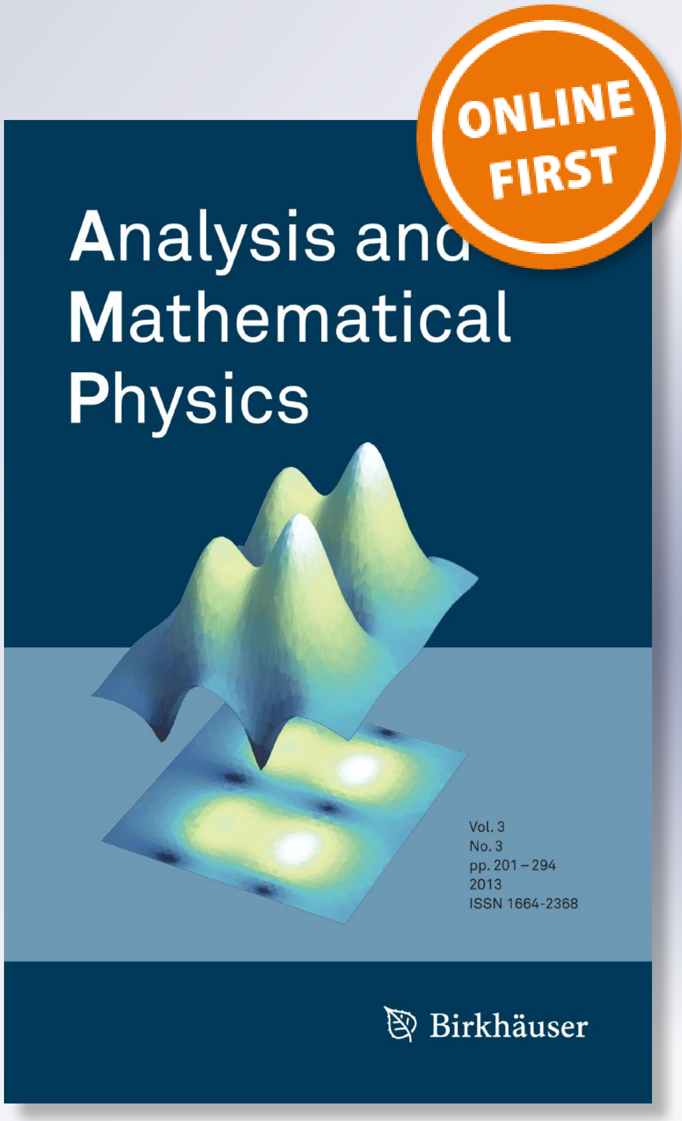
On Riemann boundary value problems for null solutions of the two dimensional Helmholtz equation

Juan Bory Reyes, Ricardo Abreu Blaya, Ramón Martin Rodríguez Dagnino & Boris Aleksandrovich Kats

Analysis and Mathematical Physics

ISSN 1664-2368

Anal.Math.Phys.
DOI 10.1007/s13324-018-0210-3



Springer

Anal.Math.Phys.
https://doi.org/10.1007/s13324-018-0210-3

On Riemann boundary value problems for null solutions of the two dimensional Helmholtz equation

Juan Bory Reyes[1] · **Ricardo Abreu Blaya**[2] · **Ramón Martin Rodríguez Dagnino**[3] · **Boris Aleksandrovich Kats**[4]

Received: 4 July 2017 / Accepted: 14 January 2018

Abstract The Riemann boundary value problem (RBVP to shorten notation) in the complex plane, for different classes of functions and curves, is still widely used in mathematical physics and engineering. For instance, in elasticity theory, hydro and aerodynamics, shell theory, quantum mechanics, theory of orthogonal polynomials, and so on. In this paper, we present an appropriate hyperholomorphic approach to the RBVP associated to the two dimensional Helmholtz equation in $\mathbb{R}^2$. Our analysis is based on a suitable operator calculus.

Keywords Quaternionic analysis · Helmholtz equations · Boundary value problems

Mathematics Subject Classification Primary 30G35

✉ Juan Bory Reyes
juanboryreyes@yahoo.com

Ricardo Abreu Blaya
rabreu@facinf.uho.edu.cu

Ramón Martin Rodríguez Dagnino
rmrodrig@itesm.mx

Boris Aleksandrovich Kats
katsboris877@gmail.com

1 SEPI-ESIME-Zacatenco, Instituto Politécnico Nacional, Mexico, CD-MX, Mexico

2 Facultad de Informática y Matemática, Universidad de Holguín, Holguín, Cuba

3 School of Engineering and Sciences, Tecnológico de Monterrey, Monterrey, NL, Mexico

4 Kazan Federal University, Kazan, Russian Federation

Published online: 20 January 2018

1 Introduction

The algebra of quaternions, in combination with classical and modern analytic methods, give rise to the development of the so-called Quaternionic Analysis. Nowadays, this is the most attractive and close generalization of complex analysis since it preserves many of its key features. A summary of developments of this function theory and its relations to physics is contained in [23,24]. Moreover, both [29,30] emphasize applications in great details to boundary value problems associated to Helmholtz equation.

The theory of quaternion-valued hyperholomorphic functions (i.e., null solutions to the two dimensional Helmholtz equation) becomes, in recent years, a powerful mathematical tool for applications in potential theory [19,21] and in physical problems with elliptic geometries [32,33]. Detailed treatment of this theory, which is in the same relation to the two dimensional Helmholtz equation as the usual one-dimensional complex analysis is to the Laplace equation in $\mathbb{R}^2$, can be found in [4,5,16,18,20,42–44], see also [30, Appendix 4]. Because of the close connection to the Helmholtz equation, its applications have been made in electromagnetic radiation, seismology, acoustics, and also quantum mechanics.

An important question that is not addressed in the aforementioned works, and which is the main focus of the present paper, is the Riemann boundary value problem for hyperholomorphic functions. This generalization, besides its own undoubted pure mathematical interest, seems reasonable to be expected that it would be important for concrete applied problems.

The classical Riemann boundary value problem for holomorphic functions on piecewise-smooth closed Jordan curves γ have been studied for many researchers so far, see for example [14,31,35] for extensive treatments and discussions.

There is no need to remind the reader of the theoretical significance and success of this theory, it has been proven to be a desired effectiveness for solving large classes of boundary value problems, including among others, the Dirichlet, Neumann and Schwarz, which are either special cases of Riemann boundary value problems or closely linked to them. Also, is closely connected with the theory of singular integral equations [14,35] and has a wide range of applications in other fields, such as in the theory of cracks and elasticity [31,35], in quantum mechanics and of statistical physics [9] as well as in the theory of linear and of nonlinear partial differential equations [13] and in the theory of orthogonal polynomials and in asymptotic analysis [11].

The classical Riemann boundary value problem may be formulated as follows:

Let us assume that two $\mathbb{C}$-valued functions G and g belonging to the Hölder function space are given on a piecewise-smooth closed Jordan curve γ, and we need to find a function Φ with the following properties: a) it is holomorphic in both the internal and external (infinite) domains uniquely determined by γ; b) it is representable by a Cauchy type integral with a Hölder density and has limit values $\Phi^{\pm}$ everywhere on γ such that on γ satisfies the condition

$$\Phi^{+} = G \cdot \Phi^{-} + g. \tag{1.1}$$

The aim of this paper is to treat, instead of the above statement, a generalized Riemann boundary value problem for hyperholomorphic functions on closed Jordan rectifiable curves in $\mathbb{R}^2$. The main focus is on the solvability conditions, where a very important role is played by the boundedness assumed for a principal-value integral operator (reminiscent of the singular Cauchy's integral), whose kernel is a certain fundamental solution of Helmholtz equation.

We obtain the main result under a stated growth condition at infinity by using the Cauchy formula for hyperholomorphic functions and the so called method of regularization of quasi-solutions, which is offered by B. A. Kats [25,26] for solving of holomorphic RBVP on non-rectifiable curves. We prove that the RBVP for hyperholomorphic functions is solvable by combining the Plemelj–Sokhotski formulas with a sort of Liouville theorem given here. The proof of uniqueness is based on versions of the Painleve theorem.

2 Preliminaries

In this section we collect definitions and basic results that will be used constantly through the rest of this paper.

We shall denote by $\mathbb{H}(\mathbb{R})$ and $\mathbb{H}(\mathbb{C})$ the sets of real and complex quaternions (=biquaternions), respectively. Then, each quaternion a can be represented in the form $a = \sum_{k=0}^{3} a_k i_k$ where $\{a_k\} \subset \mathbb{R}$ for real quaternions and $\{a_k\} \subset \mathbb{C}$ for complex quaternions, i_0 is the multiplicative unit and $\{i_k | k = 1, 2, 3\}$ are the quaternionic imaginary units. As usual, we denote by i the imaginary unit in $\mathbb{C}$. By definition

$$i \cdot i_k = i_k \cdot i, \quad k = 0, 1, 2, 3.$$

The quaternionic conjugation $\overline{a}$ is defined by $\overline{a} := a_0 - \vec{a}$, where $\vec{a} := \sum_{k=1}^{3} a_k i_k$. The module of a quaternion a coincides with its Euclidean norm : $|a| = \|a\|_{\mathbb{R}^8}$. In particular, for $a \in \mathbb{H}(\mathbb{R})$ we have $|a| = \|a\|_{\mathbb{R}^4}$ and $|a|^2 = a \cdot \overline{a} = \overline{a} \cdot a$ whereas for a complex quaternion $|a|^2 \neq a \cdot \overline{a}$. Moreover, for a, b belonging to $\mathbb{H}(\mathbb{R})$ there holds: $|a \cdot b| = |a| \cdot |b|$ which is an extremely important property while working with real quaternions. However, for $a, b \in \mathbb{H}(\mathbb{C})$ there only holds the inequality $|ab| \leqslant \sqrt{2} \cdot |a| \cdot |b|$. If $a \in \mathbb{H}(\mathbb{C})$ is invertible, we will denote as a^{-1} its inverse.

Let Ω denotes a Jordan domain in $\mathbb{R}^2 \cong \mathbb{C}$, and let us define $\Omega^+ := \Omega$ and $\Omega^- := \mathbb{C} \setminus \overline{\Omega^+}$. Furthermore, we assume the boundary Γ of Ω to be, until further notice, a closed Jordan rectifiable curve.

We shall consider $\mathbb{H}(\mathbb{C})$-valued functions defined in Ω:

$$f : \Omega \longrightarrow \mathbb{H}(\mathbb{C}).$$

For a suitable subset $\mathbf{E} \subset \mathbb{R}^2$ we define, in usual component-wise meaning, the following classes of functions: $C^s(\mathbf{E}; \mathbb{H}(\mathbb{C}))$, $s \in \mathbb{N} \cup \{0\}$, of s times continuously differentiable functions; $L_p(\mathbf{E}; \mathbb{H}(\mathbb{C}))$, $p > 1$ of p-integrable functions; and $C^{0,\nu}(\mathbf{E}; \mathbb{H}(\mathbb{C}))$, $0 < \nu \leq 1$ of Hölder continuous functions, respectively.

The Hölder space $C^{0,\nu}(\mathbf{E}; \mathbb{H}(\mathbb{C}))$ equipped with the norm

$$|f|_{\nu,\mathbf{E}} := \max_{(t,\tau)\in\mathbf{E}} |f(t,\tau)| + \sup_{\substack{(t_1,\tau_1),(t_2,\tau_2)\in\mathbf{E}\\ (t_1,\tau_1)\neq(t_2,\tau_2)}} \frac{|f(t_1,\tau_1) - f(t_2,\tau_2)|}{|(t_1,\tau_1) - (t_2,\tau_2)|^{\nu}},$$

is a Banach space.

Let $\lambda \in \mathbb{C} \setminus \{0\}$, and let α denote an arbitrary fixed square root of λ in $\mathbb{C}$. This λ generates the two dimensional Helmholtz operator with a quaternionic wave number, which acts on $C^2(\Omega; \mathbb{H}(\mathbb{C}))$ and must be left and right:

$$_{\lambda}\Delta := \Delta_{\mathbb{R}^2} +^{\lambda} M, \tag{2.1}$$

$$\Delta_{\lambda} := \Delta_{\mathbb{R}^2} + M^{\lambda},$$

where $\Delta_{\mathbb{R}^2} := \partial_1^2 + \partial_2^2$, $\partial_k := \frac{\partial}{\partial x_k}$, $M^{\lambda}[f] := f\lambda$, ${}^{\lambda}M[f] := \lambda f$.

Finally, we discuss one more piece of notation. The following partial differential operators:

$$\begin{aligned} &{}_{st}\partial := i_1 \cdot \partial_1 + i_2 \cdot \partial_2;\ {}_{st}\overline{\partial} := \overline{i_1} \cdot \partial_1 + \overline{i_2} \cdot \partial_2;\\ &\partial_{st} := \partial_1 \circ M^{i_1} + \partial_2 \circ M^{i_2};\ \overline{\partial}_{st} := \partial_1 \circ M^{\overline{i}_1} + \partial_2 \circ M^{\overline{i}_2}. \end{aligned}$$

Thus,

$${}_{st}\partial^2 = \partial_{st}^2 = -\Delta_{\mathbb{R}^2}.$$

Set

$${}_{\alpha}\partial := \partial_{st} +^{\alpha}M;\ \partial_{\alpha} := {}_{st}\partial + M^{\alpha}.$$

Then we have the following factorizations of the Helmholtz operator:

$$\Delta_{\lambda} = -\partial_{\alpha} \circ \partial_{-\alpha} = -\partial_{-\alpha} \circ \partial_{\alpha}. \tag{2.2}$$

A function $f \in C^1(\Omega, \mathbb{H}(\mathbb{C}))$ is said to be hyperholomorphic if it satisfies the equation $\partial_{\alpha} f \equiv 0$ in Ω.

Furthermore, the set of hyperholomorphic functions in Ω will be denoted by $\mathcal{M}_{\alpha}(\Omega, \mathbb{H}(\mathbb{C}))$. We will use the letter c to denote a generic positive constant, and reserve the notation $z := (x, y)$ for a typical point of $\mathbb{R}^2 \setminus \{0\}$.

Let $\theta_{\alpha}(z)$ denote the fundamental solution of Δ_{λ}:

$$\theta_{\alpha}(z) := (-1)\frac{i}{4} H_0^{(1)}(\alpha|z|),$$

where $\{\alpha \in \mathbb{C} \setminus \{0\} : \mathrm{Im}(\alpha) \geq 0\}$ and $H_0^{(1)}$ is the usual zero-order Hankel function of the first kind. For a more detailed discussion of this fundamental solution see the excellent treatment in [15, pp. 59–74].

By taking into account (2.2), and with these clarifications, the quaternionic Cauchy kernel K_α is defined as a fundamental solution of the operator ${}_\alpha\partial$, and it can be calculated by the formula

$$\mathcal{K}_{st,\alpha}(z) = -\partial_{-\alpha}\,\theta_\alpha(z), \quad z \in \mathbb{R}^2 \setminus \{0\}. \tag{2.3}$$

Hence, it can be obtained explicitly as

$$\mathcal{K}_{st,\alpha}(z) = (-1)\frac{i\alpha}{4}\left(H_1^{(1)}(\alpha|z|)\frac{z}{|z|} + H_0^{(1)}(\alpha|z|)\right). \tag{2.4}$$

The Hankel functions can be expanded (at the origin) into the series, see [7,49]:

$$H_0^{(1)}(t) = \left(1 - (-1)\frac{2i}{\pi}(\ln\frac{t}{2} + \chi)\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k}}{2^{2k}(k!)^2} + \frac{2i}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}t^{2k}}{2^{2k}(k!)^2}\sum_{m=1}^{k}\frac{1}{m}, \tag{2.5}$$

$$H_1^{(1)}(t) = \left(1 - (-1)\frac{2i}{\pi}(\ln\frac{t}{2} + \chi)\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k+1}}{2^{2k+1}k!(k+1)!} + (-1)\left(\frac{2i}{\pi t}\right)$$
$$+ \frac{i}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}t^{2k+1}}{2^{2k+1}k!(k+1)!}\left(\sum_{m=1}^{k+1}\frac{1}{m} + \sum_{m=1}^{k}\frac{1}{m}\right), \tag{2.6}$$

where χ is the Euler's number.

We should remark that Eq. (2.6) is slightly different from the corresponding equation in [16–18,20] since they are developing these series expansions in accord to equations 8.402, 8.403 and 8.405 of [22]. However, we are using the most common definitions (Sect. 3.6 of [49], Equations 14.76 and 14.77 of [7])

$$H_n^{(1)}(t) = J_n(t) + iY_n(t).$$

The Bessel function of the second kind, $Y_n(z)$, is defined in Sect. 3.53 of [49] and equation 14.61 of [7], and it is slightly different from the function $N_n(z)$ used in [22].

Before stating the main results of this work, we will perform some necessary estimations. If $\mathrm{Im}(\alpha) > 0$ then $\mathcal{K}_{st,\alpha}$ decays exponentially at infnity. Furthermore, if $\mathcal{E}_{st,\alpha}(z)$ has a mathematical form such that

$$\mathcal{K}_{st,\alpha}(z) = \mathcal{E}_{st,\alpha}(z)\exp\{i\alpha|z|\}$$

then

$$|\mathcal{E}_{st,\alpha}(z)| = O\left(\frac{1}{|z|^{\frac{1}{2}}}\right), \quad \text{as } |z| \to \infty.$$

This fact follows from the definition of $\theta_\alpha(z)$ in addition to a straightforward computation, see [34, Section 3]. For such a purpose, they are also used the classical asymptotic expansions at infinity (in the sense of Poincaré; cf., e.g., [45, (9.13.1), p. 166]) of Hankel functions that are given by

$$H_0^{(1)}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} e^{i(z-\pi/4)} \left[1 + \sum_{m=1}^{\infty} (-1)^m \frac{(0,m)}{(2iz)^m}\right]$$
$$H_1^{(1)}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} e^{i(z-3\pi/4)} \left[1 + \sum_{m=1}^{\infty} (-1)^m \frac{(1,m)}{(2iz)^m}\right]$$

as $|z| \to \infty$, where

$$(\nu, m) = \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)\cdots(4\nu^2 - (2m-1)^2)}{2^{2m} m!}.$$

See also [7], Sect. 14.6.

We highlight the important fact, by Rellich's lemma, that any solution f of the Helmholtz equation (2.1) for non-zero real α which satisfies $f(z) = O(|z|^{-\frac{1}{2}})$ in a (connected) neighborhood of infinity in $\mathbb{R}^2$ must vanish identically (the Liouville theorem for hyperholomorphic functions). For further details see [34, Lemma 8.1]

3 Hyperholomorphic Cauchy integrals

3.1 Some results in complex analysis and historical facts

The classical RBVP theory in the plane was developed based on three basic results related to the Cauchy integral. Namely, the Plemelj–Sokhotski formulas, estimates in L_p or Hölder norms of the corresponding boundary values, as well as the Plemelj–Privalov theorem, which states the boundedness of the Cauchy singular integral in the Hölder classes on closed Jordan curves, see [6] for more details. These results directly apply to the study of the solvability condition of the Riemann problem with Hölder class data.

As it was pointed out in [47] many linear operators occurring in analysis enjoy one or both of the following properties: to map the space of Hölder continuous functions with order μ to the one of order ν for appropriated μ and ν, and the same for the Lebesgue space for exponent p to one for exponent q, for appropriated p, q.

Suppose γ is a closed rectifiable Jordan curve in the complex plane. G. David [10] has shown that the Cauchy singular integral operator

$$\mathcal{S}f : L^p(\gamma; \mathbb{C}) \longrightarrow L^p(\gamma; \mathbb{C}), \tag{3.1}$$

is bounded if and only if γ is Ahlfors–David-regular (Carleson curve). The later means that there is a $c > 0$ so that for all $z \in \gamma$ and for all $r > 0$ the arc-length measure of $\gamma \cap B(z, r)$ is at most cr, where $B(z, r)$ stands for the closed disk with center

z and radius r. The class of Ahlfors-David-regular curves includes among other the piece-wise Liapunov, K-curves, Lipschitz, etc.

As a matter of fact, the necessity of the Ahlfors–David-regularity for (3.1) was also proved by Salimov [41] using different techniques.

The classical Plemelj–Privalov theorem in complex analysis deals with the boundedness of the Cauchy singular integral operator in Hölder spaces, i.e., the implication

$$f \in C^{0,\nu}(\gamma;\mathbb{C}) \Rightarrow \mathcal{S}f \in C^{0,\nu}(\gamma;\mathbb{C})) \qquad 0 < \nu < 1, \tag{3.2}$$

as a bounded operator for closed rectifiable Jordan curves.

This theorem is well-known, see [35,37,38]. A geometric condition that completely characterizes the class of all closed rectifiable Jordan curves on which the Plemelj–Privalov theorem is valid is given in [40] in terms of the planar measure of boundary strips of sets constructed from the curve.

As far as we know, the first time that (3.2) was denominated as the Plemelj–Privalov theorem was in a not well-known paper of Babaev and Salaev [8].

3.2 The hyperholomorphic functions case

We refer to [1,3] for a recent overview of the historical developments of the higher dimensional Plemelj–Privalov theorem. In particular, this result is discussed in [3, Theorem 2.4] in the context of real Clifford algebras-valued functions in Jordan domain of $\mathbb{R}^n$. Moreover, these authors presented the maximal class of surfaces (curves when $n = 2$) for the validity of the Plemejl–Privalov theorem in Clifford analysis.

The Cauchy kernel $\mathcal{K}_{st,\alpha}$ generates, as usual, two important hyperholomophic integrals:

- The Cauchy type integral

$$K_\alpha[f](x,y) := -\int_\Gamma \mathcal{K}_{st,\alpha}(x-u, y-v) n_{st}(u,v) f(u,v)\, d\Gamma_{(u,v)}, \quad (x,y) \in \mathbb{R}^2 \setminus \Gamma, \tag{3.3}$$

where $n_{st}(u,v) = (n_1(u,v), n_2(u,v))$ denotes the unit outward normal vector to Γ at the point (u,v).

- The Teodorescu transform

$$T_\alpha[f](x,y) := \int_\Omega \mathcal{K}_{st,\alpha}(x-u, y-v) f(u,v)\, du \wedge dv, \quad (x,y) \in \mathbb{R}^2.$$

We will also need the following two well-known results in order to prove the main theorems of this paper.

Theorem 1 [43] *Let $\Omega \subset \mathbb{R}^2$ be a domain and let its boundary Γ be a closed rectifiable Jordan curve. Assume that $f \in C^1(\Omega, \mathbb{H}(\mathbb{C})) \cap C^0(\overline{\Omega}, \mathbb{H}(\mathbb{C}))$, then*

$$f(x,y) = K_\alpha[f](x,y) + T_\alpha \cdot \partial_\alpha[f](x,y), \; (x,y) \in \Omega.$$

Theorem 2 [43] *Let $\Omega \subset \mathbb{R}^2$ be a domain and let its boundary Γ be a closed rectifiable Jordan curve. If $f \in \mathcal{M}_\alpha(\Omega, \mathbb{H}(\mathbb{C}) \cap C^1(\overline{\Omega}, \mathbb{H}(\mathbb{C}))$, then*

$$f(x, y) = K_\alpha[f](x, y), \quad (x, y) \in \Omega.$$

The hyperholomorphic Cauchy integral allows us to prove the following theorem, which is an analogy of the Painleve theorem for holomorphic functions (see [36]).

Theorem 3 *Let Ω be a domain of $\mathbb{R}^2$ containing a rectifiable curve Γ. If function f is hyperholomorphic in $\Omega \setminus \Gamma$ and continuous in Ω, then it is hyperhomorphic in Ω.*

The proof of this result repeats the proof of the Painleve theorem for holomorphic functions.

The proof of next theorem follows from Theorem 2 and the behavior of $\mathcal{K}_{st,\alpha}(z)$ as $|z| \to \infty$. It is a suitable analogue of the complex Liouville theorem. Here and throughout the rest of the paper we assume that α takes only non-zero real values.

Theorem 4 *Let $f \in \mathcal{M}_\alpha(\mathbb{R}^2, \mathbb{H}(\mathbb{C}))$ and*

$$|f| = O\left(\frac{1}{|z|^{\frac{1}{2}}}\right), \quad as \ |z| \to \infty,$$

then $f = 0$ in $\mathbb{R}^2$.

Based on the Cauchy type integral $K_\alpha[f]$, which is well defined for any $f \in C^{0,\nu}(\Gamma; \mathbb{H}(\mathbb{C}))$, a singular Cauchy integral was introduced in [18,20] by taking boundary limits. As a matter of fact, there are intimate connections between both integrals, which is expressed by Plemelj–Sokhotski formulas. So, the investigation of boundary values of Cauchy-type integrals requires the study of the corresponding properties of the singular integral.

A recent work presenting the Plemelj–Sokhotski formulas in the two dimensional hyperholomorphic theory setting is [39], where the boundary Γ is regarded as a Liapunov curve (i.e., the angle formed by the tangent to the curve satisfies Hölder's condition). It should be noted that, in contrast to this work in Liapunov curves, Plemelj–Sokhotski formulas for more general class of curves include both a curvilinear and an area integral.

Theorem 5 [16,18,20] *Let Ω be a bounded domain with Ahlfors–David-regular boundary and let $f \in C^{0,\nu}(\Gamma; \mathbb{H}(\mathbb{C}))$, $0 < \nu < 1$. Then the boundary values*

$$K_\alpha^{\pm}[f](t, \tau) := \lim_{\Omega^{\pm} \ni (x,y) \to (t,\tau) \in \Gamma} K_\alpha[f](x, y)$$

of the Cauchy type integral $K_\alpha[f]$ are given by

$$\begin{aligned} K_\alpha^{+}[f](t, \tau) &= (I_\alpha(t, \tau) + 1) f(t, \tau) + F_\alpha[f - f(t, \tau)](t, \tau) \\ &=: \frac{1}{2}(f(t, \tau) + \mathcal{S}_\alpha[f](t, \tau)) =: P_\alpha^{+}[f](t, \tau), \end{aligned}$$

$$K_\alpha^-[f](t,\tau) = (I_\alpha(t,\tau))f(t,\tau) + F_\alpha[f - f(t,\tau)](t,\tau)$$
$$=: \frac{1}{2}(-f(t,\tau) + \mathcal{S}_\alpha[f](t,\tau)) =: P_\alpha^-[f](t,\tau),$$

for all $(t,\tau) \in \Gamma$, *which take the form of the usual Plemelj–Sokhotski formulas and where* $F_\alpha[f]$ *is the singular Cauchy integral given by*

$$F_\alpha[f](t,\tau) := -\int_\Gamma \mathcal{K}_{st,\alpha}(t-u,\tau-v)n_{st}(u,v)[f(u,v) - f(t,\tau)]\,d\Gamma_{(u,v)}$$

and

$$I_\alpha(t,\tau) := -\alpha \int_\Omega \mathcal{K}_{st,\alpha}(t-u,\tau-v)\,du \wedge dv.$$

Remark 3.1 It is assumed that $f \in C^{0,\nu}(\Gamma; \mathbb{H}(\mathbb{C}))$ in Theorem 5, hence all integrals must be interpreted in the Riemann sense (proper or improper). If now $f \in L_p(\Gamma; \mathbb{H}(\mathbb{C}))$ then one has to understand $\mathcal{K}_{st,\alpha}[f]$ as a Lebesgue integral, and the necessary changes can be easily made. For instance, the limits in Theorem 5 exist almost everywhere on Γ with respect to the Lebesgue measure. An L_p-formulation of this theorem follows from the standard Calderon-Zygmund theory and by recalling that $C^{0,\nu}(\Gamma; \mathbb{H}(\mathbb{C}))$ is dense in $L_p(\Gamma; \mathbb{H}(\mathbb{C}))$ according to classical arguments.

The next theorem shows a result similar to Plemelj–Privalov theorem in the hyperholomorphic framework.

Theorem 6 [16,42] *Let* Γ *be a piecewise-Liapunov closed Jordan curve. Then* $\mathcal{S}_\alpha$ *is a bounded operator acting on* $C^{0,\nu}(\Gamma; \mathbb{H}(\mathbb{C}))$, *i.e.,*

$$f \in C^{0,\nu}(\Gamma; \mathbb{H}(\mathbb{C})) \Rightarrow \mathcal{S}_\alpha f \in C^{0,\nu}(\Gamma; \mathbb{H}(\mathbb{C}))) \qquad 0 < \nu < 1, \tag{3.4}$$

as a bounded operator.

There is certain evidence that Theorem 6 may be extended to a setting larger than the piecewise Liapunov curves, in particular the Ahlfors–David regular curves, see [17, Corollary 2]. Here the main difficulties arise from finding a Hölder estimation for all terms of the Cauchy integral boundary values, i.e., including the estimation of I_α.

We should notice that for $\alpha = 0$, the complete characterization of the class of rectifiable Jordan curves for what Theorem 6 holds is given in [3, Theorem 2.4, for $n = 2$].

Through the remainder of our work we take it for granted that the relation (3.4) holds, and leave the geometric properties of the curves at a second importance level. This approach will allow us to make the exposition in a more general setting.

Observe now that, condition (3.4) and Plemelj–Sokhotski formulas of Theorem 5 combine to obtain the following result, which will be needed in the sequel.

Theorem 7 *Let* $0 < \nu < 1$*. There exist a real constant* $c_\Gamma = c_\Gamma(\alpha)$ *such that for any* $f \in C^{0,\nu}(\Gamma; \mathbb{H}(\mathbb{C}))$

$$|P_\alpha^-[f]|_{\nu,\Gamma} \leq c_\Gamma |f|_{\nu,\Gamma}. \tag{3.5}$$

4 Riemann problems for hyperholomorphic functions

In a hyperholomorphic scenario the RBVP may be described as follows: Find Φ in $\mathcal{M}_\alpha(\Omega^\pm, \mathbb{H}(\mathbb{C}))$, which is continuous up to the boundary Γ, vanishes at infinity and such that their boundary values $\Phi^\pm$ on the curve Γ satisfies the condition

$$\Phi^+(t,\tau) = \Phi^-(t,\tau) \cdot G(t,\tau) + g(t,\tau), \ \forall (t,\tau) \in \Gamma, \tag{4.1}$$

where $G(t,\tau)$ and $g(t,\tau)$ are given functions in $C^{0,\nu}(\Gamma; \mathbb{H}(\mathbb{C}))$.

The simplest case of the RBVP is called the jump problem

$$\Phi^+(t,\tau) - \Phi^-(t,\tau) = g(t,\tau), \ \forall (t,\tau) \in \Gamma, \tag{4.2}$$

i.e., the RBVP with $G \equiv 1$. Its solution for sufficiently smooth curve is given by a Cauchy type integral.

Theorem 8 *If* Γ *is Ahlfors–David regular and* $g \in C^{0,\nu}(\Gamma; \mathbb{H}(\mathbb{C}))$*, then the jump problem* (4.2) *has a unique solution equals to* $K_\alpha[g]$.

Proof The function $K_\alpha[g]$ satisfies the boundary condition (4.2) by Theorem 5, and it is a unique solution by virtue of the Painleve and Liouville theorems for hyperholomorphic functions. □

The next step to deal with a RBVP is to consider a RBVP with constant G.

Theorem 9 *If* Γ *is Ahlfors–David regular,* G *is a constant invertible complex quaternion and* $g \in C^{0,\nu}(\Gamma; \mathbb{H}(\mathbb{C}))$*, then the problem* (4.1) *has a unique solution.*

Proof This result reduces to the preceding theorem by introducing a new desired function

$$\Psi(x,y) = \begin{cases} \Phi(x,y), & (x,y) \in \Omega^+ \\ \Phi(x,y)G, & (x,y) \in \Omega^-. \end{cases}$$

Let us establish an explicit reduction of the problem (4.1) to an equivalent singular integral equation.

We may assume that the solution of (4.1) is of the form

$$\Phi(x,y) = -\int_\Gamma \mathcal{K}_{st,\alpha}(x-u, y-v) n_{st}(u,v)\varphi(u,v)\, d\Gamma_{(u,v)}, \tag{4.3}$$

where $\varphi \in C^{0,\nu}(\Gamma; \mathbb{H}(\mathbb{C}))$. Then, the function φ should satisfy the following singular integral equation:

$$P_\alpha^+[\varphi](t,\tau) = P_\alpha^-[\varphi](t,\tau)G(t,\tau) + g(t,\tau), \ (t,\tau) \in \Gamma. \tag{4.4}$$

Conversely, if $\varphi \in C^{0,\nu}(\Gamma; \mathbb{H}(\mathbb{C}))$ represents a solution of (4.4), then (4.3) is a solution of (4.1). Combining these results and the Plemelj–Sokhotski formulas in Theorem 5 gives

$$\varphi(t,\tau) = \Phi^+(t,\tau) - \Phi^-(t,\tau).$$

Here is another manner of stating (4.4).

$$\varphi(t,\tau) = P_\alpha^-[\varphi](t,\tau)(G(t,\tau) - 1) + g(t,\tau), \ (t,\tau) \in \Gamma. \tag{4.5}$$

□

Theorem 10 *Let Γ be a piecewise-Liapunov curve, and G and g are functions in $C^{0,\nu}(\Gamma; \mathbb{H}(\mathbb{C}))$. If the value $|G-1|_{\nu,\Gamma}$ is sufficiently small, then the problem (4.1) has a unique solution.*

Proof Theorem 7 implies that the integral operator, defined by the first term of the right hand side of (4.5), is a contractive operator mapping $C^{0,\nu}(\Gamma; \mathbb{H}(\mathbb{C}))$ into itself. Therefore, there exists a unique solution of (4.4) and thus a unique one of (4.1). □

After combining the last two theorems we obtain

Theorem 11 *Let $G = G_0G_1$, where G_0 is an invertible constant quaternion and G_1 is a function from $C^{0,\nu}(\Gamma; \mathbb{H}(\mathbb{C}))$ such that $|G-1|_{\nu,\Gamma}$ is sufficiently small, and $g \in C^{0,\nu}(\Gamma; \mathbb{H}(\mathbb{C}))$. If Γ is a piecewise Liapunov curve, then the problem (4.1) has a unique solution.*

Proof The introduction of the new desired function Ψ equating Φ in Ω^+ and ΦG_0 in Ω^- reduces the problem to the case considered in the previous theorem. □

Remark 4.1 The uniqueness of a solution in all these theorems means that we treat the case of the null Gakhov index [14].

Remark 4.2 At the level of Ahlfors–David-regular curves we conjecture that the scope of our results obtained on Hölder spaces may be extended to the much larger class of Lebesgue p-integrable functions in order to solve the Riemann boundary value problem with L_p data.

Then we consider the case of a non-rectifiable curve Γ. In this situation we cannot use a curvilinear integral along Γ. But the main ideas of the paper [25] keep their validity. Let Γ be a non-rectifiable closed curve, $g \in C^{0,\nu}(\Gamma; \mathbb{H}(\mathbb{C}))$. We consider a function $F : \mathbb{C} \mapsto \mathbb{H}(\mathbb{C})$ to be a quasi-solution of the jump problem (4.2) if it is continuously differentiable in $\mathbb{C}\setminus\Gamma$, and it has limit values on Γ from both sides related by the equality $F^+ - F^- = g$, its support is compact, and its first partial derivatives are integrable. Now, let us consider the function

$$\Phi(x,y) := F(x,y) - T_\alpha[\partial_\alpha F](x,y).$$

This function is hyperholomorphic in $\mathbb{C} \setminus \Gamma$. If $\partial_\alpha F \in L^p$, $p > 2$, then $T_\alpha[\partial_\alpha F]$ is continuous in $\mathbb{R}^2$, and Φ is a solution of the problem (4.2). Thus, the problem is solvable if it has a quasi-solution F with derivative $\partial_\alpha F$ integrable with a degree $p > 2$.

In the paper [4] a quasi-solution is built in the form $F = \tilde{g}\chi^+$, where $\tilde{g}$ is the Whitney extension of g (see, for instance, [46]), and χ^+ is a characteristic function of domain Ω^+. As a result, there were proved that the jump problem for hyperholomorphic functions on non-rectifiable curve is solvable if

$$\nu > \frac{1}{2} dm(\Gamma). \tag{4.6}$$

Here $dm(\Gamma)$ stands for the upper Minkowski dimension of the curve Γ (see, for instance, [48]). It is just the same condition which ensures solvability of the jump problem for holomorphic functions (see [25,26]).

Recently D. B. Katz introduced a new metric characteristic for non-rectifiable curves (see [27,28]). It is called Marcinkiewicz exponents.

Definition 1 Let $B^{\pm}(t,r) := \Omega^{\pm} \cap \{z : |z-t| \le r\}$, $t \in \Gamma$, $I_p^{\pm}(\Gamma,t,r) := \int_{B^{\pm}(t,r)} \frac{dxdy}{dist^p(z,E)}$. We put $\mathfrak{m}^{\pm}(\Gamma;t) := \sup\{p : \lim_{r\to 0} I_p^{\pm}(\Gamma,t,r) < \infty\}$, and $\mathfrak{m}^*(\Gamma) := \inf\{\max\{\mathfrak{m}^+(\Gamma;t), \mathfrak{m}^-(\Gamma;t)\} : t \in \Gamma\}$. The values $\mathfrak{m}^{\pm}(\Gamma;t)$ and $\mathfrak{m}^*(\Gamma)$ are called Marcinkiewicz exponents.

D. B. Kats built a quasi-solution F of the jump problem for holomorphic functions such that $\overline{\partial} F \in L^p$, $p > 2$, under the assumption

$$\nu > 1 - \frac{1}{2}\mathfrak{m}^*(\Gamma), \tag{4.7}$$

and we must remark that this assumption is weaker than (4.6). Analogous construction gives for the hyperholomorphic jump problem a quasi-solution F such that $\partial_\alpha F \in L^p$, $p > 2$. Thus, the following result is valid

Theorem 12 *The jump problem* (4.2) *for hyperholomorphic function is solvable if* $g \in C^{0,\nu}(\Gamma; \mathbb{H}(\mathbb{C}))$ *and the Hölder exponent* ν *is related with Marcinkiewicz exponent by relation* (4.7).

This theorem sharpens the mentioned result of [4]. Clearly, it implies an analog result to Theorem 9 for non-rectifiable curves.

Remark 4.3 The uniqueness of solution of the RBVP on non-rectifiable curves is more complicated. As shown by E. P. Dolzhenko [12], the non-rectifiable analog of the Painleve theorem fulfills for holomorphic function f only under the additional assumption $f^{\pm} \in C^{0,\nu}(\Gamma; \mathbb{H}(\mathbb{C}))$ for sufficiently large ν. The analog of this result for Clifford analysis is proved in [2]. We have evidence to support a conjecture that the Dolzhenko theorem is valid for hyperholomorphic functions as well.

Acknowledgements J. Bory Reyes was partially supported by Instituto Politécnico Nacional in the framework of SIP programs.

Compliance with ethical standards

Conflict of interest The authors declare that they have no competing interests.

References

1. Abreu Blaya, R., Bory Reyes, J.: The Plemelj–Privalov theorem in variable exponent Clifford analysis. Georgian Math. J. **19**(3), 401–415 (2012)
2. Abreu Blaya, R., Bory Reyes, J., Peña Peña, D.: Jump problem and removable singularities for monogenic functions. J. Geom. Anal. **17**(1), 1–14 (2007)
3. Abreu Blaya, R., Bory Reyes, J., Moreno García, T.: The Plemelj–Privalov theorem in Clifford analysis. C. R. Acad. Sci. Paris Ser. I **347**, 223–226 (2009)
4. Abreu Blaya, R., Bory Reyes, J., Rodríguez Dagnino, R.M.: Boundary value problems for hyperholomorphic solutions of two dimensional Helmholtz equation in a fractal domain. Appl. Math. Comput. **261**, 183–191 (2015)
5. Abreu Blaya, R., Bory Reyes, J., Rodríguez Dagnino, R.M.: 2D Quaternionic time harmonic Maxwell system in elliptic coordinates. Adv. Appl. Clifford algebras **25**(2), 255–270 (2015)
6. Abreu Blaya, R., Bory Reyes, J., Kats, B.A.: Cauchy Integral and the Singular Integral Operator over closed Jordan curves. Monatsh. Math. **176**(1), 1–15 (2015)
7. Arfken, G.B., Weber, H.J., Harris, F.H.: Mathematical Methods for Physicists, 7th edn. Academic Press, Elsevier, Waltham (2013)
8. Babaev, A.A., Salaev, V.V.: An analogue to the theorem of Plemelj–Privalov for the case of non-smooth curves and its applications. Sov. Math., Dokl. **6**, 389–391, 1965; translation from Dokl. Akad. Nauk SSSR **161**, 267–269, 1965; Errata. Ibid. **7**(3), vi (1966), resp. 168, 494, (1966)
9. Chelkak, D., Smirnov, S.: Universality in the 2D Ising model and conformal invariance of Fermionic observables. Inventiones mathematicae **189**(3), 515–580 (2012)
10. David, G.: Op*é*rateurs int*é*graux singuliers sur certaines courbes du plan complexe (Singular integral operators over certain curves in the complex plane. Ann. Sci. Ecole Norm. Sup. (4) **17**(1), 15789 (1984)
11. Deift, P.: Orthogonal polynomials and random matrices: a Riemann–Hilbert approach. American Mathematical Society, Providence, Rhode Island (2000)
12. Dolzhenko, E.P.: On the removal of singularities of analytic functions. Am. Math. Soc. Transl. **97**, 3341 (1970)
13. Fokas, A.S.: A unified approach to boundary value problems. University of Cambridge, Cambridge (2008)
14. Gakhov, F.D.: Boundary Value Problems. Dover Publications, Inc., New York (1990)
15. Garnir, H.G.: Les Problèmes aux Limites de la Physique Mathèmatique. Birkhaüser, Basel (1958)
16. Gerus, O.F., Schneider, B., Shapiro, M.: On boundary properties of α-hyper- holomorphic functions in domains of $\mathbb{R}^2$ with piecewise Liapunov boundary. In: Proceedings of the Third International ISAAC Congress Progress in Analysis (August 20–25, 2001, Berlin, Germany), World Scientific, Berlin, pp. 375–382 (2003)
17. Gerus, O.F.: An estimate for the modulus of continuity of a quaternion singular Cauchy integral. (Ukrainian, English) Ukr. Mat. Zh. **62**(10), 1428–1435 (2010); translation in Ukr. Math. J. **62**(10), 1657–1665, (2010)
18. Gerus, O.F., Shapiro, M.: On the boundary values of a quaternionic generalization of the Cauchy-type integral in $\mathbb{R}^2$ for rectifiable curves. J. Nat. Geom. **24**(1–2), 120–136 (2003)
19. Gerus, O.F., Shapiro, M.: On boundary properties of metaharmonic simple and double layer potentials on rectifiable curves in $\mathbb{R}^2$. Zb. Pr. Inst. Mat. NAN Ukr. **1**(3), 67–76 (2004)
20. Gerus, O.F., Shapiro, M.: On a Cauchy-type integral related to the Helmholtz operator in the plane. Bol. Soc. Mat. Mex. III. Ser. **10**(1), 63–82 (2004)
21. Gerus, O.F., Kutrunov, V.N., Shapiro, M.: On the spectra of some integral operators related to the potential theory in the plane. Math. Methods Appl. Sci. **33**(14), 1685–1691 (2010)
22. Gradshteyn, I.S., Ryzhik, I.M.: Tables of Integrals, Series, and Products. Academic Press, Inc., London (1980)

23. Gürlebeck, K., Sprössig, W.: Quaternionic analysis and elliptic boundary value problems. Birkhäusser, Boston (1990)
24. Gürlebeck, K., Sprössig, W.: Quaternionic and Clifford Calculus for Physicists and Engineers. Wiley, Cichester (1997)
25. Kats, B.A.: Riemann boundary value problem on non-rectifiable Jordan curve. Doklady AN USSR **267**(4), 789–792 (1982)
26. Kats, B.A.: The Riemann boundary value problem on non-rectifiable curves and related questions. Complex Var Elliptic Equ **59**(8), 1053–1069 (2014)
27. Katz, D.B.: Marcinkiewicz exponents with applications in boundary value problems. Izvestija vuzov. Matem. **3**, 68–71 (2014)
28. Katz, D.B.: Local Marcinkiewicz exponents with application. Uchenye Zapiski Kazan Univ. Ser. Phys.-Math. nauki **156**(4), 31–38 (2014)
29. Kravchenko, V.V.: Applied Quaternionic Analysis. Heldemann Verlag, Berlin (2003)
30. Kravchenko, V., Shapiro, M.: Integral Representations for Spatial Models of Mathematical Physics. Pitman Research Notes in Mathematics, vol. 351, p. 256. Harlow, Addison Wesley Longman (1996)
31. Lu, J.K.: Boundary Value Problems for Analytic Functions. World Scientific, Singapore (1993)
32. Luna-Elizarrarás, M.E., Rodríguez-Dagnino, R.M., Shapiro, M.: On a version of quaternionic function theory related to Mathieu functions. In: Simos, Theodore E. et al. (ed.), Numerical analysis and applied mathematics. International conference of numerical analysis and applied mathematics, Corfu, Greece, 16–20 September 2007. Melville, NY: American Institute of Physics (AIP). AIP Conference Proceedings 936, 761–763 (2007)
33. Luna-Elizarrarás, M.E., Pérez-de la Rosa, M.A., Rodríguez-Dagnino, R.M., Shapiro, M.: On quaternionic analysis for the Schrödinger operator with a particular potential and its relation with the Mathieu functions. Math. Methods Appl. Sci. **36**(9), 1080–1094 (2013)
34. McIntosh, A., Mitrea, M.: Clifford algebras and Maxwell's equations in Lipschitz domains. Math. Methods Appl. Sci. **22**(18), 1599–1620 (1999)
35. Muskhelishvili, N.I.: Singular integral equations. Boundary problems of function theory and their application to mathematical physics, Dover Publications Inc., New York, (1992), Translated from the second (1946) Russian edition and with a preface by J. R. M. Radok, Corrected reprint of the 1953 English translation
36. Painleve, P.: Sur les lignes singulieres des fonctions analytiques, Paris (1887)
37. Plemelj, J.: Ein Erganzungssatz zur Cauchy'schen Integraldarstellung analytischer, Funktionen Randwerte betreffend Monatsh. für Math. und Phys B. **19**(S), 205–210 (1908)
38. Privaloff, I.: Sur les fonctions conjuguees. Bull. Soc. Math. France **44**(23), 100–03 (1916)
39. Rocha-Chávez, R., Shapiro, M., Tovar, L.: On the Hilbert operator for α-hyperholomorphic function theory in $\mathbb{R}^2$. Complex Var. Theory Appl. **43**(1), 1–18 (2000)
40. Salaev, V.V., Guseǐnov, E.G., Seǐfullaev, R.K.: The Plemelj–Privalov theorem. Dokl. Akad. Nauk SSSR 315 (4) (1990) 790.793; Translation in Soviet Math. Dokl. **42**(3), 849–852 (1991)
41. Salimov, T.S.: A singular Cauchy integral in H_ω spaces. Theory of functions and approximations, Part 2 (Saratov: 130–134, (1983) Saratov. Gos. Univ, Saratov (1982)
42. Schneider, B., Kavaklioglu, Ö.: Poincar$\acute{e}$-Bertrand formula on a piecewise Liapunov curve in two-dimensional. Appl. Math. Comput. **202**(2), 814–819 (2008)
43. Shapiro, M., Tovar, L. M.: On a class of integral representations related to the two-dimensional Helmholtz operator. In: Ramírez de Arellano, E. et al. (ed.) Operator theory for complex and hypercomplex analysis. Proceedings of a conference, Mexico City, Mexico, December 12–17, 1994. Providence, RI: American Mathematical Society. Contemp. Math. **212**, 229–244 (1998)
44. Shapiro, M., Tovar, L.M.: Two-dimensional Helmholtz operator and its hyperholomorlhic solutions. J. Nat. Geom. **11**, 77–100 (1997)
45. Spain, B., Smith, M.G.: Functions of Mathematical Physics. Van Nostrand Reinhold Co., London, New York (1970)
46. Stein, E.M.: Singular Integrals and Differential Properties of Functions. Princeton University Press, Princeton (1970)
47. Stein, M.E., Zygmund, A.: Boundedness of translation invariant operators on Hölder spaces and L^p-spaces. Ann. Math. **2**(85), 337–349 (1967)
48. Tricot, C.: Curves and Fractal Dimension. Springer, New York (1995)
49. Watson, G.N.: A Treatise on the Theory of Bessel Functions, 2nd edn. Cambridge University Press, Cambridge (1995)