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Finite Element Method and Partial Area Method in One Diffraction Problem

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Abstract—In this work, the boundary value problem for the Helmholtz equation in a half-band, corresponding to the physical problem of diffraction of TE-polarized electromagnetic wave on the wall of a resonator with a hole in a semi-infinite waveguide, is solved by the finite element method. The obtained solution is compared with the solution obtained earlier by the method of partial domains. Good correspondence between the solutions obtained by two different methods is shown. The absolute difference between the solutions was calculated. The dependence of the absolute difference on the triangulation parameter in the finite element method is given for a fixed ISLAE truncation parameter in the partial domain method.

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1. INTRODUCTION

In the most general formulation, the diffraction problem consists in finding solutions of Maxwell equations that satisfy certain boundary conditions. For the uniqueness of the solution, these conditions are usually supplemented with the Sommerfeld radiation conditions and conditions that take into account the behavior of the electromagnetic field in the vicinity of sharp edges.

Assume that the desired field does not depend on one of the coordinates. In this case, the system of Maxwell equations splits into two independent subsystems, which give two types of partial solutions. These solutions are called TE and TM polarized and satisfy the first and second boundary value problems for the two-dimensional Helmholtz equation, respectively. Thus, boundary value problems and conjugation problems for the system of Maxwell equations are reduced to boundary value problems and conjugation problems for the Helmholtz equation. Statements of various boundary value problems for equations of elliptic type, including the Helmholtz equation, are considered in [1].

Various methods are used for the exact and numerical solving boundary value problems. For example, in the book [2] to solve diffraction problems in waveguide structures, the geometries of which are obtained by various modifications of a branched waveguide, the methods of partial domains and Wiener–Hopf are successfully applied. In the first case, the solution of the problem is sought in the form of an expansion in a Fourier series in terms of the orthonormal set of eigenfunctions of the corresponding Sturm–Liouville problem with unknown coefficients. Further, from the conditions for matching fields at common interfaces between media, it is easy to obtain series equations. By projecting these functional equations onto the corresponding orthogonal sets of functions, one can reduce the original problem to an infinite set of linear algebraic equations (LSAE) with respect to unknown coefficients. For the problem with a branched waveguide, the ISLAE solution can be found explicitly by the direct matrix inversion method. The asymptotics of the found coefficients is shown, which ensures the fulfillment of the conditions on the sharp edge for the potential function.

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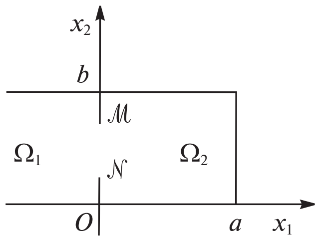

Fig. 1. Semi-infinite waveguide with diaphragm.

Note that different ways of projecting onto orthogonal sets of functions lead to different ISLAEs. For example, as it is shown in [3], direct projection of series equations leads to ISLAEs with the properties of convolution-type equations, which significantly limits the possibility of using simple methods such as the reduction method to solve them. Therefore, when developing numerical methods for solving the problem, as well as when proving existence and uniqueness theorems for solutions, it is advisable to carry out regularization. This can be done by various methods, for example, [4] uses the method of integral-series identities to regularize the summation equations and reduce them to ISLAE, and [5] and [6] use the semi-inversion and Riemann—Hilbert methods, which are reduced to the selection of the main singular part in the summation equation and its analytical inversion.

The study of diffraction boundary value problems in a rigorous mathematical formulation and eigenvalue problems is given in [7].

In the papers [8—11], the problems of diffraction in infinite and semi-infinite waveguide structures with various inhomogeneities generating resonant phenomena are studied.

In this work, the [12—14] finite element method is used to solve and study the boundary value problem for the Helmholtz equation in a half-band, which corresponds to the physical problem of electromagnetic wave diffraction in a semi-infinite waveguide with a diaphragm located in front of the plug. A comparison is made with the solution obtained earlier in [15] using the partial domain method.

2. THE DIFFRACTION PROBLEM

Consider a semi-infinite rectangular waveguide with a diaphragm located in front of the plug in the cross-sectional plane $z = 0$ (Fig. 1), and separating the waveguide into regions $I(z < 0)$ and $II(0 < z < c)$. Assume that the medium is homogeneous and isotropic everywhere. The vectors of TE-polarized electromagnetic field can be represented as

$$\mathbf{E} = (0, 0, u(x_1, x_2)), \quad \mathbf{H} = \left(\frac{\partial u(x_1, x_2)}{\partial x_2}, \frac{\partial u(x_1, x_2)}{\partial x_1}, 0\right).$$

The functions $u_0(x_2) = u(0, x_2)$ and $u_1(x_2) = (\partial u/\partial x_2)(0, x_2)$ will be called the zero and the first traces of the function $u(x)$.

We will look for a solution of the Helmholtz equations

$$\Delta u^i + \kappa^2 u^i = 0, \quad u^i = u^i(x), \quad x = (x_1, x_2) \in \Omega_1 \cup \Omega_2 \tag{1}$$

with boundary conditions on metal

$$u^i\big|_{\Gamma_i} = 0, \quad \Gamma_i = \partial\Omega_i \backslash \mathcal{N} \tag{2}$$

and interface conditions on the hole

$$[u]|_{\mathcal{N}} = 0, \tag{3}$$

$$\left[\frac{\partial u}{\partial x_1}\right]\bigg|_{\mathcal{N}} = 0. \tag{4}$$

Square brackets $[\cdot]$ denote the difference between traces to the left and to the right of $\mathcal{N}$

$$[u]|_{\mathcal{N}} = u^1(0, x_2) - u^2(0, x_2) = 0. \tag{5}$$

We will seek the solution (1) in the region Ω_1 in the form of the sum of the incident $u^+(x)$ and reflected $u^-(x)$ waves

$$u^1(x) = u^+(x) + u^-(x).$$

Thus, the conditions (2)–(4) can be rewritten as: $x \in \mathcal{M}$

$$u_0^+ + u_0^- = 0, \tag{6}$$

$$u_0^2 = 0, \tag{7}$$

$x \in \mathcal{N}$

$$u_0^+ + u_0^- = u_0^2, \tag{8}$$

$$u_1^+ + u_1^- = u_1^2. \tag{9}$$

We will look for the oncoming wave $u^+(x)$ as a wave coming from "infinity" in the form $u^+(x) = s_l(x_2)e^{i\gamma_l x_1}$, where $s_l(x) = \sqrt{\frac{2}{a}} \sin \frac{\pi n x_2}{a}$. The reflected wave, according to Steklov's theorem, will be sought as an expansion in terms of eigenfunctions of the Sturm–Liouville problem for the Laplace operator in the form

$$u^-(x) = \sum_{n=1}^{\infty} a_n s_n(x_2) e^{-i\gamma_n x_1}$$

and the transmitted wave in the form

$$u^2(x) = \sum_{n=1}^{\infty} b_n s_n(x_2) \left(e^{i\gamma_n x_1} - e^{i\gamma_n c}\, e^{-i\gamma_n (x_1 - c)}\right).$$

Note that the traces of a positively oriented wave are related by the relation

$$u_1(x_2) = \int_0^b u_0(t) K(t, x_2)\, dt$$

but negatively oriented

$$u_1(x_2) = -\int_0^b u_0(t) K(t, x_2)\, dt,$$

where

$$K(t, x_2) = \sum_{n=1}^{+\infty} i\gamma_n s_n(t) s_n(x_2)$$

Lemma 1. @The zero and first traces of an non-oriented wave $u^2(x)$ on the hole $\mathcal{N}$ are related by the relation

$$u_1^2(x_2) = 2u_1^+(x_2) - \int_{\mathcal{N}} u^2(x_1, t) K(t, x_2) dt, \quad x \in \mathcal{N}. \tag{10}$$

From (9) follows

$$u_1^2(x_2) = -\int_{\mathcal{N}} u_0^-(t) K(t, x_2) dt - \int_{\mathcal{M}} u_0^-(t) K(t, x_2) dt + u_1^+(x_2)$$

taking into account (8) and (6) we get (10).

In addition, using the method of integral series identities, we can prove the following theorem

Theorem 1. @The two-dimensional problem of diffraction of TE-polarized electromagnetic wave by the diaphragm in a semi-infinite waveguide is reduced to ISLAE

$$\gamma_k b_k - \sum_{n=1}^{\infty} b_n \left(1 - e^{2i\gamma_n c}\right) \sum_{m=1}^{\infty} \left(\frac{\gamma_m}{1 - e^{2i\gamma_m c}}\right) J_{nm} I_{mk} = \gamma_l I_{lk}, \quad k = 1, 2, ... \tag{11}$$

@where $I_{mk} = \int_{\mathcal{M}} s_n(t) s_m(t)\, dt$, $J_{nm} = \int_{\mathcal{N}} s_n(t) s_m(t)\, dt$.

3. VARIATIONAL STATEMENT OF THE PROBLEM

Here and below, we will assume that $u(x) = u^2(x)$ and $\Omega = \Omega_2$. Potential function $u(x)$ we will seek as a solution to the boundary value problem for the Helmholtz equation

$$\Delta u + \kappa^2 u = 0, \quad x = (x_1, x_2) \in \Omega \tag{12}$$

with boundary conditions on $\partial\Omega = \mathcal{M} \cup \mathcal{N}$ as

$$u(x) = 0, \quad x \in \mathcal{M} = \partial\Omega \backslash \mathcal{N}, \tag{13}$$

$$\nabla u \cdot n = 2u^0 - \int\limits_{\mathcal{N}} u(x_1, t) K(t, x_2) dt, \quad x \in \mathcal{N}, \tag{14}$$

where $K(t, x_2) = \sum_{m=1}^{M} \gamma_m s_m(t) s_m(x_2)$. Let us pass to the variational formulation of the problem. Let's multiply (12) on the $v(x)$ and integrate over the region Ω

$$\int\limits_{\Omega} \left(v\Delta u + \kappa^2 vu\right)\, dx = 0. \tag{15}$$

Applying the Ostrogradsky–Gauss formula to the first integrand (15), we obtain

$$\int\limits_{\Omega} \left(\nabla u \cdot \nabla v - \kappa^2 uv\right) dx - \int\limits_{\mathcal{N}} v \nabla u \cdot n\, dx = 0, \tag{16}$$

or

$$\begin{aligned} \int\limits_{\Omega} \left(\nabla u \cdot \nabla v - \kappa^2 uv\right) dx + \sum_{m=1}^{M} \gamma_m \int\limits_{\mathcal{N}} v(0, x_2) s_m(x_2)\, dx_2 \int\limits_{\mathcal{N}} u(0, t) s_m(t)\, dt \\ = 2\gamma_l \int\limits_{\mathcal{N}} v(0, x_2) s_l(x_2) dx_2. \end{aligned} \tag{17}$$

Function

$$u \in V^0 = \left\{u \in H^1(\Omega) :\, u(x) = 0,\, x \in \mathcal{M}\right\}, \tag{18}$$

where $H^1(\Omega)$ is Sobolev space, will be called an approximate solution of the (12)–(14) problem if it satisfies the integral identity (17) for any function $v \in V^0$.

The solution (17) will be sought in the form

$$u(x) = \sum_{i=1}^{n} u_i \varphi_i(x) = \sum_{i \in i_n} u_i \varphi_i(x) + \sum_{i \in i_d} u_i \varphi_i(x), \tag{19}$$

where u_i are the desired expansion coefficients. We substitute (19) into (17) and obtain a system of linear algebraic equations with respect to the required coefficients u_i

$$\sum_{i \in i_n} a_{ij} u_i = \phi_j, \quad j \in i_n, \tag{20}$$

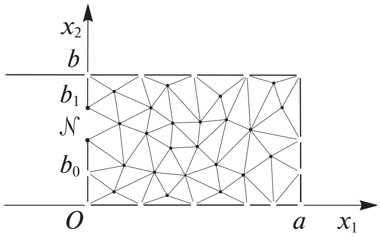

Fig. 2. Triangulation of Ω region using P_1 grid.

where

$$a_{ij} = \int\limits_{\Omega} \left(\nabla\varphi_i \cdot \nabla\varphi_j - \kappa^2\varphi_i\varphi_j\right) dx + \sum_{m=1}^{M} \gamma_m \int\limits_{\mathcal{N}} \varphi_j(0, x_2)s_m(x_2)\, dx_2 \int\limits_{\mathcal{N}} \varphi_i(0, t)s_m(t)\, dt,$$

$$\phi_j = 2\gamma_l \int\limits_{\mathcal{N}} \varphi_j(0, x_2)s_l(x_2)\, dx_2. \tag{21}$$

For the numerical implementation of the finite element method of the (21) system, we will use a triangular P_1 grid and the method of conformal linear triangular elements. We choose the basis functions of the reference triangle with vertices at the bars $\hat{a}_1 = (0, 0)$, $\hat{a}_2 = (1, 0)$, $\hat{a}_3 = (0, 1)$ in the form

$$\hat{\varphi}_1(\hat{x}) = 1 - \hat{x}_1 - \hat{x}_2, \quad \hat{\varphi}_2(\hat{x}) = \hat{x}_1, \quad \hat{\varphi}_3(\hat{x}) = \hat{x}_2.$$

Note that the basis functions are chosen so that the condition

$$\hat{\varphi}_i(\hat{a}_j) = \delta_{ij}, \quad i, j = 1, 2, 3.$$

An example of triangulation of the region Ω using the P_1 grid is shown in Fig. 2, where the punctured (white) points show the set i_d, for which the Dirichlet condition (13) is satisfied, and the black ones show the set of points i_n for which the condition (14) is satisfied

4. COMPUTATIONAL EXPERIMENT

Consider a truncated (reduced) system of linear algebraic equations obtained from the system (11) by replacing infinite sums by finite ones

$$\gamma_k b_k - \sum_{n=1}^{N} b_n \left(1 - e^{2i\gamma_n c}\right) \sum_{m=1}^{N} \left(\frac{\gamma_m}{1 - e^{2i\gamma_m c}}\right) J_{nm} I_{mk} = \gamma_l I_{lk}, \quad k = 1, 2, ..., N, \tag{22}$$

where N is the truncation parameter. Integrals J_{nm} and I_{nm}, which are related by the relation $I_{nm} = \delta_{nm} - J_{nm}$, we will calculate explicitly by the formulas

$$J_{nm} = \begin{cases} \frac{\sin(n-m)\pi a_1 - \sin(n-m)\pi a_0}{(n-m)\pi} - \frac{\sin(n+m)\pi a_1 - \sin(n+m)\pi a_0}{(n+m)\pi}, & n \neq m, \\ a_1 - a_0 - \frac{\sin(n+m)\pi a_1 - \sin(n+m)\pi a_0}{(n+m)\pi}, & n = m. \end{cases}$$

All further calculations will be carried out at the resonant frequency k_{11} of a square resonator with the size $a = b = 1$. As is known, these resonant frequencies coincide with the eigenvalues $\kappa_{nm} = \sqrt{(\pi n)^2 + (\pi m)^2}$. We choose the hole in the form $\mathcal{N} = (0.3, 0.7)$.

Solving the truncated system (22) we find the unknown coefficients $\{b_n\}_{n=1}^{N}$ of the potential function

$$u_N(x) = \sum_{n=1}^{N} b_n s_n(x_2) \left(e^{i\gamma_n x_1} - e^{i\gamma_n c}\, e^{-i\gamma_n (x_1 - a)}\right). \tag{23}$$

The absolute difference between the solutions $u_{20}(x)$ and $u_{25}(x)$ calculated by the formula $\varepsilon(x) = |u_{25}(x) - u_{20}(x)|$ less than 10^{-16}. Thus, it is enough to restrict ourselves to the truncation parameter

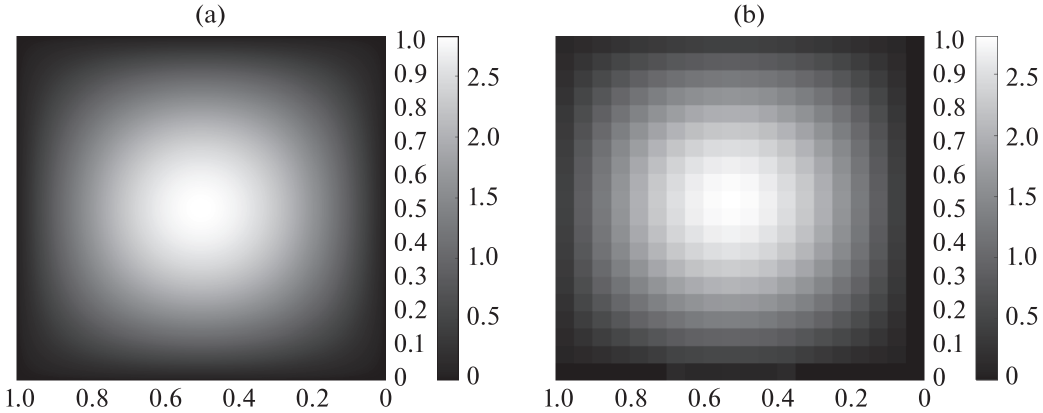

Fig. 3. a) Solution $u_N(x)$ of the boundary value problem (1)–(4) by the partial domain method for $N = 20$ b) solution $u_h(x)$ of the boundary value problem (12)–(14) by the finite element method for $h = 1/20$.

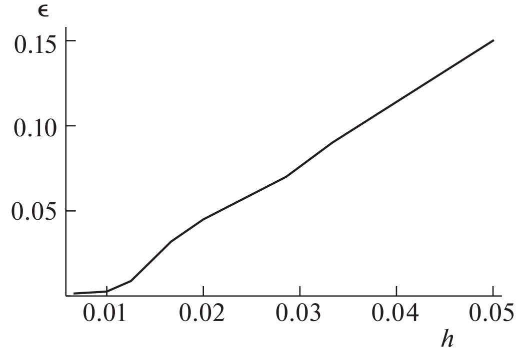

Fig. 4. Dependence of the absolute difference on a sharp edge on the partition diameter in the finite element method.

$N = 20$. Such a good result, in particular, is confirmed in [2], where in the problem of diffraction in a branched waveguide, the relative error due to reduction ($N = 20$) and calculated by the formula

$$\delta_N = \left| \frac{u_\infty - u_N}{u_N} \right|$$

does not exceed 1%.

The solution of the boundary value problem (12)–(14) will be denoted by $u_h(x)$, where $h-$ is the triangulation parameter (triangle size) in the finite element method.

Fig. 3 shows solutions $u_N(x)$ and $u_h(x)$ of problems (1)–(4) and (1)–(14) respectively. As you can see by eye, these solutions are quite similar, but for a more accurate comparison, let us set the absolute difference in the form

$$\varepsilon(h, x) = |u_{20}(x) - u_h(x)|, \quad x \in \bar{\Omega}. \tag{24}$$

The Fig. 4 shows the dependence $\varepsilon(h, x)$ on the sharp edges of the diaphragm for various h. As can be seen, with a decrease in h, i.e., a refinement of the partition diameter, the absolute difference also decreases, which indicates the reliability of the solutions obtained.

Graphs of $\varepsilon(h, x)$ for some h are shown in the Fig. 5.

Note that in order to achieve the required accuracy of the solution in the required regions (for example, near sharp edges), h must be reduced, which, in turn, leads to an exponential increase in the computation time. In such cases, it is advisable to solve the problem on a grid with refinement in the required area.

An example of such a grid is shown in Fig. 6 a), where the partition diameter $h(x)$ is given as $h(x) = 0.01 + 0.2\sqrt{(x_1 - 0.5)^2 + x_2^2}$. The corresponding solution to the problem is shown in Fig. 6 b).

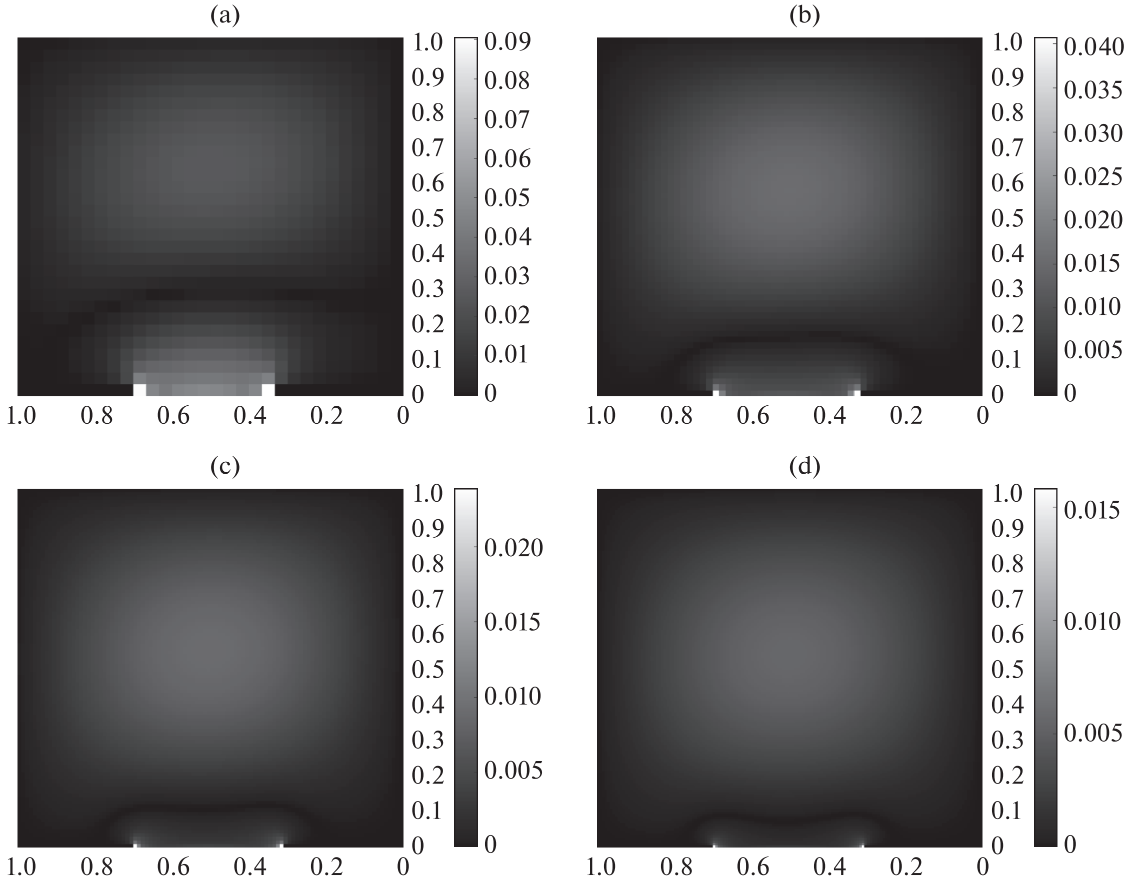

Fig. 5. Absolute difference of solutions $\varepsilon(x)$: a) $h = 1/30$, b) $h = 1/60$, c) $h = 1/100$, d) $h = 1/150$.

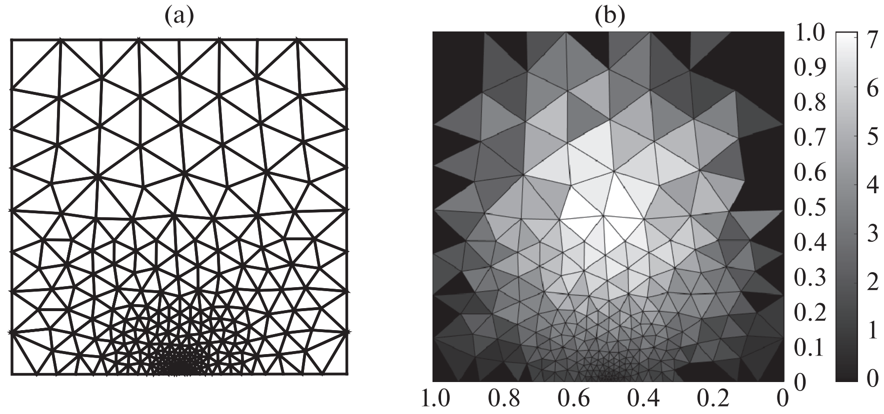

Fig. 6. a) P_1 mesh with mesh refinement in the area of the hole b) solution $u(x)$ of the problem on a condensed mesh.

5. CONCLUSION

In this work, the boundary value problem for the Helmholtz equation in a half-band is solved by the finite element method, which corresponds to the physical problem of diffraction TE-polarized electromagnetic wave on the wall of a resonator with a hole in front of the plug in a semi-infinite waveguide.

The obtained solution is compared with the solution obtained earlier by the method of partial regions. The absolute difference of the solutions obtained by the two methods is calculated, on the basis of which the good consistency of the given solutions is shown. The decrease in the absolute difference during mesh refinement in the finite element method with a fixed ISLAE truncation parameter in the partial domain method is empirically shown. The solution of the problem with a fine mesh, in the region of sharp edges, is given.

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