Periodic Motion

Periodic motion of some source object is necessary to produce a sustained musical sound (i.e., one with definite pitch and quality). For example, to produce a standard musical A (440 Hz), the source object must sustain periodic motion at 440 vibrations per second with a tolerance of less than 1 Hz -- the normal human ear can detect the difference between 440 Hz and 441 Hz. The conditions necessary for periodic motion are

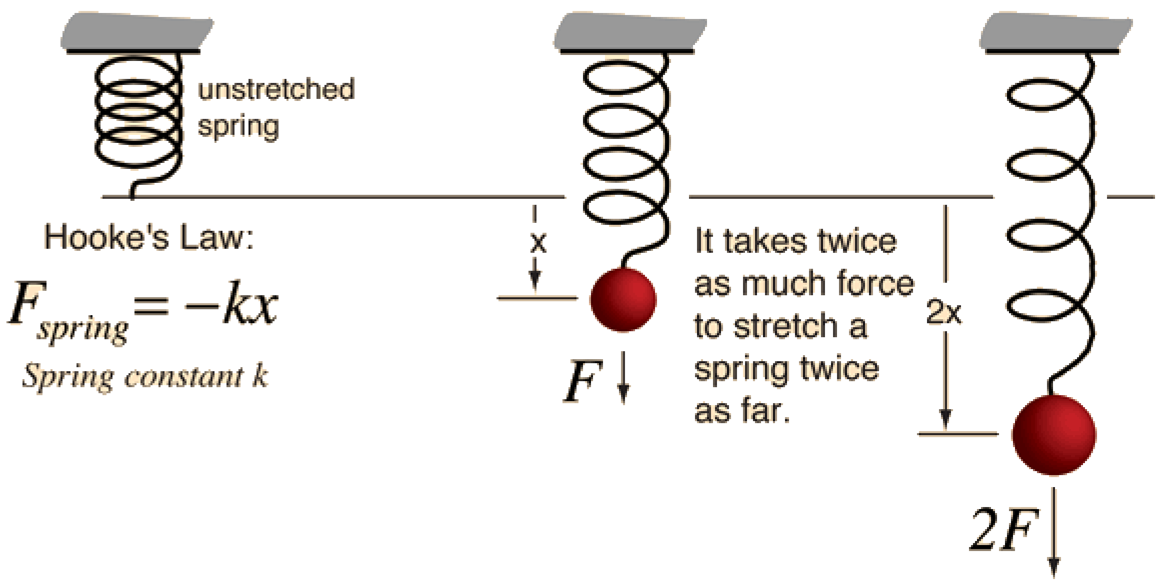
1. ***elasticity*** - the capacity to return precisely to the original configuration after being distorted.
    - a definite equilibrium configuration
    - a restoring force to bring the system back to equilibrium

2. A source of energy.

Periodic Motion

Elasticity is the property of an object or material which causes it to be restored to its original shape after distortion. It is said to be more elastic if it restores itself more precisely to its original configuration. A spring is an example of an elastic object - when stretched, it exerts a restoring force which tends to bring it back to its original length. This restoring force is generally proportional to the amount of stretch, as described by ***Hooke's Law***.

One of the properties of elasticity is that it takes about twice as much force to stretch a spring twice as far. That linear dependence of displacement upon stretching force is called Hooke's law.



Description of Periodic Motion

Motion which repeats itself precisely can be described with the following terms:

- **Period**: the time required to complete a full cycle, **T** in seconds/cycle
- **Frequency**: the number of cycles per second, **f** in 1/seconds or Hertz (Hz)
- **Amplitude**: the maximum displacement from equilibrium **A**

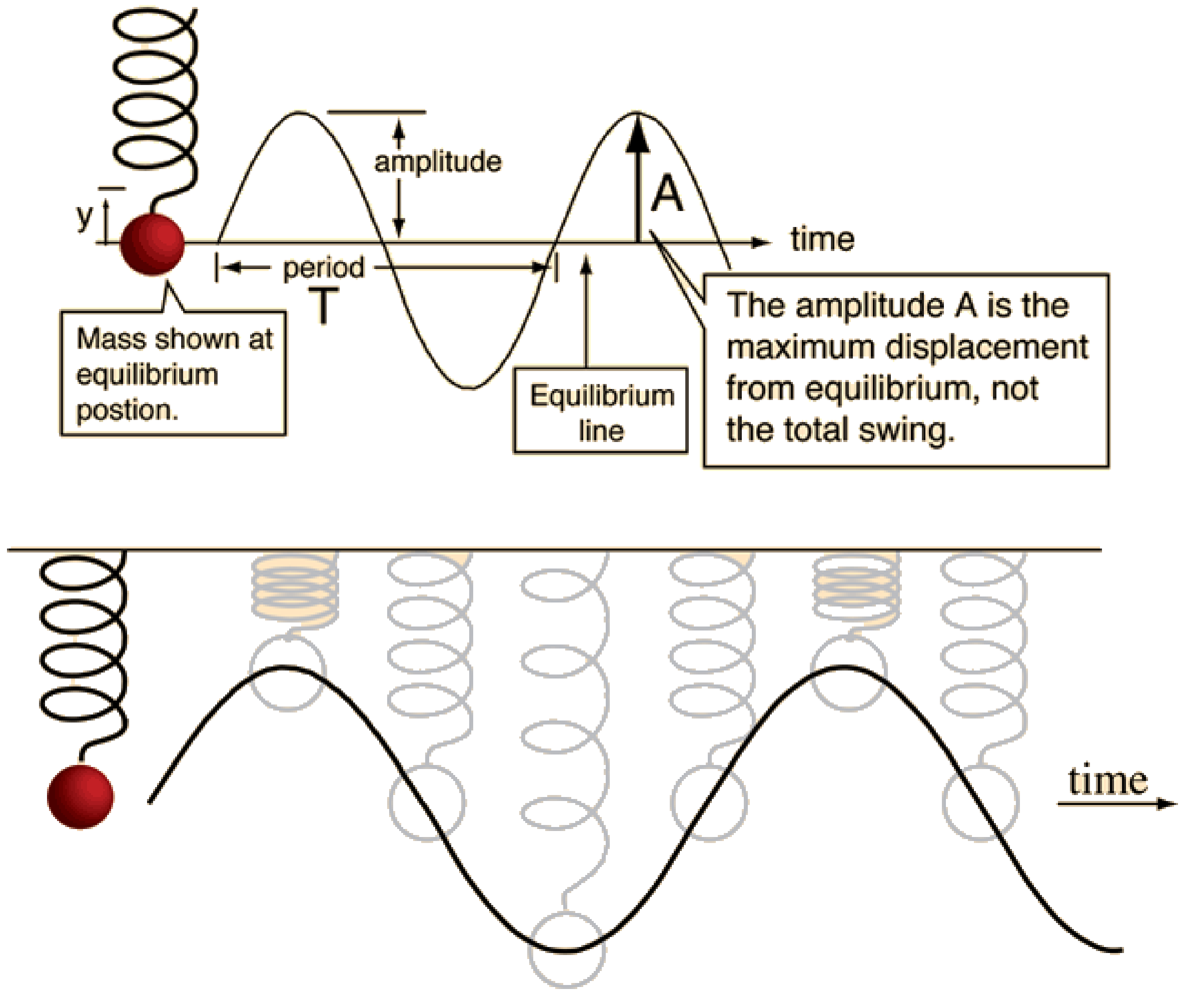
and if the periodic motion is in the form of a traveling wave, one needs also

- Velocity of propagation: **v**
- Wavelength: repeat distance of wave **λ**.

In a plot of periodic motion as a function of time, the period can be seen as the repeat time for the motion. The frequency is the reciprocal of the period.

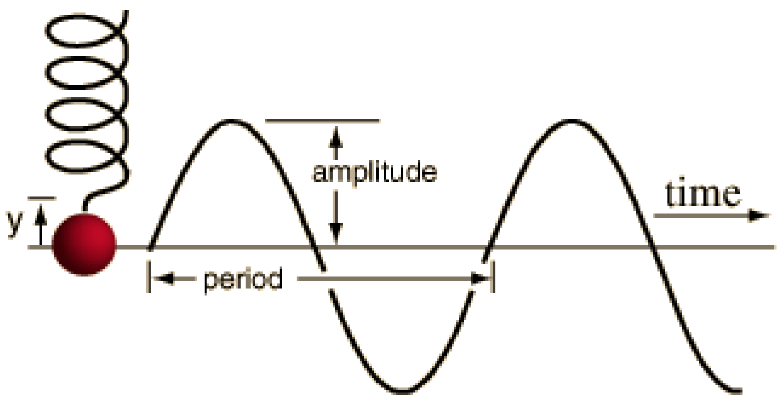
$f = \frac{1}{T}$, f - frequency $\qquad$ $T = \frac{1}{f}$, T - period

Description of Periodic Motion



Simple Harmonic Motion

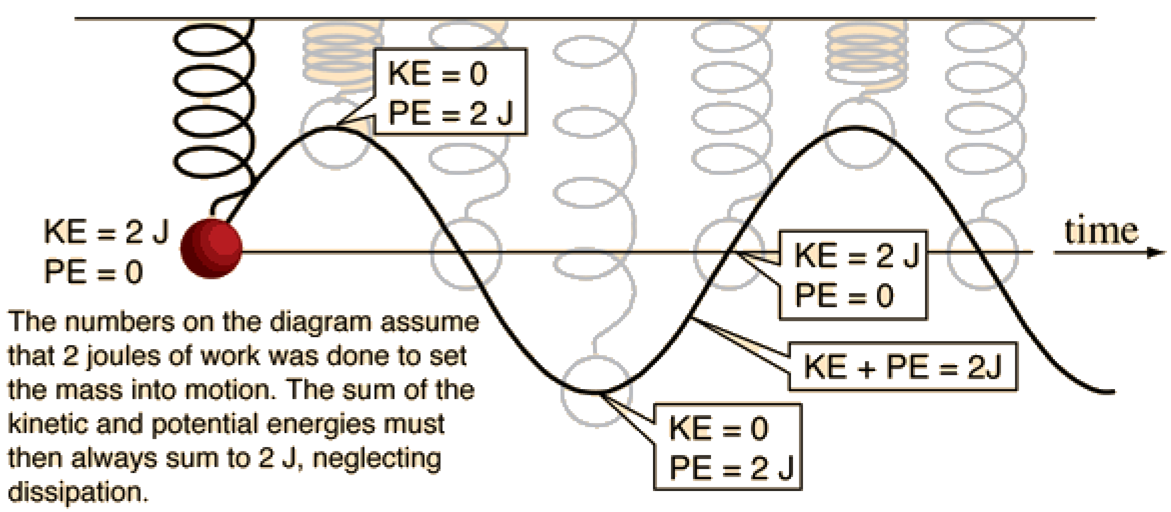
When a mass is acted upon by an elastic force which tends to bring it back to its equilibrium configuration, and when that force is proportional to the distance from equilibrium (e.g., doubles when the distance from equilibrium doubles, a Hooke's Law force), then the object will undergo simple harmonic motion when released.



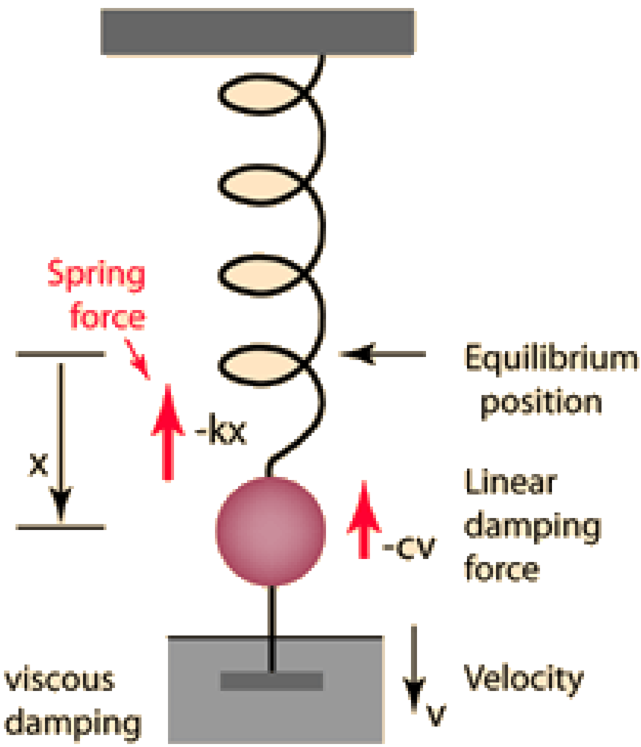
A mass on a spring is the standard example of such periodic motion. If the displacement of the mass is plotted as a function of time, it will trace out a pure sine wave.

Energy of an Oscillator

A mass on a spring transforms energy back and forth between kinetic and potential energy. If there were no dissipation, conservation of energy would dictate that the motion would continue forever. For any real vibrating object, the implication of the conservation of energy principle is that the vibrator will continue the transformation from kinetic to potential energy until all the energy is transferred into some other form. To set the object into motion, a net external force must do work on the mass to initially stretch the spring, that amount of work being 2 joules in the example below.



Damped Harmonic Oscillator



The Newton's 2nd Law motion equation is

$$ma + cv + kx = 0$$

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

This is in the form of a homogeneous second order differential equation and has a solution of the form

$$x = e^{\lambda t}$$

Substituting this form gives an auxiliary equation for λ

$$m\lambda^2 + c\lambda + k = 0$$

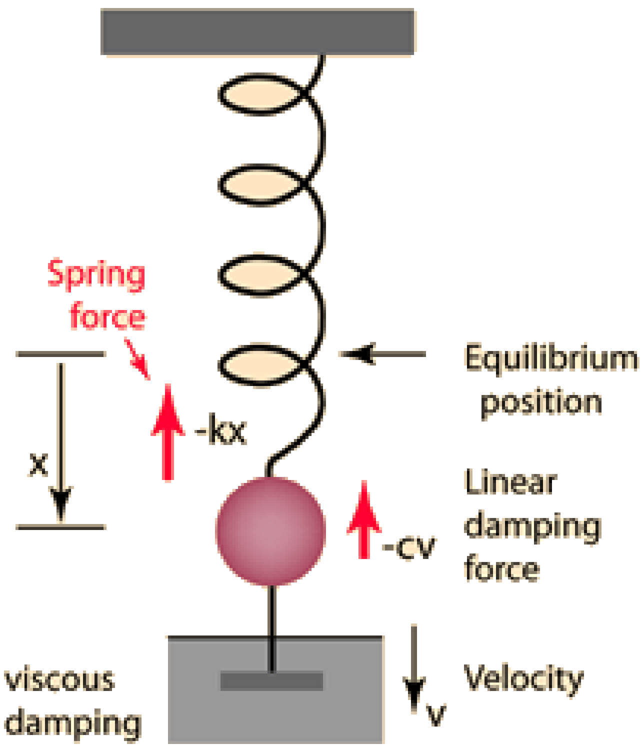
The roots of the quadratic auxiliary equation are

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

The three resulting cases for the damped oscillator are

$c^2 - 4mk > 0$	Overdamped
$c^2 - 4mk = 0$	Critical damping
$c^2 - 4mk < 0$	Underdamped

Damping Coefficient



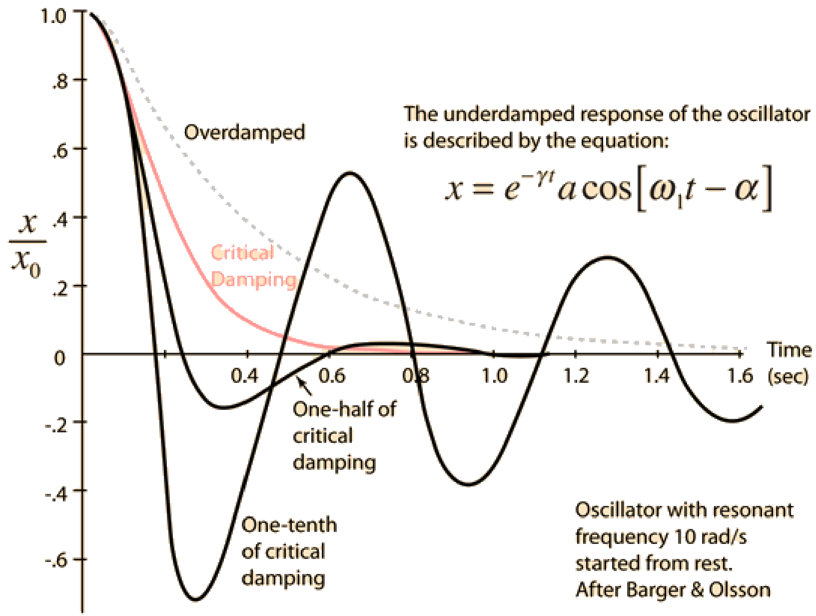
When a damped oscillator is subject to a damping force which is linearly dependent upon the velocity, such as viscous damping, the oscillation will have exponential decay terms which depend upon a damping coefficient. If the damping force is of the form

$$F_{damping} = -cv$$

then the damping coefficient is given by

$$\gamma = \frac{c}{2m}$$

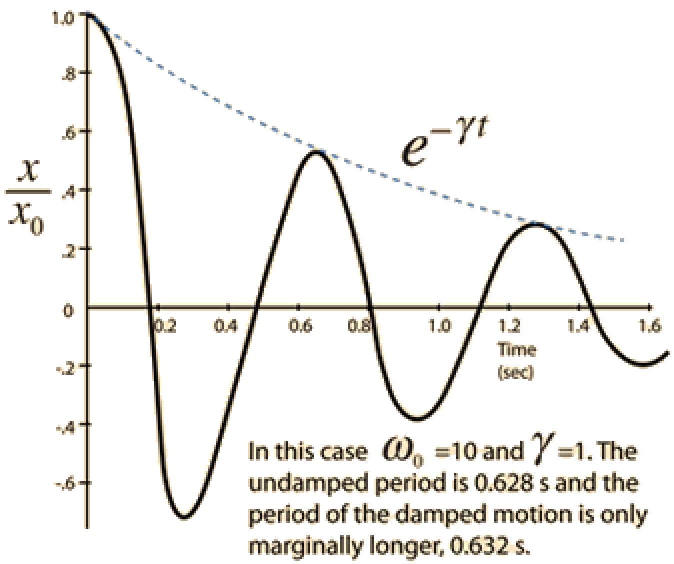
Underdamped Oscillator



For any value of the damping coefficient γ less than the critical damping factor the mass will overshoot the zero point and oscillate about x=0. The behavior is shown for one-half and one-tenth of the critical damping factor. Also shown is an example of the overdamped case with twice the critical damping factor.

Note that these examples are for the same specific initial conditions, i.e., a release from rest at a position x0. For other initial conditions, the curves would look different, but the behavior with time would still decay according to the damping factor.

Underdamped Oscillator



When a damped oscillator is underdamped, it approaches zero faster than in the case of critical damping, but oscillates about that zero.

The equation is that of an exponentially decaying sinusoid.

$$x = e^{-\gamma t} a \cos[\omega_1 t - \alpha]$$

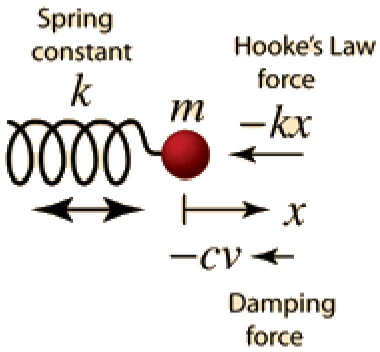
The damping coefficient is less than the undamped resonant frequency . The sinusoid frequency is given by

$$\omega_1 = \sqrt{{\omega_1}^2 - \gamma^2}$$

but the motion is not strictly periodic

Driven Oscillator

If a damped oscillator is driven by an external force, the solution to the motion equation has two parts, a transient part and a steady-state part, which must be used together to fit the physical boundary conditions of the problem.



The motion equation is of the form

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_0\cos(\omega t + \varphi_d)$$

and has a general solution

$$x(t) = x_{transient} + x_{steady\ state}$$

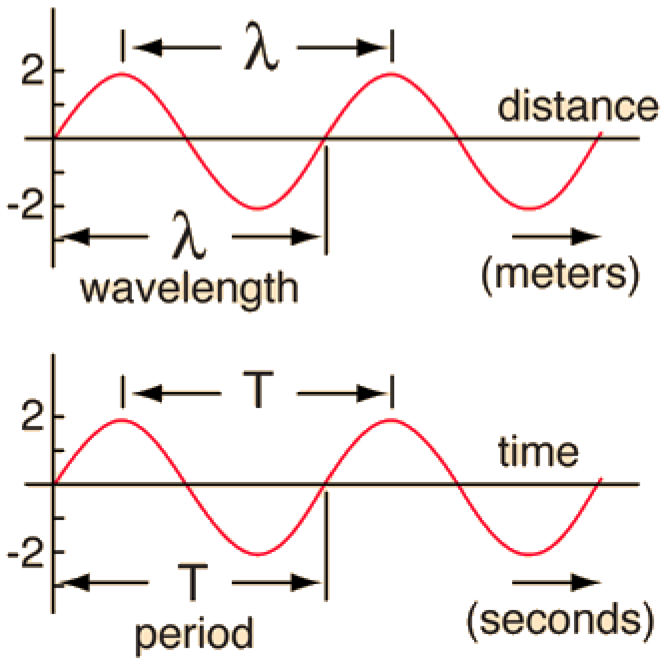
In the underdamped case this solution takes the form

Transient solution — Steady-state solution

$$x(t) = A_h e^{-\gamma t}\sin(\omega' t + \varphi_h) + A\cos(\omega t - \varphi)$$

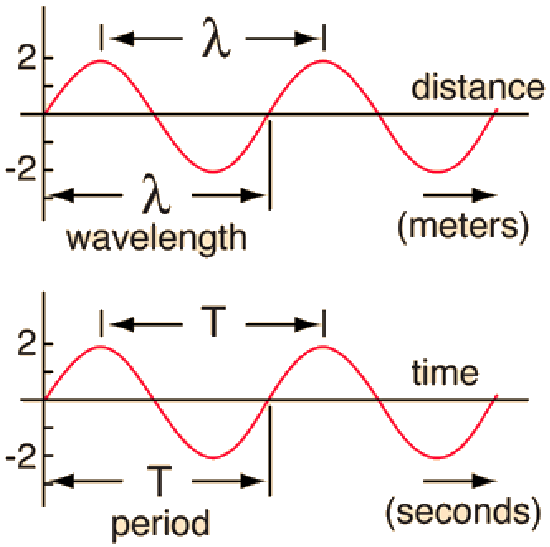
Determined by initial position and velocity — Determined by driving force

Wave Graphs

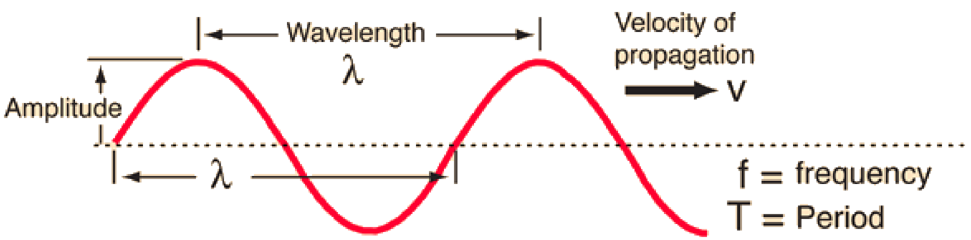


Waves may be graphed as a function of time or distance. A single frequency wave will appear as a sine wave in either case. From the distance graph the wavelength may be determined. From the time graph, the period and frequency can be obtained. From both together, the wave speed can be determined.

Traveling Waves



Waves may be graphed as a function of time or distance. A single frequency wave will appear as a sine wave in either case. From the distance graph the wavelength may be determined. From the time graph, the period and frequency can be obtained. From both together, the wave speed can be determined.



The motion relationship "distance = velocity x time" is the key to the basic wave relationship. With the wavelength as distance, this relationship becomes λ = vT. Then using f=1/T gives the standard wave relationship

$$v = f\lambda$$

Wave velocity = **frequency** x **wavelength**

This is a general wave relationship which applies to sound and light waves, other electromagnetic waves, and waves in mechanical media.

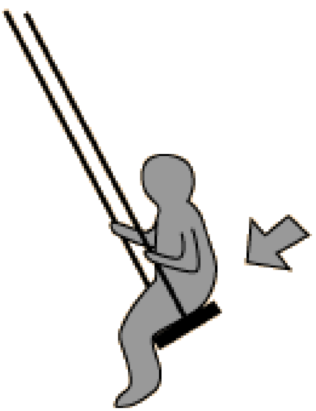
Resonance

In sound applications, a resonant frequency is a natural frequency of vibration determined by the physical parameters of the vibrating object. This same basic idea of physically determined natural frequencies applies throughout physics in mechanics, electricity and magnetism, and even throughout the realm of modern physics.

It is easy to get an object to vibrate at its resonant frequencies, hard at other frequencies.

A child's playground swing is an example of a pendulum, a resonant system with only one resonant frequency. With a tiny push on the swing each time it comes back to you, you can continue to build up the amplitude of swing. If you try to force it to swing a twice that frequency, you will find it very difficult.

Swinging a child in a playground swing is an easy job because you are helped by its natural frequency.

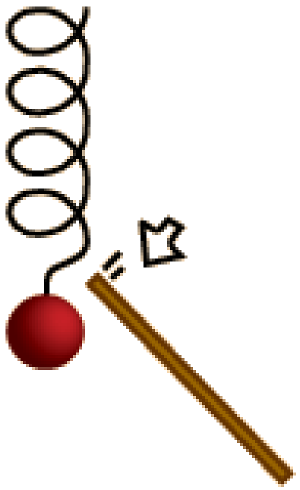


But can you swing it at some other frequency?

Picking out resonant frequencies

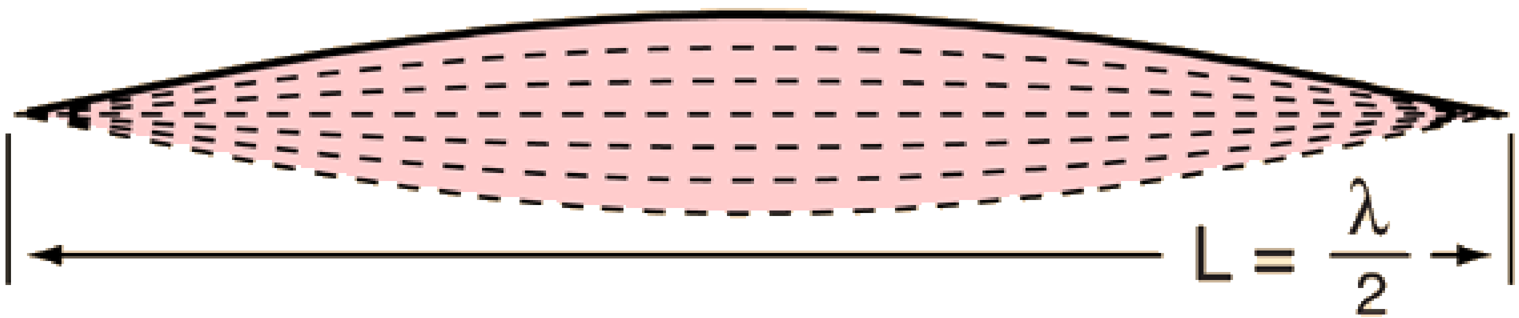
A vibrating object will pick out its resonant frequencies from a complex excitation and vibrate at those frequencies, essentially "filtering out" other frequencies present in the excitation.

If you just whack a mass on a spring with a stick, the initial motion may be complex, but the main response will be to bob up and down at its natural frequency. The blow with the stick is a complex excitation with many frequency components, but the spring picks out its natural frequency and responds to that.



Vibrating String

The fundamental vibrational mode of a stretched string is such that the wavelength is twice the length of the string.



Applying the basic wave relationship gives an expression for the fundamental frequency:

$$f_1 = \frac{v_{wave\ on\ string}}{2L}$$

Since the wave velocity is given by $v = \sqrt{\frac{T}{m/L}}$, the frequency expression can be put in the form:

$$f_1 = \frac{\sqrt{\frac{T}{m/L}}}{2L}$$

T = string tension

m = string mass

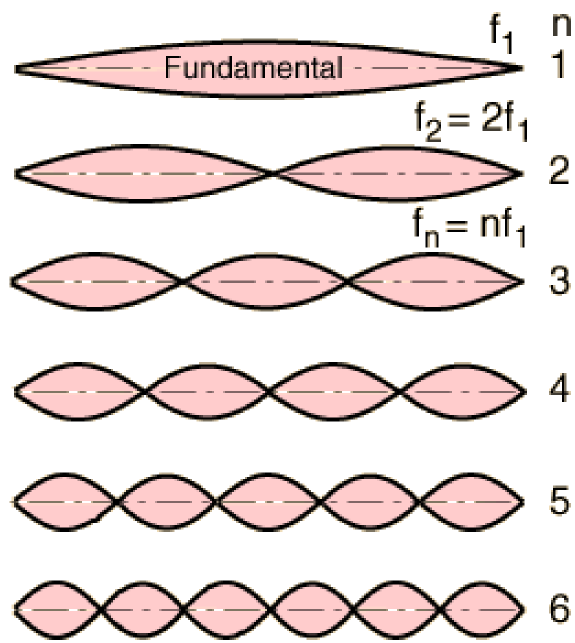
L = string length

Harmonics

An ideal vibrating string will vibrate with its fundamental frequency and all harmonics of that frequency. The position of nodes and antinodes is just the opposite of those for an open air column.

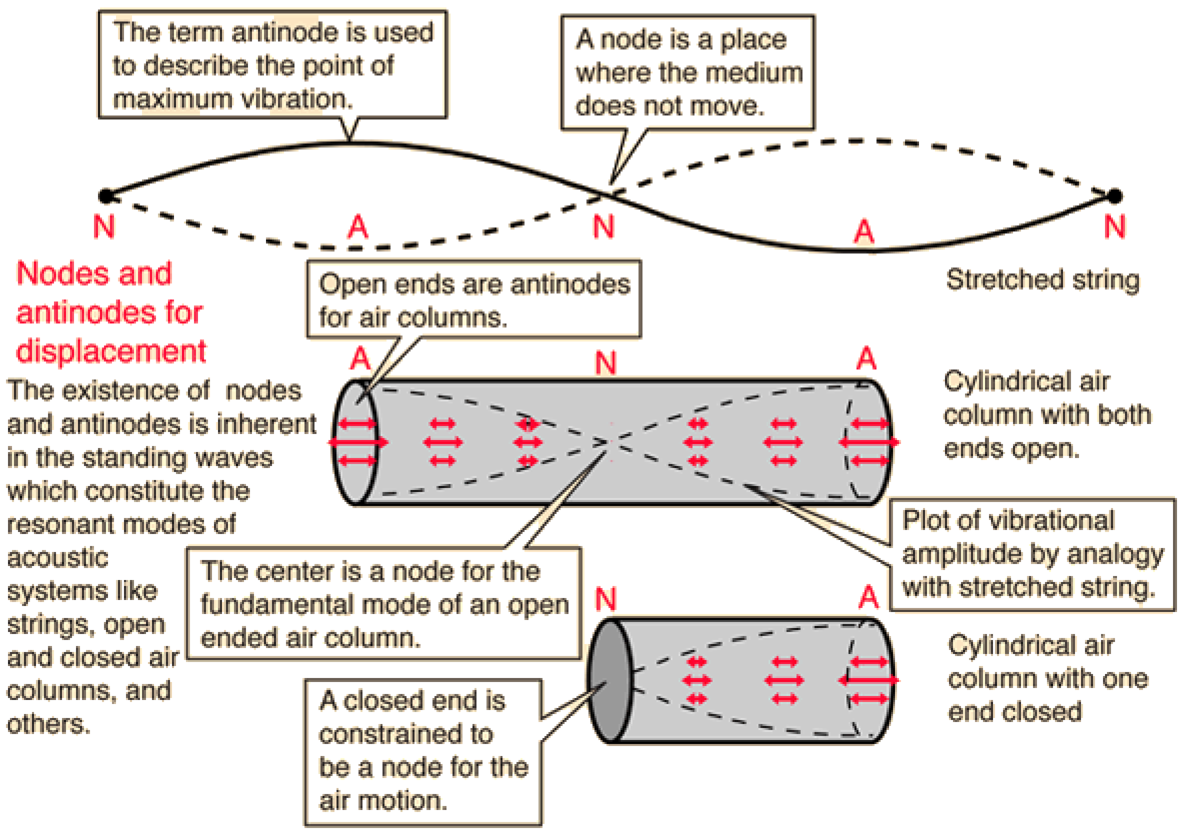
The fundamental frequency can be calculated from

$$f_1 = \frac{v_{wave\ on\ string}}{2L}$$

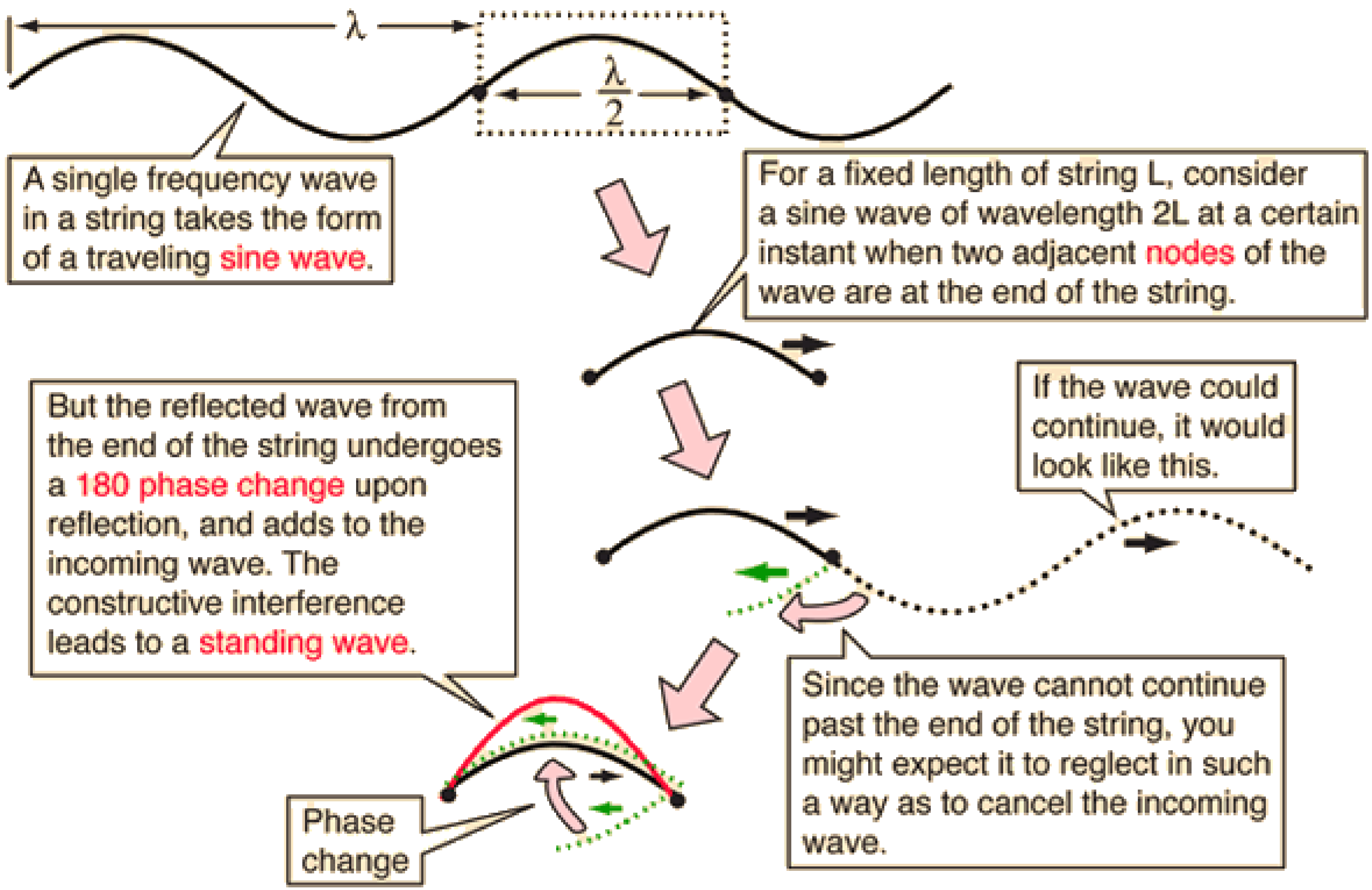


Nodes and Antinodes

The standing waves produced by wave motion in strings or air columns can be used to establish the values for wavelength, frequency and speed for the waves in accordance with the wave relationship, $v = f\lambda$.



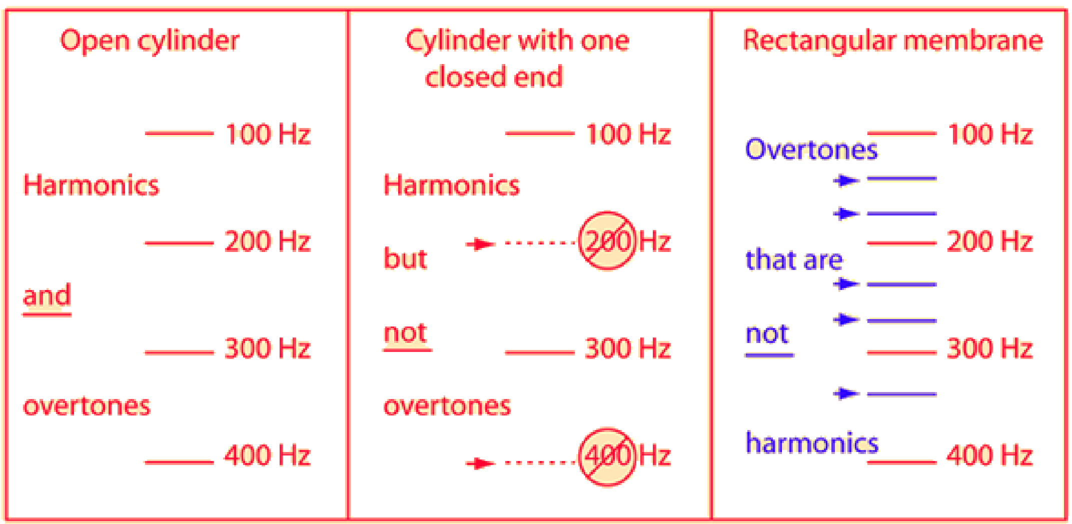
Steps to Produce String Resonance



Overtones and Harmonics

The term harmonic has a precise meaning - that of an integer (whole number) multiple of the fundamental frequency of a vibrating object. The term overtone is used to refer to any resonant frequency above the fundamental frequency - an overtone may or may not be a harmonic. Many of the instruments of the orchestra, those utilizing strings or air columns, produce the fundamental frequency and harmonics. Their overtones can be said to be harmonic. Other sound sources such as the membranes or other percussive sources may have resonant frequencies which are not whole number multiples of their fundamental frequencies. They are said to have non-harmonic overtones.

All harmonics are overtones for an open air column or a string



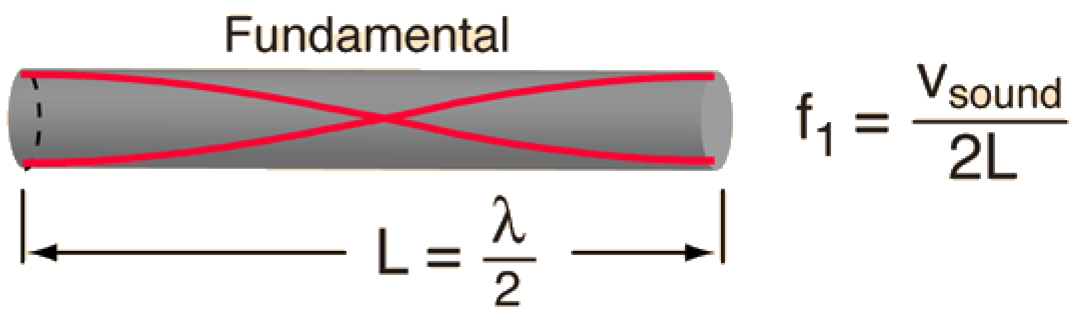
Closed air columns produce only odd harmonics.

A rectangular membrane produces harmonics, but also some other overtones.

Air Column Resonance

The resonant frequencies of air columns depend upon the speed of sound in air as well as the length and geometry of the air column. Longitudinal pressure waves reflect from either closed or open ends to set up standing wave patterns. Important in the visualization of these standing waves is the location of the nodes and antinodes of pressure and displacement for the air in the columns.

A cylindrical air column with both ends open will vibrate with a fundamental mode such that the air column length is one half the wavelength of the sound wave. Each end of the column must be an antinode for the air

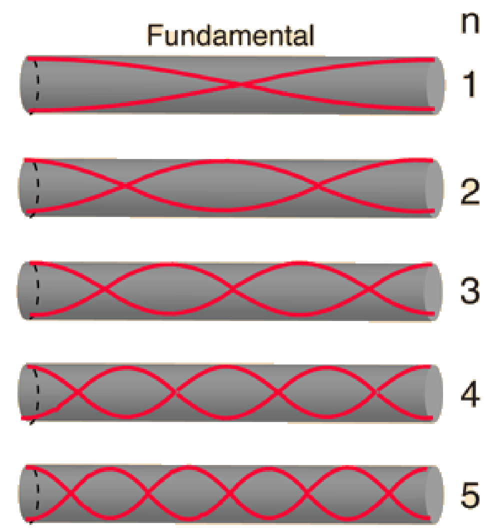


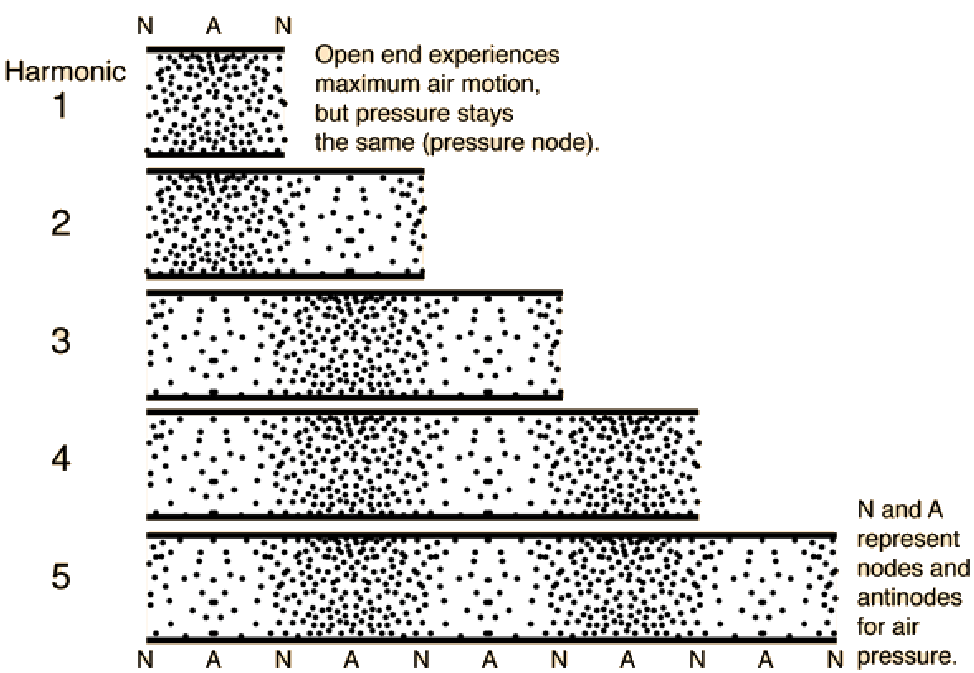
motion since the ends are open to the atmosphere and cannot produce significant pressure changes. For the fundamental mode, there is one node at the center.

The open air column can produce all harmonics. Open cylinders are employed musically in the flute, the recorder, and the open organ pipe.

Harmonics of Open Air Column

An open cylindrical air column can produce all harmonics of the fundamental. The positions of the nodes and antinodes are reversed compared to those of a vibrating string, but both systems can produce all harmonics. The sinusoidal patterns indicate the displacement nodes and antinodes for the harmonics. A pressure node corresponds to a displacement antinode, and the harmonic patterns can also be visualized in terms of air pressure or density patterns.





This is a depiction of air pressure and density variations for first five standing wave modes of an open cylinder. The ends are constrained to be nodes of pressure, being essentially at atmospheric pressure. A half cycle later, all the high pressure points would be low pressure points, and vice versa.