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Dynamics of Solitons of the Generalized Nonlinear Schrödinger Equation in an Inhomogeneous and Nonstationary Medium: Evolution and Interaction

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Abstract—The stability and dynamics of the interaction of soliton-like solutions of the generalized nonlinear Schrödinger (NLS) equation describing the dynamics of the envelope of modulated nonlinear waves and pulses (including the phenomenon of wave collapse and the self-focusing of wave beams) in plasma (including space one), as well as in nonlinear optical systems, have been studied with allowance for the inhomogeneity and nonstationarity of the distribution environment. The equation is also used in other areas of physics, such as the theory of superconductivity and low-temperature physics, small-amplitude gravitational waves on the surface of a deep inviscid fluid, etc. It should be noted that the studied equation is not completely integrable, and its analytical solutions are generally unknown (except, perhaps, for smooth solutions of the solitary wave type). However, approaches that were developed earlier for other equations (the generalized Kadomtsev–Petviashvili equation and the three-dimensional NLS equation with the derivative of the nonlinear term) of the Belashov–Karpman system makes it possible to analyze the stability of possible solutions of the these equations and to conduct a numerical study of the dynamics of soliton interaction. This approach is implemented in the study. Sufficient conditions for the stability of two- and three-dimensional soliton-like solutions are obtained analytically, and the cases of stable and unstable (with the formation of breathers) evolution of pulses of various shapes, as well as the interaction of two- and three-pulse structures, which leads to the formation of stable and unstable solutions, were studied numerically. The results can be useful in numerous applications for the physics of ionospheric and magnetospheric plasma and in many other areas of physics.

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1. INTRODUCTION. BASIC EQUATIONS

If in the Belashov–Karpman (BK) system (Belashov and Vladimirov, 2005; Belashov et al., 2018a)

$$\partial_t u + \hat{A}(t,u)u = f, \quad f = \sigma\int_{-\infty}^{x} \Delta_\perp u \mathrm{d}x + f', \qquad \Delta_\perp = \partial_y^2 + \partial_z^2 \tag{1}$$

the operator has the form $\hat{A}(t,u) = i[\gamma|u|^2 - \beta\partial_x^2] + \alpha/2$, then we have a three-dimensional (3D) generalized nonlinear Schrödinger equation (3-GNLS) (Belashov et al., 2018b):

$$\partial_t u + i\gamma|u|^2 u - i\beta\partial_x^2 u + (\alpha/2)u = \sigma\int_{-\infty}^{x} \Delta_\perp u \mathrm{d}x + f', \tag{2}$$

where $\alpha, \beta, \gamma = \varphi(t,x,y,z)$, $f' = f'(t,x,y,z)$, and $(\alpha/2)u$ describes the dissipative effects, and u is the envelope of the wave packet (pulse). The 3-GNLS equation (2) describes the dynamics of the envelope of modulated nonlinear waves and pulses (wave packets) in dispersive media and has many important applications in plasma physics (e.g., it describes the propagation of the Langmuir waves in a hot plasma) and nonlinear optics (the propagation of light pulses in crystals, optical fibers, and plane optical waveguides); it describes, in particular, such phenomena as turbulence, wave collapse, and optical self-focusing. Equation (2) is also used in other areas of physics, such as the theory of superconductivity and low temperature physics (in particular, the ordinary NLS equation is a simplified 1D form of the Ginzburg–Landau equation (Ginzburg and Landau, 1950), which they first introduced in 1950 when describing superconductivity), low-amplitude gravitational waves on the surface of a deep inviscid fluid, etc. Note that 3D equation (2) is not completely integrable, and its analytical solutions are not known in general case (except, perhaps, for smooth solutions of the solitary wave type). However, under the previously developed approaches (Belashov, 1991, 1999) for other equations of the BK system (the generalized Kadomtsev–Petviashvili (GKP) equation, when in system (1) $\hat{A}(t,u) = \alpha u\partial_x - \partial_x^2(\nu - \beta\partial_x - \gamma\partial_x^3)$ and the 3D nonlinear Schrödinger equation with the derivative of the nonlinear term (3-DNLS), if the operator in (1) is $\hat{A}(t,u) = 3s|p|^2 u^2\partial_x - \partial_x^2(i\lambda + \nu))$, we can study the stability of possible solutions of the 3-GNLS equation. In this case, the dynamics of the interaction of soliton-like structures of the GNLS equation can be studied numerically with the methods developed by Belashov and Vladimirov (2005). Solving of this problem is the goal of this work.

2. STUDY OF SOLUTIONS' STABILITY

Let us write equation (2) with $\alpha = 0$ (the 3-NLS equation) in the Hamiltonian form

$$\partial_t u = \partial_x(\delta H/\delta u), \tag{3}$$

where the Hamiltonian, which has a sense of the energy of the system, is $H = \int_{-\infty}^{\infty}\left[\frac{\gamma}{2}|u|^4 + \beta uu^*\partial_x\varphi + \frac{1}{2}\sigma(\nabla_\perp\partial_x w)^2\right]d\mathbf{r}$, $\partial_x^2 w = u$, $\varphi = \arg(u)$.

Analyzing the Hamiltonian transformation properties as detailed for the equations of the BK system earlier (Belashov, 1991, 1999; Belashov and Vladimirov, 2005) and as first used for much simpler cases of the "classical" Kadomtsev–Petviashvili equation (Kuznetsov and Musher, 1986) and the NLS equation (Zakharov and Kuznetsov, 2012), we studied the stability of 2D and 3D solutions of Eq. (2). In this case, the problem for equation (3) is formulated as the variational equation $\delta(H + \upsilon P_x) = 0$, $P_x = \frac{1}{2}\int u^2\, d\mathbf{r}$, the sense of that is that all finite solutions of Eq. (3) are stationary points of the Hamiltonian H for the fixed value of the momentum projection P_x. According to the Lyapunov stability theorem, points that correspond to the minimum or maximum of the Hamiltonian H, are absolutely stable in a dynamic system. If the extremum is local, locally stable solutions will correspond to it.

Consider deformations H conserving the momentum projection P_x:

$$u(x, r_\perp) \to \zeta^{-1/2}\eta^{-1}u(x/\zeta, \mathbf{r}_\perp/\eta), \quad \zeta, \eta \in \mathrm{C}.$$

The Hamiltonian takes the form $H(\zeta, \eta) = a\zeta^{-1}\eta^{-2} + b\zeta^{-1} - c\zeta^2\eta^{-2}$ with coefficients

$$a = (\gamma/2)\int|u|^4\, d\mathbf{r}, \quad b = \beta\int uu^*\partial_x\varphi d\mathbf{r},$$
$$c = (\sigma/2)\int(\nabla_\perp\partial_x w)^2 d\mathbf{r}. \tag{4}$$

From the necessary conditions for the extremum $\partial_\zeta H = 0$, $\partial_\eta H = 0$, we immediately find its coordinates

$$\zeta_0 = -ac^{-1}, \quad \eta_0 = \left[-ab^{-1}\left(1 + a^2c^{-2}\right)\right]^{1/2},$$

where $b < 0$ if $\eta \in \mathrm{R} \subset \mathrm{C}$, because $a > 0$, $c > 0$ by definition, and $b > 0$ if $\eta \in \mathrm{C}$. The sufficient conditions for a minimum at the point (ζ_i, η_j) are

$$\begin{vmatrix}\partial_\zeta^2 H(\zeta_i,\eta_j) & \partial_{\zeta\eta}^2 H(\zeta_i,\eta_j)\\ \partial_{\eta\zeta}^2 H(\zeta_i,\eta_j) & \partial_\eta^2 H(\zeta_i,\eta_j)\end{vmatrix} > 0,$$
$$\partial_\zeta^2 H(\zeta_i,\eta_j) > 0.$$

Solving this set of inequalities, we find that $a/c < d = (2\sqrt{2})^{-1}\sqrt{13 + \sqrt{185}}$ for waves at $b < 0$ (positive nonlinearity). From this, it follows that $H > -3bd/(1 + 2d^2)$, i.e., the Hamiltonian is bounded from below. At $b > 0$ (negative nonlinearity), the replacement $b \to -b$ is equivalent to replacing $y \to -iy$, $z \to -iz$ and $H < -3bd/(1 + 2d^2)$, i.e., the Hamiltonian is not bounded from below (bounded from above).

Thus, we have proven the possibility of the existence of the stable 3D-solutions in the 3-NLS model and obtained the conditions of their stability, i.e., we determined the ranges of values of the equation coefficients (variable in the time and space characteristics of the environment) at which 3D solitons are stable.

3. NUMERICAL STUDY OF THE EVOLUTION AND DYNAMICS OF INTERACTION OF THE GNLS SOLITONS

We carried out a numerical study of the evolution and dynamics of the interaction of soliton-like structures of the 3-GNLS equation using the methods developed and described in detail earlier (Belashov and Vladimirov, 2005). The simulation results for the general case of a heterogeneous and nonstationary medium confirm the conclusions made based on an analytical consideration of the problem. By way of illustration, Figs. 1 and 2 show the results obtained at $\sigma = 0$ (1D case) for the initial conditions in the form of a soliton-like envelope pulse:

$$u(x,0) = A\exp(-x^2/l);$$
$$u(x,0) = A\exp[-(x-5)^2/l] + A\exp[-(x+5)^2/l],$$

respectively, in the simplest case of the equations NLS with β, γ = const (stationary media); $\alpha, f = 0$; at negative nonlinearity, $\beta > 0$. In this case, $b > 0$ and the Hamiltonian $H > 3bd/(1 + 2d^2)$; hence, the stability condition for negative nonlinearity $H < 3bd/(1 + 2d^2)$ does not hold, and, as can be seen from the figures, we observe scattering of the envelope pulses with time.

Figures 3 and 4 give examples when the coefficient $\gamma \neq 0$ and the stability condition for positive nonlinearity $H > 3bd/(1 + 2d^2) > 0$ is not satisfied and when the condition for negative nonlinearity $H < 0 < -3bd/(1 + 2d^2)$ is satisfied, respectively.

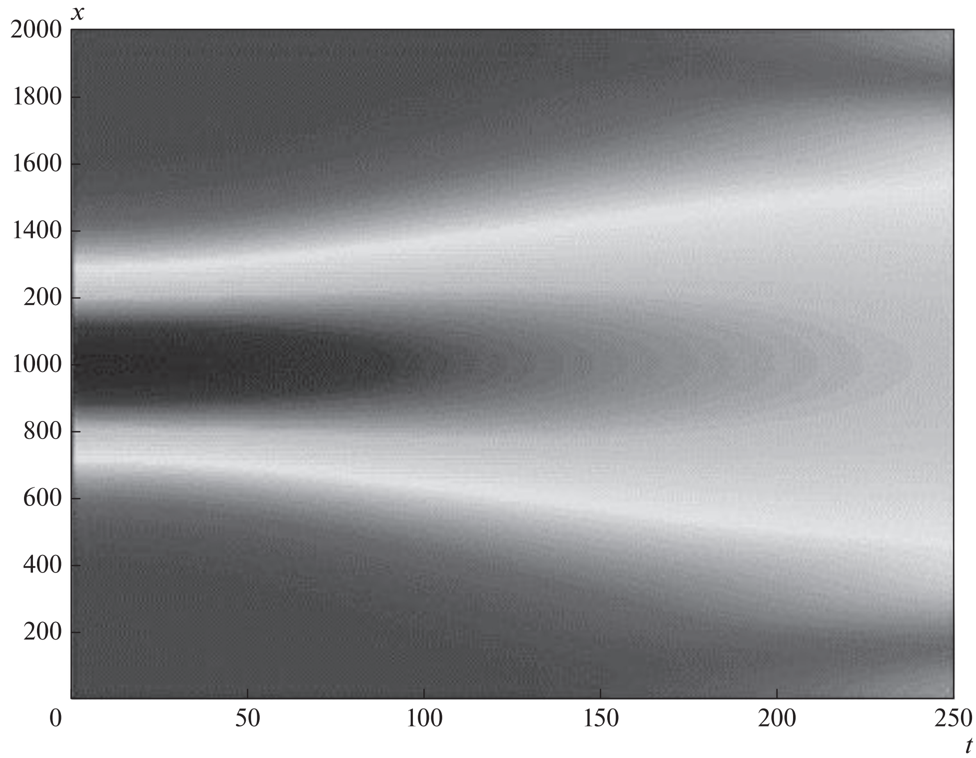

Fig. 1. Evolution of the Gaussian envelope pulse at $A = 2$, $l = 2$; $\beta = 0.5$, $\gamma = 0$.

In the first case, we observe the scattering of the envelope pulse with time. In the second, the solution is stabilized and a soliton is formed from the initial pulse.

Figure 5 shows the result for the same situation as in Fig. 4, for the case of negative nonlinearity ($\beta < 0$, $\gamma < 0$ and a, $b < 0$ in formulas (4)), but here we have a case of strong negative nonlinearity when $\gamma = -1$ and the appearance of quasi-stable powerful soliton-like breather-type pulsations is observed.

In numerical experiments for the GNLS equation (2) with $\sigma = 0$ for $\beta, \gamma = \varphi(t)$; $\alpha, f = 0$, we found that the quasi-stable evolution of the initial Gaussian pulse can be observed when in a nonstationary medium with negative nonlinearity the stability condition $H < -3bd/(1 + 2d^2)$ is satisfied. In this case, one can observe pulsations with a shift of the pulse during its evolution in the x direction (fig. 6).

Figure 7 shows two examples of the results of the evolution of a Gaussian pulse in a nonstationary medium with negative nonlinearity when the stability condition $H < -3bd/(1 + 2d^2)$ is fulfilled. As a result of the evolution, in this case, the occurrence of powerful stable pulsations of the breather type from the initial solitary pulse is observed.

Figures 8 and 9 show examples of the interaction of soliton-like initial pulses of the forms

$$u(x,0) = A[\mathrm{sch}(x) + \mathrm{sch}(x - s/2) + \mathrm{sch}(x + s/2)],$$
$$u(x,0) = A[\mathrm{sch}(x - s/2) + \mathrm{sch}(x + s/2)] \quad (5)$$

at negative nonlinearity within the GNLS model, respectively. In the first case, the stability condition is not satisfied; at the first stage, we observe the occurrence of one powerful pulse from a three-pulse initial disturbance and, then its decay into two pulses of small amplitude with time. In the second case, the stability condition is satisfied, and a stable evolution of the two-pulse disturbance takes place.

It is interesting to note that when at a particular moment of time the field becomes non-stationary, the stability of the multipulse disturbance may be violated, and the evolution process becomes unstable with small pulsations. Such a case can be seen in Figure 10 (compare to the case shown in Figure 9).

Numerical experiments also showed that at weak negative nonlinearity when the stability condition is satisfied, a transition from stable evolution to regime of stable pulsations (breathers) is observed at decreasing the initial distance s between pulses in (5) (Belashov et al., 2019a; Belashov et al., 2019).

Detailed numerical studies of the evolution and interaction problems for the 2D and 3D pulses in the 3-GNLS model were discussed earlier (Belashov et al., 2018b, 2019a, b; Belashov and Kharshiladze, 2019; Belashov et al., 2019, 2020).

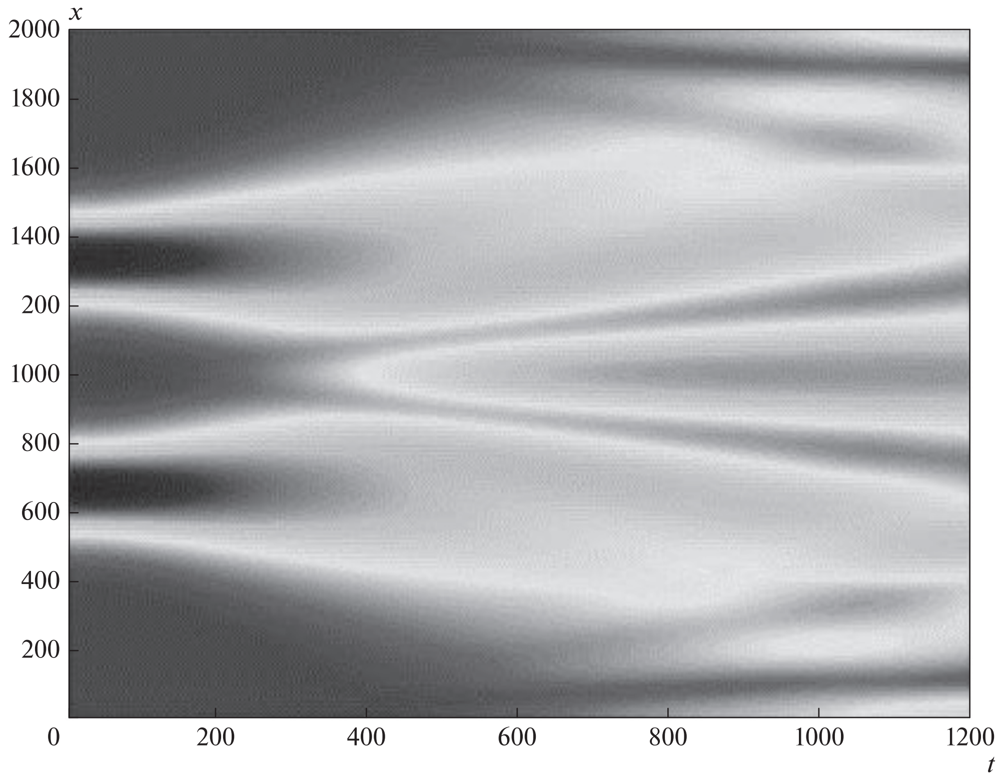

Fig. 2. Evolution of a Gaussian two-pulse envelope disturbance at $A = 1$, $l = 4$; $\beta = 0.5$, $\gamma = 0$.

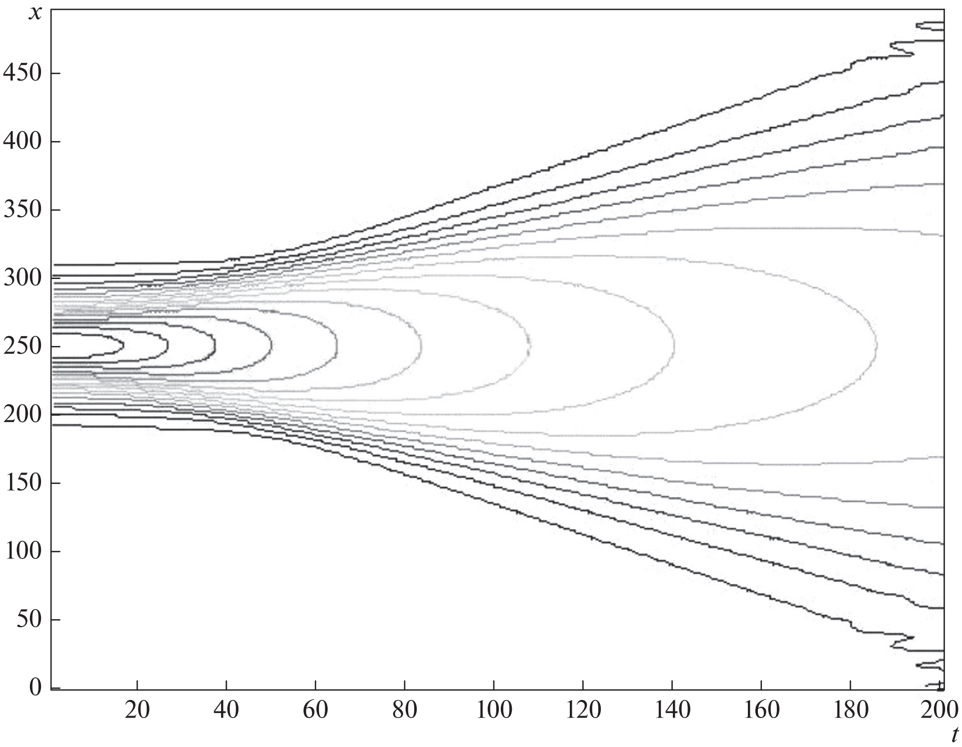

Fig. 3. Evolution of the Gaussian envelope pulse at $A = 1$, $l = 4$; $\beta < 0$, $\gamma > 0$ (positive nonlinearity).

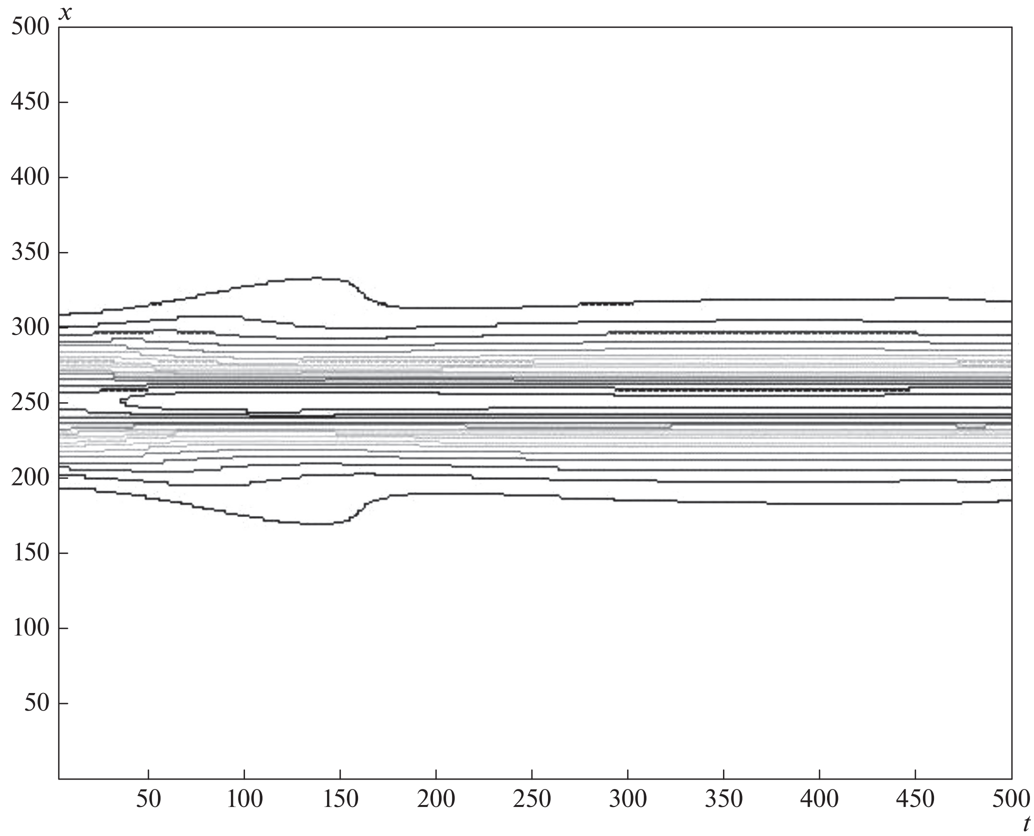

Fig. 4. Evolution of the Gaussian envelope pulse at $A = 1$, $l = 4$; $\beta < 0$, $\gamma = -0.5 < 0$ (negative nonlinearity).

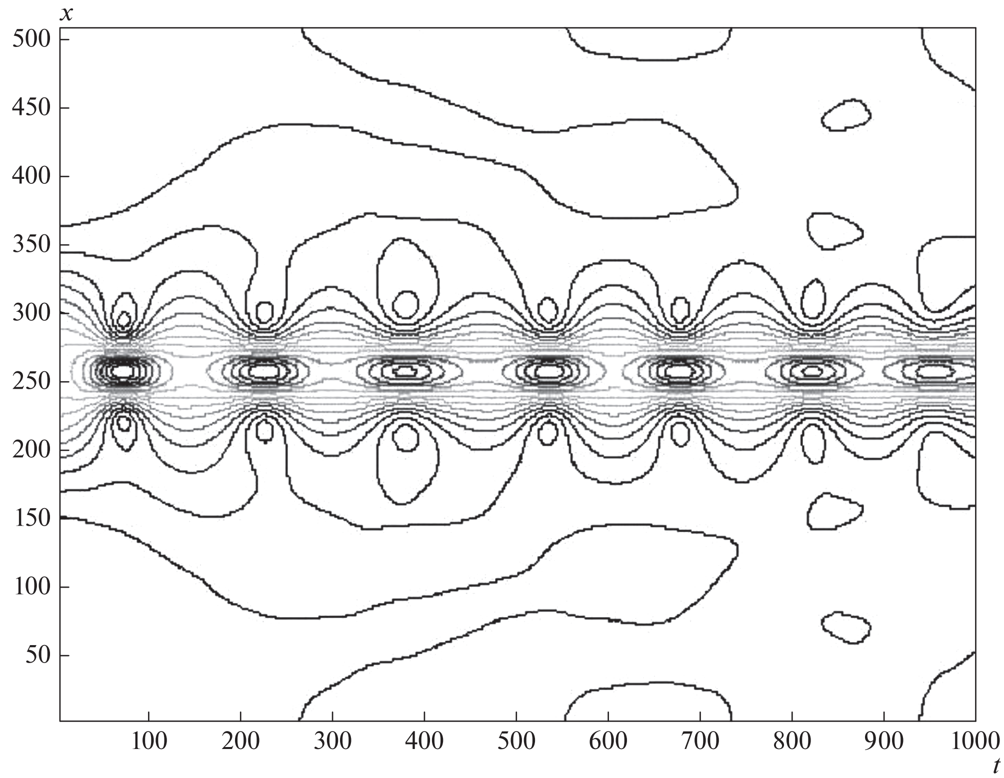

Fig. 5. Evolution of the Gaussian envelope pulse at $A = 1$, $l = 4$; $\beta < 0$, $\gamma = -1$.

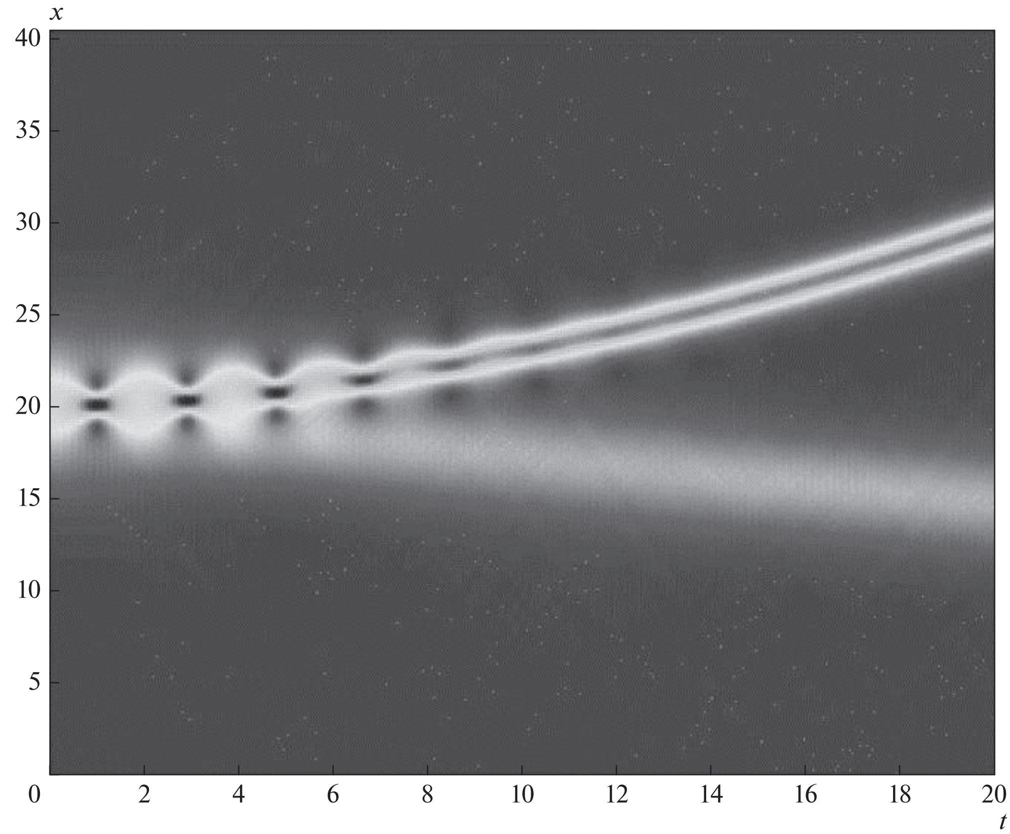

Fig. 6. Evolution of the Gaussian envelope pulse at $A = 1$, $l = 4$; $\beta(t) = -0.5\,(1 + \sin 0.1\pi t)$, $\gamma = -1$.

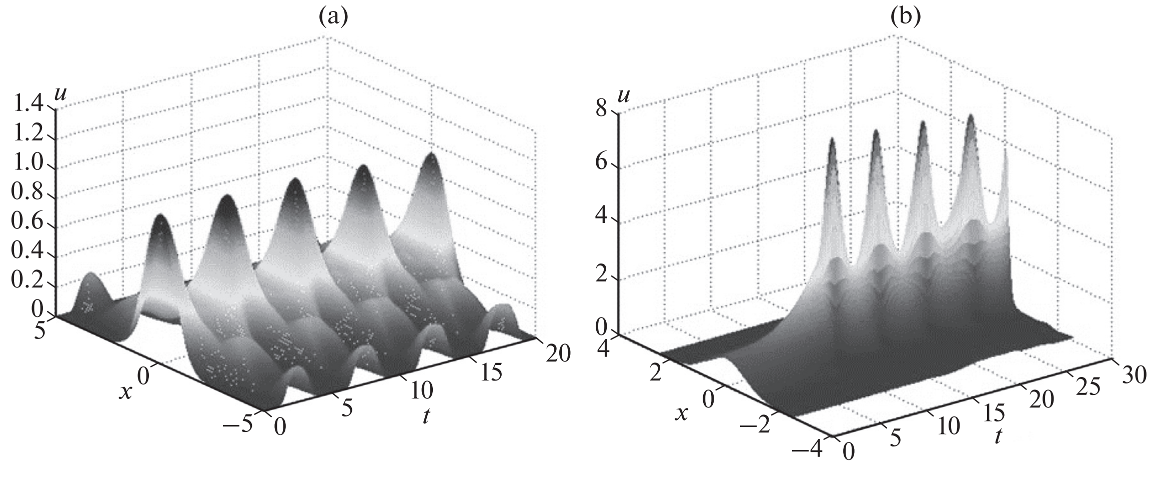

Fig. 7. Evolution of the Gaussian envelope pulse in a nonstationary medium at $\alpha, f = 0$: (*a*) $\beta = 0.5$, $\gamma = -1 + 0.01\sin 2\pi t$; (*b*) $\gamma = -1$, $\beta(t) = -0.5$ at $t \leq 5$ and $\beta(t) = 0.5\,(1 + 0.2\sin 2\pi t)$ at $t > 5$; cases of negative nonlinearity.

4. CONCLUSIONS

Summarizing the results, we note the following.

1. We discussed the problem of the evolution and dynamics of multidimensional solutions of the generalized NLS (GNLS) equation as a particular case of the BK system, namely, the stability of 3D solutions of the 3-GNLS equation and the dynamics of stable and unstable solutions of the NLS equation in stationary and nonstationary media.

2. In this paper, conditions dividing the classes of stable and unstable soliton-like solutions of the GNLS

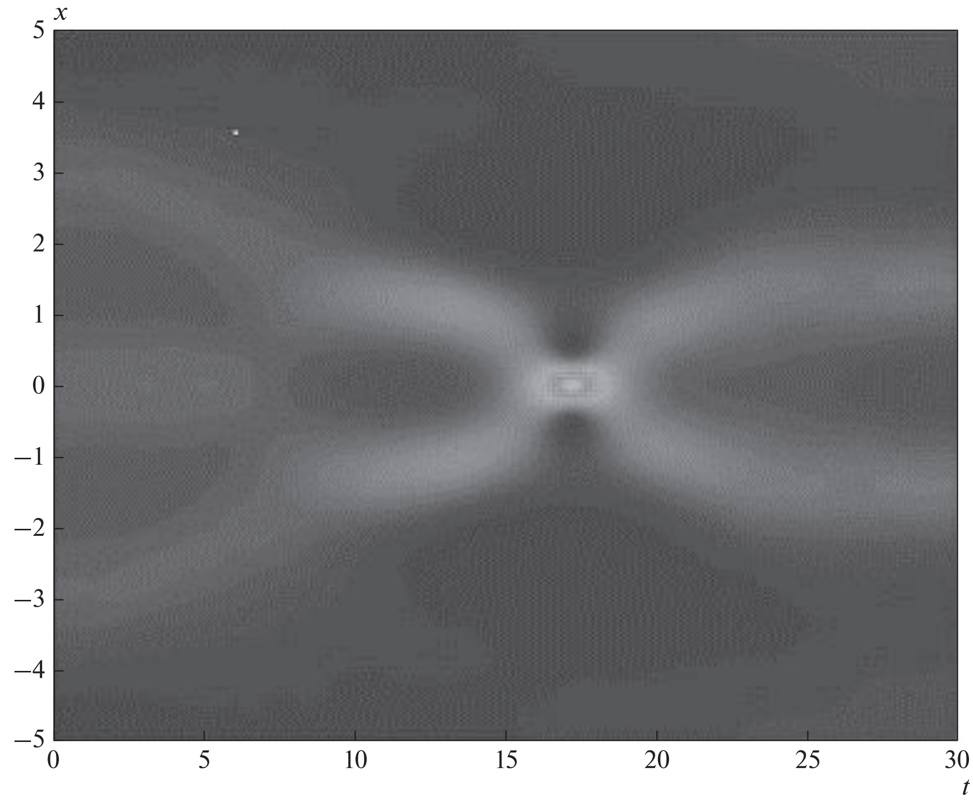

Fig. 8. Interaction of three GNLS pulses (stationary medium) at $\gamma = -1$, $\beta = 0.25$; case of weak negative nonlinearity.

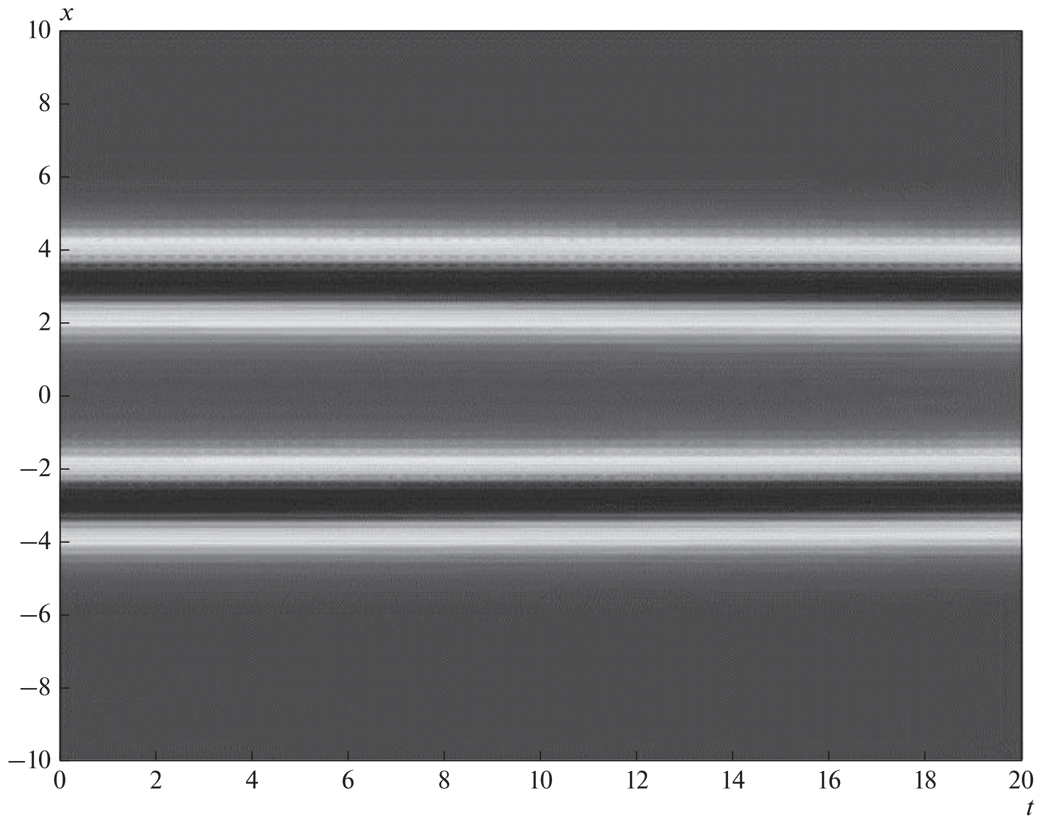

Fig. 9. Lack of the GNLS pulses interaction (stationary medium) at $\gamma = -1$, $\beta = 0.05$; the case of negative nonlinearity.

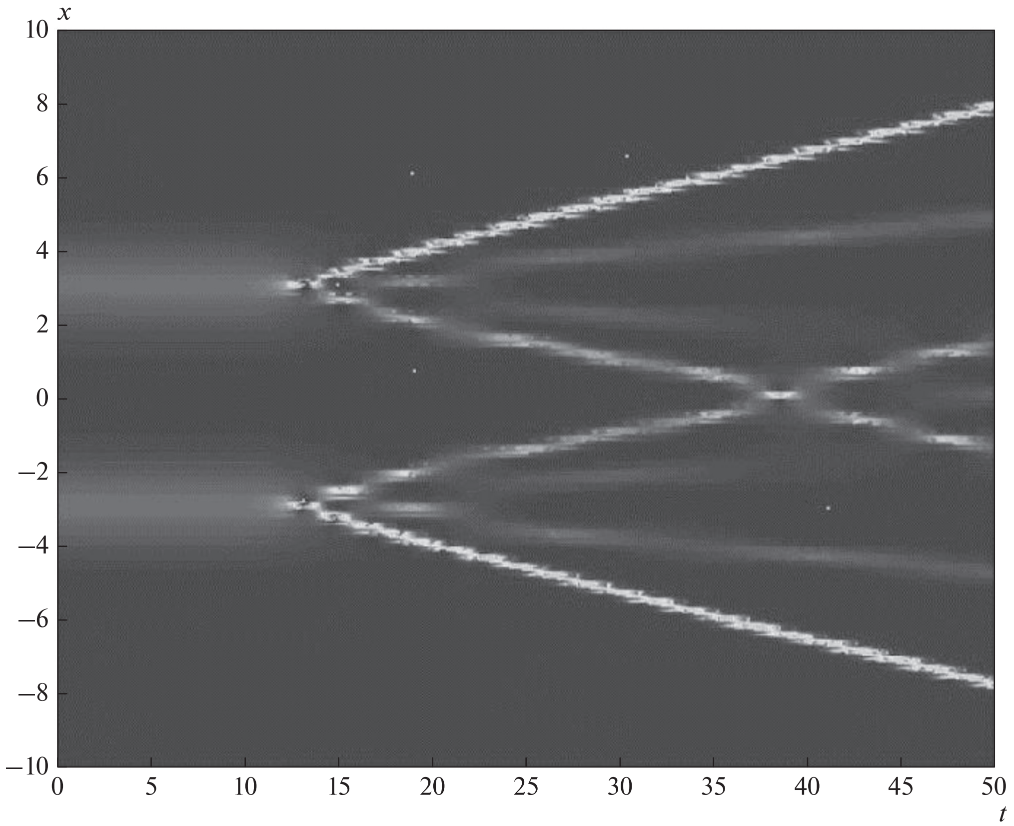

Fig. 10. Evolution of a two-pulse disturbance in a nonstationary medium at $\gamma = -1$, $\beta(t)=0.5$ for $t \leq 15$ and $\beta(t) = 0.5\,(1 + 0.2\sin(2\pi t/15))$ for $t > 15$; negative nonlinearity.

equation were obtained analytically, and sufficient conditions of stability of multidimensional solutions were found.

3. It was shown that, even in the simplest 1D case, the GNLS equation has a wide class of stable or quasi-stable solutions of types of solitons and breathers, and also unstable pulsing solutions that dissipate with time.

4. The obtained analytical results were confirmed by a numerical study of cases of the stable and unstable (with the formation of breathers) evolution of pulses of various shapes, as well as the interaction of two- and three-pulse structures leading to formation of stable and unstable solutions.

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