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Conformal Radius: At the Interface of Traditions

A. V. Kazantsev*

(Submitted by A. M. Elizarov)

Kazan (Volga Region) Federal University, ul. Kremlevskaya 18, Kazan, 420008 Russia

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Abstract—H. Behnke's and E. Peschl's definition of *plänarkonvexitat* leads to the Epstein-type inequalities when applies to the Hartogs domains in $\mathbb{C}^2$. One-parameter series of such inequalities reveals the following rigidity phenomenon: the set of the parameters with contensive inequalities is exactly the segment which center corresponds to the well-known Nehari ball. The latter plays the crucial role in the forming the Gakhov class of all holomorphic and locally univalent functions in the unit disk with no more than one-pointed null-sets of the gradients of their conformal radii. The sufficient condition for the piercing of the Nehari sphere out of the Gakhov class is found. We deduce such a condition along the lines of the subordination approach to the proof of Haegi's theorem about the inclusion of any convex holomorphic function into the Gakhov class.

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This sketch was intended as a (quite superficial, but sufficiently post-modernistic) attempt to grope for the hidden interdependence between two "worlds" generically connected with the ideas of E. Peschl: linear convexity (of the domains in $\mathbb{C}^2$ or $\mathbb{P}^2$; [1]) and differential invariants (of the coverings over $\mathbb{C}$; [2, 3]). As the tools of such a "romanticizing" the results of the works [4–7] are used; the "underlying landscape" is provided by the paper [8] executed along the Gakhov tradition [9] which has been "engrafted upon a trunk of the conformal radius" by L.A. Aksent'ev in [10]. The note [10] absorbed also the ideas and findings of Ju.E. Hohlov, S.R. Nasyrov, F.F. Mayer and M.I. Kinder (see [11–15]).

The Hartogs domains appear here as a sort of some "link" between the "worlds" above mentioned, and the naturalness of this appearance is justified by the Riemann Mapping Theorem. The latter states the existence of the function $w = F(Z, z)$ holomorphic and univalent with respect to Z in the hyperbolic domain D, and such that $F(z, z) = 0$, $F_Z(z, z) = 1$.

The function F generates the one-to-one correspondence $(Z, z) \mapsto (F(Z, z), z)$ between the product $D \times D$ and the Hartogs domain $\{(w, z) \in \mathbb{C} \times D : |w| < R(z)\}$ where $R(z)$ is the inner mapping, or *conformal radius* of the domain D at its point z ([16], Bd. 2, Abschn. 4, Kap. 2, [17]; see also [18]). The holomorphic parametrization, $f : \mathbb{D} \to D$, of the domain D by the unit disk $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ gives the well-known expression for $R(f(\zeta)) = h_f(\zeta)$,

$$h_f(\zeta) = (1 - |\zeta|^2)|f'(\zeta)|. \tag{1}$$

It is convenient to call the quantity (1) by the *hyperbolic derivative* of the function f holomorphic in $\mathbb{D}$ (see [8]). It should be noted that the logarithm of (1) is exactly the invariant α or δ from [3] or [2], respectively.

For "our version of the definition" of the linear convexity we are needed in the expressions for the real Hessian of the function of n complex variables,

$$\mathrm{Hess} r(Z)\xi = 2\sum_{j,k=1}^{n} r_{Z_j\bar{Z}_k}(Z)\xi_j\bar{\xi}_k + 2\mathrm{Re}\left\{\sum_{j,k=1}^{n} r_{Z_jZ_k}(Z)\xi_j\xi_k\right\},$$

*E-mail: avkazantsev63@gmail.com

and for the complex tangent space,

$$T_Z^{\mathbb{C}}(\partial G) = \left\{ \xi \in \mathbb{C}^n : \sum_{j=1}^{n} r_{Z_j}(Z)\xi_j = 0 \right\},$$

to the boundary ∂G of the domain $G = \{W \in \mathbb{C}^n : r(W) < 0\}$ at a point $Z \in \partial G$.

In order to avoid the "ritual" discussion on the correspondences and interdependences between the various definitions of the linear convexity together with their strengthening and weakening (however, see, for instance, [19–21]), we introduce "our" term and give its definition (suitable also in the general case) at once for the Hartogs domains in $\mathbb{C}^2$.

Let D be a Riemann surface having the hyperbolic universal covering surface, i.e. there exists a holomorphic parametrization $f : \mathbb{D} \to D$. By a Hartogs domain over D we mean a domain of the form

$$H = \{(z, w) \in D \times \mathbb{C} : |w| < \Omega(z)\} \tag{2}$$

where the function $\Omega \in C^2(D)$ is positive and satisfies the inequality $(\log \Omega)_{z\bar{z}} < 0$ in D (i.e. the domain H is strictly pseudoconvex). We shall refer to the domain H as to the *locally non-strictly linear convex domain over D* if $\mathrm{Hess} r(z, w)(\lambda, \mu) \geq 0$ for any $(z, w) \in r^{-1}(0) \cap (D \times \mathbb{C})$ and for any $(\lambda, \mu) \in T^{\mathbb{C}}_{(z,w)}(\partial H)$. Our choice of the defining inequality $r(z, w) < 0$ for H corresponds to the function $r(z, w) = \log |w| - \log \Omega(z)$, so we have

$$\frac{1}{2}\mathrm{Hess} r(z, w)(\lambda, \mu) = -(\log \Omega)_{z\bar{z}}|\lambda|^2 - \mathrm{Re}\left\{ \frac{1}{2w^2}\mu^2 + (\log \Omega)_{zz}\lambda^2 \right\}$$

and

$$T^{\mathbb{C}}_{(z,w)}(\partial H) = \left\{ (\lambda, \mu) \in \mathbb{C}^2 : \frac{\mu}{w} = 2(\log \Omega)_z \lambda \right\}.$$

Excluding the intermediate divisions by w, we rewrite our definition as follows:

$$\mathrm{Re}\left\{ \left[(\log \Omega)_{zz} + 2(\log \Omega)_z^2 \right] \lambda^2 \right\} \leq -(\log \Omega)_{z\bar{z}}|\lambda|^2, \quad z \in D, \quad \lambda \in \mathbb{C},$$

or, equivalently,

$$\left| (\log \Omega)_{zz} + 2(\log \Omega)_z^2 \right| \leq -(\log \Omega)_{z\bar{z}}, \quad z \in D. \tag{3}$$

The change

$$\Omega = \sqrt{R/e^s} \tag{4}$$

($R = R(z)$ is the conformal radius) and the subsequent reduction to the unit disk by means of $z = f(\zeta)$, $\sigma := s(f(\zeta))$, transform the estimate (3) into Epstein inequality [4]

$$\left| \sigma_{\zeta\zeta} - \sigma_\zeta^2 - \frac{1}{2}S_f(\zeta) - \frac{2\bar{\zeta}}{1 - |\zeta|^2}\sigma_\zeta \right| \leq \sigma_{\zeta\bar{\zeta}} + \frac{1}{(1 - |\zeta|^2)^2}, \quad \zeta \in \mathbb{D}, \tag{5}$$

guaranteeing the univalence of the function f under the certain additional conditions ([4], see also[5]). Here, as usual, $S_f = (f''/f')' - (f''/f')^2/2$ is the Schwarzian derivative of the function f. So, we have the following

Theorem 1. *If the domain* (2) *is locally non-strictly linear convex over D, then any holomorphic covering $f : \mathbb{D} \to D$ is Epsteinian in the sense of* (5).

Let's consider the choice $s = \log R^{1-2\beta}$ in (4) where $\beta \in \mathbb{R}$, i.e. $\Omega = R^\beta$. The substitution such an s in (3) implies $\beta > 0$, and the reduction to $\mathbb{D}$ extracts the class $\mathcal{N}(\beta)$ of all holomorphic functions $f(\zeta) = \zeta + \ldots$ in $\mathbb{D}$ satisfying the condition

$$\left| S_f(\zeta) + \left(\beta - \frac{1}{2}\right)\left(\frac{f''}{f'}(\zeta) - \frac{2\bar{\zeta}}{1 - \zeta\bar{\zeta}}\right)^2 \right| \leq \frac{2}{(1 - \zeta\bar{\zeta})^2}, \quad \zeta \in \mathbb{D}. \tag{6}$$

It should be noted that for the holomorphic functions f in $\mathbb{D}$ the estimate (6) implies the property of local univalence in the unit disk, i.e. the inequality $f'(\zeta) \neq 0$ at any point $\zeta \in \mathbb{D}$.

An interesting rigidity effect is revealed by the following

Theorem 2. *If $\beta \in [0,1]$, then the class $\mathcal{N}(\beta)$ is the linear-invariant family of the order* $\operatorname{ord}(\mathcal{N}(\beta)) \le (1-\beta)^{-1/2}$, *and contains S^0, the class of all convex functions in $\mathbb{D}$. The classes $\mathcal{N}(\beta)$ are empty when $\beta \notin [0,1]$.*

The sketch of the proof (see [8]). The non-existence of the function $f \in \mathcal{N}(\beta)$ with

$$\operatorname{ord} f := \sup_{\zeta\in\mathbb{D}} \left| -\bar{\zeta} + \frac{1-|\zeta|^2}{2}\frac{f''}{f'}(\zeta) \right| = +\infty \tag{7}$$

is established in the spirit of [7] by the convergence $\operatorname{ord} f_r \to \operatorname{ord} f$, $r \to 1-$, where $f_r(\zeta) = f(r\zeta)/r$ and $\beta \ne 1$. Assumption (7) leads to the violation of the inequality (6) writing in terms of the second and third coefficients of the linear-invariant actions on f by Möbius automorphisms of the unit disk. The above violation occurs at the expense of the relation between these coefficients ([6], Theorem 2.3a) which we shall use now to prove the "rigidity" of the series $\mathcal{N}(\beta)$ with respect to the parameter.

So, let $\beta \ne 1$, let the class $\mathcal{N}(\beta)$ contains the function f with $\operatorname{ord} f = \alpha$ ($< +\infty$ by the just proved), and let $\mathfrak{A}_\alpha$ be the universal linear-invariant family of order α. Since the intersection $\mathcal{N}(\beta) \cap \mathfrak{A}_\alpha$ is compact, there exists a function $g(\zeta) = \zeta + a_2\zeta^2 + a_3\zeta^3 + \ldots \in \mathcal{N}(\beta) \cap \mathfrak{A}_\alpha$ with $a_2 = \alpha$. Then we have $a_3 = (2\alpha^2+1)/3$ by the above mentioned Theorem 2.3a from [6]. Thus the inequality (6) implies the relation $\left|2(\beta-1)\alpha^2 + 1\right| \le 1$, whence $0 \le (1-\beta)\alpha^2 \le 1$, and, as the result, $0 \le \beta \le 1$ with regard to $\alpha \ge 1$. In passing we have established the required estimate for the order of $\mathcal{N}(\beta)$.

The inclusion $S^0 \subset \mathcal{N}(\beta)$, $\beta \in [0,1]$, may be proved, e.g., by the use of the well-known inequality $|b_3 - b_2^2| \le (1-|b_2|^2)/3$ for the functions $f(\zeta) = \zeta + b_2\zeta^2 + b_3\zeta^3 + \ldots \in S^0$ [22] (the traditional reference is [23]) providing the fulfillment of the coefficient form of (6).

Hypothesis. If the Hartogs domain (2) is linear convex over D (in the suitable version of the definition of the linear convexity), then any holomorphic covering of the Riemann surface (Riemann domain over $\mathbb{C}^n$ when $n \ge 2$) D is univalent.

Classes $\mathcal{N}(\beta)$, $\beta \in [0,1]$, inherit a number of properties from their "center" which is the well-known Nehari class $\mathcal{N}(1/2)$ [24] (see theorems 2 and 3 in [25]). In particular, we have the following

Theorem 3. *Let $\beta \in [0,1]$ and $f \in \mathcal{N}(\beta)$. If $f(\mathbb{D}) \ne strip$, then the hyperbolic derivative* (1) *of the function f has no more than one critical point in $\mathbb{D}$.*

Proof. See [8]. □

For no one to think the author is only able to "glide" through the work [8], we dwell on the problem of the uniqueness of the critical point of (1) more explicitly.

This problem occured in [16] and was issued in [17]. S. Peschl in [2] and [3] "has passed on a tangent to it", but his "descendants", [26], established the equivalence

$$\nabla h_f(\zeta) = 0 \;\Leftrightarrow\; \frac{f''(\zeta)}{f'(\zeta)} = \frac{2\bar{\zeta}}{1-|\zeta|^2}, \tag{8}$$

where the relation on the right-hand side is called the *Gakhov equation* (see [10]).

The null-set of the gradient of the function (1), i.e. the set of the form

$$M_f = \{\zeta \in \mathbb{D} : \nabla h_f(\zeta) = 0\}, \tag{9}$$

appeared in some boundary value problems of mathematical physics and PDE (see the survey [27]). However, the frontal study of the sets (9) for the functions (1) took place in the frame of the boundary value problems for the analytic functions ([9; 28, § 33]) and started by the works of Kazan mathematicians ([10–15, 29, 30]). The source of inspiration was the fact observed in [10] that the connection (8) was essentially known to Gakhov [9], who used it to prove the solvability of his version of the inverse boundary value problem (cf. canonical references [28, § 33], and [31, § 3]).

It was an epochal result [25, 30] that the set M_f is empty or a singleton for any function $f \in \mathcal{N}(1/2) \setminus \{f(\mathbb{D}) = strip\}$ (the case $f(\mathbb{D}) = strip$ corresponds to M_f is a continuum). Later on this result was reproved in the articles [32, 18] and [33], but the proof in [33] closely resembles the adapted version (without the reference!) of the segment on pp. 397–398 of the well-known paper of O. Martio and J. Sarvas [34]. The significant resemblance just mentioned is well romanticized: one of the most intriguing topics in the geometric function theory during the last 25 years (see [33, 5, 35], etc.) gains a hint at its own original sin and, moreover, its own "mysteriology": it is quite possible that an absent structure (in Eco's terminology [36]) of the intrigue could be hidden in the dramatic events at Kazan Seminar on the Geometric Function Theory in 1980's (see [37]; cf. [12] and [13] as [12] *versus* [13]).

Now we restrict our space of attention (in the sense of [38]) to some questions on a simultaneous going out of the class $\mathcal{N}(1/2)$ and of the Gakhov set, or class $\mathcal{G}$ which is the class of all holomorphic and locally univalent functions f in $\mathbb{D}$ with M_f is no more than a singleton [37]. It is quite easy to show that if $f \in \mathcal{N}(1/2) \setminus \{f(\mathbb{D}) = strip\}$ and $M_f \neq \varnothing$, then the unique element of M_f is a maximum of the function (1). Our task here is to construct a workable version of the above simultaneity. Moreover, we want to immerse this task into the atmosphere of the article [10] keeping the charisma of the Soviet "la Belle Epoque".

We use the notations $\rho_f(\zeta) = (1-|\zeta|^2)^2|S_f(\zeta)|$ and $||S_f|| = \sup_{\zeta\in\mathbb{D}} \rho_f(\zeta)$. Finiteness of $||S_f||$ is the criterion of the local univalence for holomorphic f in $\mathbb{D}$ (cf. remark after the inequality (6)). For the sake of convenience we shall work with the class H_0 of all holomorphic and locally univalent functions f in $\mathbb{D}$ normalized by $f(0) = f'(0) - 1 = 0$. As is well-known, the Nehari condition, $f \in \mathcal{N}(1/2)$, is equivalent to the estimate $||S_f|| \leq 2$ where the constant 2 is sharp for the inclusion $f \in \mathcal{G}$ ([29, 30, 18]). Therefore, our case is

$$||S_f|| > 2, \tag{10}$$

and we shall study how the set M_{f_r} lose its one-pointed status for the so-called *level set family*

$$f_r(\zeta) = f(r\zeta)/r, \tag{11}$$

where r varies over the interval $0 < r < 1$. The framework for our study is given by the following

Definition. Let us say that a family (11) with $f \in H_0$ and (10) *pierce* the Nehari sphere $\mathbb{S} = \{h \in H_0 : ||S_h|| = 2\}$ into the complement $H_0 \setminus \mathcal{G}$ to the Gakhov class $\mathcal{G}$ at the level set time $r = r_0$ through the point $a \in M_{f_{r_0}}$, if the following conditions are fulfilled simultaneously:

$$||S_{f_{r_0}}|| = \rho_{f_{r_0}}(a) = 2 \quad \text{and} \quad g'_{r_0}(a) = 0, \tag{12}$$

where $g_r(\zeta) = g(r\zeta)$, $g(\zeta) = \zeta f''(\zeta)/f'(\zeta)$.

The correctness of this definition is based on the fact that the norm $||S_{f_r}||$ is continuous and monotone increasing in $r \in (0,1)$. Inequalities $M_{f_{r_0}} \neq \varnothing$ and $f_{r_0} \neq strip$ take place due to the relation $r_0 < 1$ following from (10). The epoch-making result above mentioned (or the theorem 3 with $\beta = 1/2$) implies $M_{f_{r_0}} = \{a\}$. The descent of the conditions (12) goes back to the paper [39] (see also [18] and [8]).

When $a = 0$ $(f''(0) = 0)$, these conditions may be reformulated as follows: $r_0 = r_N(f) = \sqrt{2/|S_f(0)|}$, where $r_N(f) := \sup\{r > 0 : f \in \mathcal{N}(1/2)\}$ is the "radius of Nehariсity" of the function $f \in H_0$. In fact, the equivalences $r_0 = r_N(f) \Leftrightarrow ||S_{f_{r_0}}|| = 2$ and $r_0 = \sqrt{2/|S_f(0)|} \Leftrightarrow \rho_{f_{r_0}}(0) \equiv r_0^2|S_f(0)| = 2$ are valid, and if $a = 0$, then the relation $g'_r(a) = 0$ is fulfilled for any $r \in (0,1)$. It remains to note that the quantity $r_N(f)$ is correctly defined for every $f \in H_0$ with (10) by the virtue of the continuity and the monotonicity of $||S_{f_r}||$ in $r \in (0,1)$ again. Furthermore, $f_{r_N(f)} \in \mathcal{N}(1/2)$, and $r = r_N(f)$ is the unique root of the equation $||S_{f_r}|| = 2$ in the r-interval $(0,1)$.

The first condition (12) means that $\zeta = a$ is a point of extremum (maximum) of the function $\rho_{f_{r_0}}(\zeta)$. Acting along the lines of derivation of the Gakhov equation (see [26, 10]), we obtain the extremum necessary condition for $\rho_{f_{r_0}}(\zeta)$,

$$S'_{f_{r_0}}(\zeta)/S_{f_{r_0}}(\zeta) = 4\bar{\zeta}/(1-|\zeta|^2), \tag{13}$$

which must be satisfied at the point $\zeta = a$. When $a = 0$, the condition (13) turns into equality

$$f^{(\mathrm{IV})}(0) = 0. \tag{14}$$

An exclusion of the parameter r is possible here due to the construction of (11).

The second condition (12) means that $(\zeta, r) = (a, r_0)$ is a bifurcation point for the Gakhov equation

$$f''_r(\zeta)/f'_r(\zeta) = 2\bar{\zeta}/(1-|\zeta|^2) \tag{15}$$

corresponding to the functions of the family (11) (see [18, 39]). We need the following particular case of the main lemma from the paper [18] (or the following extract of the theorems 1 and 2 in [39]).

Lemma. *Let $f \in H_0$, and let f_r, $r \in (0,1)$, be the family (11). Let $0 < r_0 < 1$. Suppose M_{f_r} is a singleton for any $r \in (0, r_0]$ with $M_{f_{r_0}} = \{a\}$, where $a \in \mathbb{D}$. Then the equation (15) has exactly three roots in $\mathbb{D}$ for every $r > r_0$ near r_0 if and only if $g'_{r_0}(a) = 0$ when $a \neq 0$, or if and only if $r_0 = \sqrt{2/|S_f(0)|}$ when $a = 0$.*

Now we give the simple sufficient condition for the piercing when $a = 0$:

Proposition. *Let the function $f \in H_0$ satisfies the conditions (10) and $f''(0) = f^{(IV)}(0) = 0$. Suppose that for any $\xi \in \partial\mathbb{D}$ the function $|S_{f_r}(\xi)|$ increases along with $r \in (0,1)$. Then the family (11) pierce the Nehari sphere $\mathbb{S}$ into the complement $H_0 \setminus \mathcal{G}$ to the Gakhov class $\mathcal{G}$ through the origin at the level set time $r_0 = r_N(f) = \sqrt{2/|S_f(0)|}$.*

Proof. We want to show that for any $r \in (0,1)$ the function $\rho_{f_r}(\zeta)$ decreases along the radii $\{\rho\xi : \rho \in (0,1)\}$, $|\xi| = 1$. Since $|S_{f_r}(\xi)| = r^2|S_f(r\xi)|$ is the increasing function in $r \in (0,1)$, the quantity $S_f(w)$ doesn't vanish on $\mathbb{D}$. So, we have

$$2 + \mathrm{Re} w\frac{S'_f(w)}{S_f(w)} = r\frac{\partial}{\partial r}\log|S_{f_r}(\xi)| > 0,$$

where $w = r\xi$, $r \in (0,1)$ and $|\xi| = 1$. Then there exists a function φ from the Schwarz lemma, such that

$$w\frac{S'_f(w)}{S_f(w)} = \frac{4\varphi(w)}{1 - \varphi(w)}, \quad w \in \mathbb{D},$$

and $|\varphi(w)| \le |w|^2$, $w \in \mathbb{D}$, due to (14). Therefore,

$$\left|w\frac{S'_f(w)}{S_f(w)}\right| \le \frac{4|w|^2}{1 - |w|^2}, \quad w \in \mathbb{D} \setminus \{0\}. \tag{16}$$

Now we substitute $w = r\zeta$ into (16) and use the obvious "scaling lows" $S_{f_r}(\zeta) = r^2 S_f(r\zeta)$ and $S'_{f_r}(\zeta) = r^3 S'_f(r\zeta)$ to get the following inequality

$$\mathrm{Re}\xi\frac{S'_{f_r}(\rho\xi)}{S_{f_r}(\rho\xi)} < \frac{4\rho}{1 - \rho^2}, \quad \rho = |\zeta| \in (0,1), \quad |\xi| = 1, \quad r \in (0,1).$$

Thus we have proved that for any $r \in (0,1)$ the function $\rho_{f_r}(\zeta)$ decreases along the radii of $\mathbb{D}$. Hence $||S_{f_r}|| = \rho_{f_r}(0) = r^2|S_f(0)|$, $r \in (0,1)$, $r_N(f) = \sqrt{2/|S_f(0)|}$, and the above lemma imply that $f_r \in H_0 \setminus \mathcal{G}$ when $r > r_N(f)$ near $r_N(f)$. So, the level set time is $r_0 = r_N(f) = \sqrt{2/|S_f(0)|}$, and the proof is complete. □

Remark. The class of f's defining by the assumptions of the above Proposition is non-empty (see, e.g., [29]).

Let us briefly examine the effect of *déjà vu* which has been laid in the situation. Transferring the corpus of last proof to the case of the function h_f instead of ρ_f, $f \in H_0$, we shall find that if $f''(0) = 0$, then the condition

$$\frac{\partial}{\partial r}|f'_r(\xi)| > 0, \quad \xi \in \partial\mathbb{D}, \quad r \in (0,1), \tag{17}$$

implies the decrease of the function h_{f_r} along the radii of $\mathbb{D}$ for any $r \in (0,1)$, and, moreover, for $r = 1$ with the only exclusion $f(\mathbb{D}) = strip$. It is easy to see that the condition (17) is equivalent to the convexity of a function f, i.e. $f \in S^0$. So, in the proof of the above Proposition one can recognize the elements of the subordination approach used in the spirit of L.A. Aksent'ev's proof [10] of Haegi's Theorem 4 [17] asserting the inclusion which gains now the form $S^0 \setminus \{f(\mathbb{D}) = strip\} \subset \mathcal{G}$.

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