) \setminus O_\varepsilon(u))\sqrt{1-\varepsilon^2} < 1.$$

The contradiction obtained proves the assertion of Lemma 2.5. $\square$

Lemma 2.6. *If ρ is a normal state of finite rank, then every measure $\mu \in \widehat{\mu}_\rho$ is normal.*

Proof. By Lemma 2.5, there exists a compact set K consisting of finitely many points of the sphere $S_1(\mathcal{H})$ such that $\mu(O_\varepsilon(K)) = 1$ for any $\varepsilon > 0$ and any $\mu \in \widehat{\mu}_\rho$. $\square$

Lemma 2.7. *If ρ is a normal state, then every measure $\mu \in \widehat{\mu}_\rho$ is normal.*

Proof. The normal state ρ can be represented by a nonnegative trace-class operator with unit trace (see [15]). Therefore, for any $\varepsilon > 0$ there exist nonnegative numbers $p_1, \dots, p_m, p_0$, a finite orthonormal system of vectors $\{u_1, \dots, u_m\}$, and a normal state r such that

$$\rho = \sum_{k=1}^{m} p_k \rho_{u_k} + p_0 r = (1 - p_0)\rho' + p_0 r \qquad \text{and} \qquad p_0 \in \Big(0, \frac{\varepsilon}{2}\Big).$$

Then by Theorem 2.1 we have $\widehat{\mu}_\rho = (1 - p_0)\widehat{\mu}_{\rho'} + p_0\widehat{\mu}_r$, with any measure from $\widehat{\mu}_{\rho'}$ being normal by Lemma 2.6. Since $\varepsilon > 0$ is arbitrary, any measure from $\widehat{\mu}_\rho$ is also normal. □

Lemma 2.8. *If a measure $\mu \in S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$ is normal, then the state ρ_μ is normal.*

Proof. Take a number $\varepsilon > 0$. Then there exists a compact set $K \subset S_1(\mathcal{H})$ such that $\mu(O_\varepsilon(K)) > 1 - \varepsilon$. Let $\{f_1, \dots, f_m\} \subset S_1(\mathcal{H})$ be an ε-net of K, and let $\mathbf{P}$ be the orthogonal projection onto the linear subspace $\mathrm{span}(f_1, \dots, f_m)$. Then

$$\begin{aligned} \rho_\mu(\mathbf{P}) = \int\limits_{S_1(\mathcal{H})} (\mathbf{P}e, e)\, d\mu(e) &\geq \int\limits_{O_\varepsilon(K)\cap S_1(\mathcal{H})} (\mathbf{P}e, e)\, d\mu(e) \\ &\geq (1 - \varepsilon) \inf_{O_\varepsilon(K)\cap S_1(\mathcal{H})} (\mathbf{P}e, e) \geq (1 - \varepsilon)\sqrt{1 - \varepsilon^2}. \end{aligned}$$

Consequently, for any $\varepsilon > 0$ there exists a finite-dimensional orthogonal projection $\mathbf{P}$ such that $\rho_\mu(\mathbf{P}) > 1 - \varepsilon$. Hence (see [15]), the state ρ_μ is normal. □

Corollary 2.2. *If a measure $\mu \in S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$ is normal, then any measure from the class $\widehat{\mu}$ is normal.*

Proof. Let $\mu \in \widehat{\mu}$ and μ be a normal measure. By Lemma 2.8 we have $\rho_\mu \in \Sigma_{\mathrm{n}}(\mathcal{H})$. By Theorem 2.1 we have $\widehat{\mu} = \widehat{\mu}_{\rho_\mu}$. By Lemma 2.7 any measure $\mu \in \widehat{\mu}$ is normal. □

Corollary 2.3. *If the space $\mathcal{H}$ is separable, then every countably additive measure $\mu \in S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$ is normal and its barycenter ρ_μ is a normal state.*

Proof. Let $\varepsilon > 0$. Since $\mathcal{H}$ is separable, there exists a countable ε-net $\{f_k\}$ in the unit sphere $S_1(\mathcal{H})$. The measure μ is countably additive; therefore, there exists a number N such that $\mu\big(\bigcup_{j=1}^N O_\varepsilon(f_j) \cap S_1(\mathcal{H})\big) > 1 - \varepsilon$. Hence, the measure μ is normal. □

Lemmas 2.7 and 2.8 and Corollary 2.2 imply the following.

Theorem 2.4. *A state ρ is normal if and only if every measure $\mu \in \widehat{\mu}_\rho$ is normal.*

Corollary 2.4. *A state ρ is an extreme point of the set of normal states if and only if the equivalence class $\widehat{\mu}_\rho$ consists of normal measures and contains a two-valued measure. In this case, any such normal two-valued measure is either concentrated at a single point or defines a nonprincipal ultrafilter converging to a point in the norm of the space $\mathcal{H}$.*

Proof. The first part of the claim follows from Theorems 2.2 and 2.4. A two-valued measure $\mu \in \widehat{\mu}_\rho$ can be either countably additive or purely finitely additive. If a measure $\mu \in S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$ is countably additive and two-valued, then it is a measure concentrated at some point of $S_1(\mathcal{H})$ (see [18]). If a measure $\mu \in S_1^+(\mathrm{ba}(S_1(\mathcal{H})))$ is finitely additive and two-valued, then it defines the ultrafilter $\mathcal{F}_\mu = \mu^{-1}(\{1\})$. Let $\varepsilon > 0$. Since the measure is normal, there exists a compact set $K_\varepsilon \subset S_1(\mathcal{H})$, covered by a finite ε-net $\{f_1, \dots, f_m\}$, such that $\mu(K_\varepsilon) > 1 - \varepsilon$. Since the measure μ is two-valued, it follows that $\mu(K_\varepsilon) = 1$; moreover, there exists an element $f_\varepsilon \in \{f_1, \dots, f_m\}$ such that $\mu(\overline{O}_\varepsilon(f_\varepsilon)) = 1$. Consequently, there exists a sequence of nested closed balls on which the measure μ takes the unit value. Hence, the nonprincipal ultrafilter $\mathcal{F}_\mu$ converges to a point on $S_1(\mathcal{H})$ in the norm of the space $\mathcal{H}$. □

3. EXTREME POINTS OF THE UNIT BALL OF A NORMED IDEAL SPACE ON $\mathcal{H}$

In this section, we establish a unitarity criterion for an arbitrary element of a unital C*-algebra and analyze the properties of the sets of extreme points of the unit balls of NISs on $\mathcal{H}$.

Proposition 3.1. *Let $\mathcal{A}$ be a unital* C*-*algebra. For an arbitrary element $A \in \mathcal{A}$, the following conditions are equivalent*:

(i) $A \in \mathcal{A}^{-1}$ *and* $A, A^{-1} \in \mathcal{A}^1$;

(ii) $A \in \mathrm{extr}(\mathcal{A}^1)$ *and* $\mathrm{dist}(A, \mathcal{A}^{-1}) < 1$;

(iii) $A \in \mathcal{A}^{\mathrm{u}}$.

Proof. Let I be the unit of the algebra $\mathcal{A}$ and $A \in \mathcal{A}$.

(i) ⇒ (iii). It is clear that $|A|^2 \le |A| \le I$. We have

$$I = |AA^{-1}| = \sqrt{(A^{-1})^* A^* A A^{-1}} \ge (A^{-1})^* \sqrt{A^* A}\, A^{-1} = (A^{-1})^* |A| A^{-1}$$

by Hansen's inequality [12] for the operator monotone function $f(t) = \sqrt{t}$, $t \ge 0$. Multiplying both sides of the inequality $I \ge (A^{-1})^* |A| A^{-1}$ on the left by A^* and on the right by A and using the equality $(A^*)^{-1} = (A^{-1})^*$, we obtain $|A|^2 \ge |A|$. Thus, $|A|^2 \ge |A| \ge |A|^2$ and $|A|^2 = |A|$; i.e., the element $|A|$ is an orthogonal projection. Since $|A|$ is an invertible projection, we have $|A| = I$. In a similar way one can show that $|A^*| = I$.

(iii) ⇒ (i). For any element $A \in \mathcal{A}^{\mathrm{u}}$, we have $A^{-1} = A^*$; therefore, $\|A^{-1}\| = \|A^*\| = \|A\|$.

(ii) ⇒ (iii). For $A \in \mathrm{extr}(\mathcal{A}^1)$, we have $A \in \mathcal{A}^{\mathrm{u}}$ if and only if $\mathrm{dist}(A, \mathcal{A}^{-1}) < 1$ (see [6, proposition]).

(iii) ⇒ (ii). We have the equality $\mathrm{extr}(\mathcal{A}^1) = \{V \in \mathcal{A} \colon (I - V^*V)\mathcal{A}(I - VV^*) = \{0\}\}$ (see [17, Ch. I, Theorem 10.2(ii)]). ☐

Lemma 3.1. *Let $\langle \mathcal{E}, \|\cdot\|_{\mathcal{E}} \rangle$ be a NIS on $\mathcal{H}$, and let $A \in \mathcal{E}$ and $X, Y \in \mathcal{B}(\mathcal{H})$ be operators. Then $XAY \in \mathcal{E}$ and $\|XAY\|_{\mathcal{E}} \le \|X\| \cdot \|Y\| \cdot \|A\|_{\mathcal{E}}$.*

Proof. We have $X^*X \le \|X\|^2 I$. Hence,

$$|XA|^2 = A^* \cdot X^*X \cdot A \le A^* \cdot \|X\|^2 I \cdot A = \|X\|^2 A^* A$$

and $|XA| \le \|X\| \cdot |A|$ since the function $f(t) = \sqrt{t}$, $t \ge 0$, is operator monotone. Next, notice that $\|AY\|_{\mathcal{E}} = \|Y^*A^*\|_{\mathcal{E}}$. ☐

Proposition 3.2. *Let $\langle \mathcal{E}, \|\cdot\|_{\mathcal{E}} \rangle$ be a NIS on $\mathcal{H}$, an operator $A \in (\mathcal{B}(\mathcal{H}))^1$ be left (or right) invertible, and $A_{\mathrm{l}}^{-1} \in \mathcal{E}^1$ (respectively, $A_{\mathrm{r}}^{-1} \in \mathcal{E}^1$). Then the operator A lies in $\mathcal{E}^1$. In this case, if the operator A is invertible and $I \in \mathrm{extr}(\mathcal{E}^1)$, then $A^{-1} \in \mathrm{extr}(\mathcal{E}^1)$.*

Proof. Since the operator A^2 lies in $(\mathcal{B}(\mathcal{H}))^1$, we have $A = A^2 A_{\mathrm{l}}^{-1} \in \mathcal{E}^1$ (or $A = A_{\mathrm{r}}^{-1} A^2 \in \mathcal{E}^1$, respectively) by Lemma 3.1.

If $A^{-1} \notin \mathrm{extr}(\mathcal{E}^1)$, then $A^{-1} = \frac{1}{2}(S + T)$ with some operators $S, T \in \mathcal{E}^1$, $S \ne T$. Then $I = A^{-1}A = \frac{1}{2}(SA + TA)$, where $SA, TA \in \mathcal{E}^1$ (see Lemma 3.1). Let us show that $SA \ne TA$. Assuming that $SA = TA$, we obtain $S = SA \cdot A^{-1} = TA \cdot A^{-1} = T$, a contradiction. ☐

Proposition 3.3. *Let $\langle \mathcal{E}, \|\cdot\|_{\mathcal{E}} \rangle$ be a NIS on $\mathcal{H}$, and let an operator $A \in \mathrm{extr}(\mathcal{E}^1)$ and a positive integer $n \ge 2$ be such that the operator $B := A^{n-1}$ is invertible and $B^{-1} \in (\mathcal{B}(\mathcal{H}))^1$. If $A^n \in \mathcal{E}^1$, then $A^n \in \mathrm{extr}(\mathcal{E}^1)$.*

Proof. If $A^n \notin \mathrm{extr}(\mathcal{E}^1)$, then $A^n = \frac{1}{2}(S + T)$ with some operators $S, T \in \mathcal{E}^1$, $S \ne T$. Then $A = A^n B^{-1} = \frac{1}{2}(SB^{-1} + TB^{-1})$, where $SB^{-1}, TB^{-1} \in \mathcal{E}^1$ (see Lemma 3.1). Let us show that $SB^{-1} \ne TB^{-1}$. Assuming that $SB^{-1} = TB^{-1}$, we obtain $S = SB^{-1} \cdot B = TB^{-1} \cdot B = T$, a contradiction. ☐

Theorem 3.1. *Let $\langle \mathcal{E}, \|\cdot\|_{\mathcal{E}} \rangle$ be a NIS on $\mathcal{H}$ and an operator $A \in (\mathcal{B}(\mathcal{H}))^1 \cap \operatorname{extr}(\mathcal{E}^1)$ be invertible. Then $\mathcal{E} = \mathcal{B}(\mathcal{H})$ and*

(i) *if $A^{-1} \in \mathcal{E}^1$, then $A^{-1} \in \operatorname{extr}(\mathcal{E}^1)$;*

(ii) *if $A = U|A|$ is the polar decomposition and $U \in \mathcal{E}^1$, then $U \in \operatorname{extr}(\mathcal{E}^1)$.*

Proof. If the NIS $\langle \mathcal{E}, \|\cdot\|_{\mathcal{E}} \rangle$ contains a left (or right) invertible operator, then $I \in \mathcal{E}$ by Lemma 3.1; i.e., $\mathcal{E}$ coincides with $\mathcal{B}(\mathcal{H})$.

(i) If $A^{-1} \notin \operatorname{extr}(\mathcal{E}^1)$, then $A^{-1} = \frac{1}{2}(S+T)$ with some operators $S, T \in \mathcal{E}^1$, $S \neq T$. Then $A = AA^{-1}A = \frac{1}{2}(ASA + ATA)$, where $ASA, ATA \in \mathcal{E}^1$ (see Lemma 3.1). Let us show that $ASA \neq ATA$. Assuming that $ASA = ATA$, we obtain $S = A^{-1} \cdot ASA \cdot A^{-1} = A^{-1} \cdot ATA \cdot A^{-1} = T$, a contradiction.

(ii) Since $\mathcal{E} = \mathcal{B}(\mathcal{H})$, we have $U \in \mathcal{E}$. If $U \notin \operatorname{extr}(\mathcal{E}^1)$, then $U = \frac{1}{2}(S+T)$ with some operators $S, T \in \mathcal{E}^1$, $S \neq T$. Then $A = U|A| = \frac{1}{2}(S|A| + T|A|)$, where $S|A|, T|A| \in \mathcal{E}^1$ (see Lemma 3.1). The operator $|A|$ is also invertible. Suppose that $S|A| = T|A|$. Then $S = S|A| \cdot |A|^{-1} = T|A| \cdot |A|^{-1} = T$, a contradiction. $\square$

Theorem 3.2. *Let $\langle \mathcal{E}, \|\cdot\|_{\mathcal{E}} \rangle$ be a NIS on $\mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$. Then*

(i) *$A \in \operatorname{extr}(\mathcal{E}^1) \ \Leftrightarrow \ VAW \in \operatorname{extr}(\mathcal{E}^1)$ for all unitary operators $V, W \in \mathcal{B}(\mathcal{H})$;*

(ii) *if the space $\mathcal{H}$ is finite-dimensional, then $A \in \operatorname{extr}(\mathcal{E}^1) \ \Leftrightarrow \ |A| \in \operatorname{extr}(\mathcal{E}^1)$.*

Proof. (i) Let $A \in \operatorname{extr}(\mathcal{E}^1)$ and $VAW = \frac{1}{2}(S+T)$ with some operators $S, T \in \mathcal{E}^1$, $S \neq T$. Then

$$A = V^* \cdot VAW \cdot W^* = \frac{V^*SW^* + V^*TW^*}{2}, \qquad \text{where} \quad V^*SW^*, V^*TW^* \in \mathcal{E}^1$$

(see Lemma 3.1). Let us show that $V^*SW^* \neq V^*TW^*$. Assuming that $V^*SW^* = V^*TW^*$, we obtain $S = V \cdot V^*SW^* \cdot W = V \cdot V^*TW^* \cdot W = T$, a contradiction.

(ii) Let $A \in \operatorname{extr}(\mathcal{E}^1)$ and $U \in \mathcal{B}(\mathcal{H})$ be a unitary operator such that $A = U|A|$. If $|A| = \frac{1}{2}(S+T)$, where $S, T \in \mathcal{E}^1$, then $US, UT \in \mathcal{E}^1$ by Lemma 3.1, and $A = \frac{1}{2}(US + UT)$. Therefore, $A = US = UT$. Multiplying these equalities by U^* on the left, we obtain $|A| = S = T$; i.e., $|A| \in \operatorname{extr}(\mathcal{E}^1)$. The reverse implication can be verified similarly. $\square$

Corollary 3.1. *Let $\langle \mathcal{E}, \|\cdot\|_{\mathcal{E}} \rangle$ be a NIS on $\mathcal{H}$. If $U \in \operatorname{extr}(\mathcal{E}^1)$ for some unitary operator $U \in \mathcal{B}(\mathcal{H})$, then $V \in \operatorname{extr}(\mathcal{E}^1)$ for all unitary operators $V \in \mathcal{B}(\mathcal{H})$.*

Proof. In assertion (i) of Theorem 3.2, we set $A = U$, $V = V$, and $W = U^*$. Thus, we see that if $(\mathcal{B}(\mathcal{H}))^{\mathrm{u}} \cap \operatorname{extr}(\mathcal{E}^1) \neq \varnothing$, then $(\mathcal{B}(\mathcal{H}))^{\mathrm{u}} \subseteq \operatorname{extr}(\mathcal{E}^1)$. $\square$

If a C*-algebra $\mathcal{A}$ is unital and the set $\mathcal{A}^{-1}$ is dense in $\mathcal{A}$, then $\operatorname{extr}(\mathcal{A}^1) = \mathcal{A}^{\mathrm{u}}$ (see [6, p. 100]). Suppose that the space $\mathcal{H}$ is infinite-dimensional. The hereditary C*-subalgebra $\mathcal{C}(\mathcal{H})$ of compact operators in $\mathcal{B}(\mathcal{H})$ is not unital; therefore, $\operatorname{extr}((\mathcal{C}(\mathcal{H}))^1) = \varnothing$ by [17, Ch. I, Theorem 10.2(i)]; $\langle \mathcal{C}(\mathcal{H}), \|\cdot\| \rangle$ is a NIS on $\mathcal{H}$. For the sets of extreme points of the unit balls of some specific NISs on $\mathcal{H}$, see [10] and references therein.

The proofs of the following two statements are similar to those of Theorem 3.2 and Corollary 3.1.

Theorem 3.3. *Let $\langle \mathcal{E}, \|\cdot\|_{\mathcal{E}} \rangle$ be a NIS on $\mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$. Then*

(i) *$A \in \operatorname{extr}(S_1(\mathcal{E})) \ \Leftrightarrow \ VAW \in \operatorname{extr}(S_1(\mathcal{E}))$ for all unitary operators $V, W \in \mathcal{B}(\mathcal{H})$;*

(ii) *if the space $\mathcal{H}$ is finite-dimensional, then $A \in \operatorname{extr}(S_1(\mathcal{E})) \ \Leftrightarrow \ |A| \in \operatorname{extr}(S_1(\mathcal{E}))$.*

Corollary 3.2. *Let $\langle \mathcal{E}, \|\cdot\|_{\mathcal{E}} \rangle$ be a NIS on $\mathcal{H}$. If $U \in \operatorname{extr}(S_1(\mathcal{E}))$ for some unitary operator $U \in \mathcal{B}(\mathcal{H})$, then $V \in \operatorname{extr}(S_1(\mathcal{E}))$ for all unitary operators $V \in \mathcal{B}(\mathcal{H})$.*

4. QUANTUM CORRELATIONS GENERATED BY THE EXTREME POINTS OF THE SET OF SINGULAR STATES

Suppose that the algebra of all bounded operators $\mathcal{B}(\mathcal{H})$ is generated by some von Neumann factor $\mathcal{M}$ and its commutant $\mathcal{M}'$, so that $\mathcal{B}(\mathcal{H}) = \mathcal{M} \vee \mathcal{M}'$. Suppose that there exist two sets of resolutions of the identity, $P^{(k)} = \big(P_j^{(k)}\big)$ and $Q^{(l)} = \big(Q_m^{(l)}\big)$, that belong to the factors $\mathcal{M}$ and $\mathcal{M}'$, respectively. Fix a state $\rho \in (\mathcal{B}(\mathcal{H}))^*$. Then we can define a matrix of quantum correlations by the formula

$$\alpha_{jm}^{(kl)} = \rho\big(P_j^{(k)} Q_m^{(l)}\big). \tag{4.1}$$

Since the orthogonal projections from the sets $P_j^{(k)}$ and $Q_m^{(l)}$ pairwise commute, we can identify the resolutions of the identity $P^{(k)}$ and $Q^{(l)}$ with quantum observables with discrete spectrum. In this case, the matrix (4.1) defines quantum correlations between observables. Studying the set of such quantum correlations is a complicated mathematical problem related to the solution of old hypotheses in the theory of operator algebras [13, 2]. Our goal is to define quantum correlations in the case when ρ is a singular quantum state defined by a nonprincipal ultrafilter.

Let $\mathcal{F}$ be a nonprincipal ultrafilter on the set of positive integers $\mathbb{N}$ defined by a two-valued measure ν, let $e = (e_n)$ be an orthonormal basis in the Hilbert space $\mathcal{H}$, and let $\rho_{\mathcal{F},e}$ be an extreme point of the set of singular states which is defined by

$$\rho_{\mathcal{F},e}(A) = \int\limits_{\mathbb{N}} (Ae_n, e_n)\, d\nu(n). \tag{4.2}$$

Definition 4.1. An ultrafilter $\mathcal{F}$ is said to have a *base* consisting of sets X_n if for any $X \in \mathcal{F}$ there exists an $X_n \subset X$.

Example. Everywhere below we use the ultrafilter $\mathcal{F}$ with a base consisting of the sets $X_n = \{2^n k\colon k \in \mathbb{N}\}$, $n \in \mathbb{N}$.

Definition 4.2. Denote by $\mathcal{P}_n$ the set of all projections P for which there exists an $X \in \mathcal{F}$ such that $|(Pe_k, e_k)| = 2^{-n}$, $k \in X$.

It is easy to see that $\rho_{\mathcal{F},e}(P) = 2^{-n}$, $P \in \mathcal{P}_n$.

Proposition 4.1. *In $\mathcal{P}_n$ there exist 2^n pairwise commuting projection-valued resolutions of the identity $P^{(k)}$ and $Q^{(l)}$,*

$$\sum_{j=1}^{2^n} P_j^{(k)} = \sum_{m=1}^{2^n} Q_m^{(l)} = I, \qquad 1 \le k, l \le 2^n,$$

with the property

$$\big(P_j^{(k)} Q_m^{(l)} e_s, e_s\big) = \frac{1}{4^n}, \qquad s \in X,$$

where X is the set from Definition 4.2.

Proof. One can easily construct a family of projections with the required properties in the Hilbert space $\mathbb{C}^{4n}$. Indeed, to construct the matrices, one should use 2^n mutually unbiased orthonormal bases, so that their diagonal entries are equal to $\frac{1}{2^n}$. It is known that there exist $N+1$ families of projections of the required type in a space of dimension $N = 2^n$. The only basis for which the corresponding projection matrices have ones or zeros on their diagonals is not used. Let us demonstrate this for $n = 1$. In this case, we have to construct 4×4 matrices. Set

$$P_{1,2}^{(1)} = \begin{pmatrix} \frac{1}{2} & \pm\frac{1}{2} & 0 & 0 \\ \pm\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \pm\frac{1}{2} \\ 0 & 0 & \pm\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad Q_{1,2}^{(1)} = \begin{pmatrix} \frac{1}{2} & 0 & \pm\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \pm\frac{1}{2} \\ \pm\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \pm\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix},$$

$$P_{1,2}^{(2)} = \begin{pmatrix} \frac{1}{2} & \pm\frac{i}{2} & 0 & 0 \\ \pm\frac{i}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \pm\frac{i}{2} \\ 0 & 0 & \pm\frac{i}{2} & \frac{1}{2} \end{pmatrix}, \qquad Q_{1,2}^{(2)} = \begin{pmatrix} \frac{1}{2} & 0 & \pm\frac{i}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \pm\frac{i}{2} \\ \pm\frac{i}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \pm\frac{i}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

For spaces of larger dimension, one should take all possible tensor products of the matrices $P_j^{(l)}$ and $Q_m^{(k)}$, $1 \le j, m, l, k \le 2$. Now, in the infinite-dimensional space $\mathcal{H}$, it suffices to take the representation of the obtained resolutions of the identity that is an infinite direct sum of projections in $\mathbb{C}^{4n}$. $\square$

Consider the Gelfand–Naimark–Segal representation π associated with the state $\rho_{\mathcal{F},e}$ in the Hilbert space $\mathfrak{H}$, in which the state $\rho_{\mathcal{F},e}$ becomes a vector state, so that

$$\rho_{\mathcal{F},e}(A) = (\pi(A)\Omega, \Omega)_{\mathfrak{H}}, \qquad A \in \mathcal{B}(\mathcal{H}), \quad \Omega \in \mathfrak{H}, \quad \|\Omega\| = 1.$$

Denote by $\mathcal{M}_F$ the von Neumann algebra generated by the resolutions of the identity $\pi(P^{(k)})$, where $P^{(k)}$ are as constructed in the proof of Proposition 4.1.

Theorem 4.1. *The von Neumann algebra* $\mathcal{M}_F$ *is a factor of type* II_1, *and the state on it defined by the vector* Ω *is a trace.*

Proof. As already mentioned, the sets of projections $P^{(k)}$ can be identified with quantum observables. Such observables can be obtained by taking a linear combination of projections with some coefficients that define the spectrum of the observable. In particular, if the spectrum is given by the numbers $\lambda = \epsilon_1 \dots \epsilon_k$ with ϵ_j chosen from among one of the two values of $\sqrt{-1}$, then as an observable we obtain an operator unitarily equivalent to the tensor product of k Pauli matrices in the space $\mathbb{C}^2$. Thus, we can assume that the factor $\mathcal{M}_F$ is generated not by resolutions of the identity but by unitary operators that are tensor products of Pauli operators. The product (composition) of such operators does not take us beyond this class. It remains to notice that the restriction of the state $\rho_{\mathcal{F},e}$ to an operator unitarily equivalent to the tensor product of Pauli matrices in the above sense gives zero if at least one cell contains a matrix unitarily equivalent to σ_x or σ_y, and gives ± 1 otherwise. Such a state is a trace state. The factor property of the algebra $\mathcal{M}_F$ is inherited from the similar property in a finite-dimensional space. $\square$

Remark 4.1. A similar statement can be proved for the factor $\mathcal{M}'_F$ generated by the resolutions of the identity $Q^{(l)}$.

Remark 4.2. In [4, 5], the authors conjectured that all extreme points of the set of states have the form (4.2). A disproof of this conjecture assuming the continuum hypothesis was given in [1].

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CONFLICT OF INTEREST

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