Dissimilarity-Based Correlation of Movements and Events on Circular Scales of Space and Time

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Abstract. Circular scales appear in many applications related to a comparative analysis of the timing of events, wind directions, animals and vehicle movement directions, etc. The paper introduces a new non-statistical correlation function on circular scales based on a recently proposed approach to constructing correlation functions (association measures) using (dis)similarity measures on the set with an involutive operation. An involutive negation and a dissimilarity function satisfying required properties on a circular set are introduced and used for constructing the new correlation function. This correlation function can measure correlation both between two grades of circular scale and between sets of measurements of two circular variables.

Keywords: Circular Scale, Correlation Coefficient, Correlation Function, Similarity Function, Negation on Circular Scale.

1 Introduction

The paper introduces the dissimilarity-based correlation function on circular scales. Such scales appear in many applications related to the comparative analysis of time events, animals and vehicle movements, wind directions in weather analysis and forecasting, etc. [12-19]. It was proposed several correlation coefficients on a circular scale. Traditionally these coefficients based on a statistical approach to the calculation of correlation between circular variables given by sets of measurements of these variables [12-16, 18]. Recently it was introduced a general approach to the construction of a correlation functions (association measures) on sets with involution (reflection) operation using (dis)similarity functions [2-4, 9, 10]. Depending on the domain, an involution operation can denote a negation, complement, multiplication on -1, and another operation mapping elements of the domain into "opposite elements." In this approach, the correlation between the two objects is positive if they are "similar" and negative if they are "opposite." An "object" can denote an element of a set, a variable given by the set of measurements, a time series, a vector, etc. As usual, when the generalization of some theory includes the old one, the new approach to the construction of correlation coefficients includes as particular cases the classical correlation and association coefficients (Pearson's, Spearman's, Kendall's, Yule's Q, etc.) [2, 3, 8-10]. Based on (dis)similarity-based approach, the correlation functions (association measures) have been introduced on the set [0,1] of probability and membership values, on the set of fuzzy sets, on the set of subintervals of [0,1], for bipolar rating profiles, and for binary data [4-7, 11].

Based on this new approach to the construction of correlation functions, the paper introduces the reflection (negation) operation involutively mapping the grades of circular scale into "opposite" grades. Further, the dissimilarity function satisfying some necessary properties on the set of the grades of the circular scale is introduced. Finally, using this dissimilarity function, a new correlation function on a circular scale is constructed. Some examples of calculation of the new dissimilarity and correlation functions are given.

The paper has the following structure. Section 2 gives a short introduction to circular scales. Section 3 gives basic definitions of (dis)similarity and correlation functions and describes the method of construction of correlation function from these functions. Section 4 introduces an involutive negation, new dissimilarity, and correlation functions on circular scales and provides some examples. Sections 5 and 6 contain short discussions of how the new correlation functions can be used for calculation of correlation between variables given by sets of measurements on circular scales, and how circular scales can be considered as relative and dynamic scales in the analysis of movements of vehicles. Section 7 presents conclusions and discussion of future work.

2 Circular Scales

Consider some examples of circular scales. A 24-hour clock called a military time contains 24 grades $(1, 2, \dots, 12, 13, \dots, 24)$ where 24 is also considered as 0. Such a scale can have more grades if it includes minutes. It is also used circular (round or circle) calendars with 12 months or a circular calendar with seven days of the week. Another example gives the scale of angular measurements from $0℃$ to $360℃$. In such a circular scale, $360℃$ can be considered as $0℃$. Angles can be measured in radians. There are also used several types of the compass rose circular scales: 8-point compass rose (windrose) with the grades (N, NO, O, SO, S, SW, W, NW), see Fig. 1 [17], 16-point or 32-point compass rose, an ancient 12-wind rose, etc.

In the following section, we consider basic definitions of dissimilarity and correlation functions that will be introduced later on circular scales.

3 Similarity and Correlation Functions

Correlation functions (association measures) were introduced in [2-4] on a set with involutive operation as functions satisfying some basic properties of Pearson's product-moment correlation coefficient. It was shown [2, 3, 8-11] that most of the known correlation and association coefficients satisfy the properties of correlation functions and can be constructed by methods proposed in [2-4, 9, 10] using some similarity or dissimilarity functions. Formally, (dis)similarity functions are fuzzy relations, and many properties of fuzzy relations can be considered for these functions [1, 8, 10, 20]. In this section, we consider some related definitions and methods that will be used further for the construction of the correlation function on a circular scale.

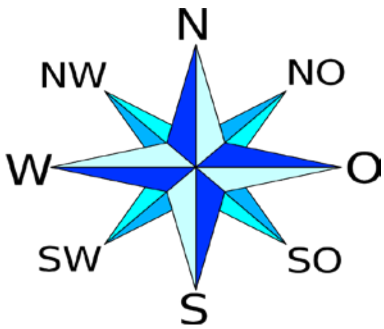

Fig. 1. 8-Point Compass Rose, adopted from [17].

Let Ω be a nonempty set with involutive operation $N(x)$ called *reflection* or *negation* such that for any element *x* in Ω the reflection of *x* also belongs to Ω, and double reflection of *x* equals to *x*:

$$N\big(N(x)\big) = x. \qquad \text{(involutivity)}$$

Such involutive operation usually considered for many types of data: the complement of sets or fuzzy sets, the negation in binary or fuzzy logic, the multiplication on -1 on the set of real numbers, etc. If for some *x* in Ω it is fulfilled

$$N(x) = x,$$

then such element is called a *fixed point* of the reflection *N*. The set of all fixed points of N in Ω denoted by $FP(N,\Omega)$ or FP. For example, on the set of real numbers, the operation of multiplication on -1 is an involutive operation: $N(x) = (-1)\cdot x = -x$, due to $N\big(N(x)\big) = N(-x) = -(-x) = x$. This operation has a fixed point $x = 0$ because $N(0) = -0 = 0$. But the set $F \setminus \{0\}$ of real numbers without 0 has not fixed points with respect to this involutive operation. Note that not all sets have fixed points of a reflection operation defined on the set. For example, the negation operation in binary logic defined on the set of truth values $\Omega = \{0,1\}$ has not fixed points.

Definition 1 [3, 8]. Let Ω be a set with a reflection operation *N*, and *V* be a subset of $\Omega \setminus FP(N)$ closed under operation *N*, i.e., for any *x* in *V* its reflection $N(x)$ belongs to *V*. The *correlation function (association measure)* on *V* is a function $A(x,y)$ such that for any *x* and *y* in *V* it takes values in [-1,1], and satisfies the following properties:

A1. $A(x,y) = A(y,x)$, (symmetry)

A2. $A(x,x) = 1$, (reflexivity)

A3. $A(x, N(y)) = -A(x, y)$. (inverse relationship)

It is shown that correlation functions satisfy the following properties fulfilled for all *x* and *y* in *V*:

$$A(x, N(x)) = -1, \quad \text{(opposite elements)}$$

$$A(N(x), N(y)) = A(x, y), \quad \text{(co-symmetry)}$$

$$A(x, N(y)) = A(N(x), y). \quad \text{(co-symmetry-II)}$$

As it follows from the definition, the correlation between the elements of Ω and possible fixed points x_{Fp} of the reflection operation N on Ω is not defined. Depending on the possible applications of correlation functions, these correlations can be defined as follows: $A(x, x_{FP}) = 0$ or $A(x, x_{FP}) = 1$.

Correlation functions can be obtained from similarity and dissimilarity functions [3, 8] defined as follows.

A function $S(x, y)$ is called a *similarity function* on Ω if for all *x*,*y* in Ω it takes values in [0,1] and satisfies the following properties:

$$S(x, y) = S(y, x), \quad \text{(symmetry)}$$

$$S(x, x) = 1. \quad \text{(reflexivity)}$$

A function $D(x, y)$ is called a *dissimilarity function* on Ω if for all *x*,*y* in Ω it takes values in [0,1] and satisfies the following properties:

$$D(x, y) = D(y, x), \quad \text{(symmetry)}$$

$$D(x, x) = 0. \quad \text{(irreflexivity)}$$

Similarity and dissimilarity functions are dual concepts. They can be easily obtained one from another as follows:

$$S(x, y) = 1 - D(y, x), \quad D(x, y) = 1 - S(y, x).$$

Such similarity and dissimilarity functions are called *complementary*. Generally, they will be referred to as *(dis)similarity functions*, and they can be considered together [8], but for a specific domain, depending on the method of their construction, it is sufficient to consider only one of these functions. On circular scales, we will obtain dissimilarity function from a distance; hence we will consider further a dissimilarity function.

A non-negative real-valued function $d(x, y)$ of elements of Ω will be referred to as a *distance* if for all *x*, y in Ω it satisfies the following properties:

$$d(x, y) = d(y, x), \quad \text{(symmetry)}$$

$$d(x, x) = 0. \quad \text{(irreflexivity)}$$

If for some positive real value M for all *x*, y in Ω it is fulfilled $d(x, y) \leq M$, then the function

$$D(x,y) = \frac{d(x,y)}{M},$$

will be a dissimilarity function taking values in the interval [0,1].

Let *N* be a reflection operation on a set Ω and $FP(N,\Omega)$ be a set of all fixed points of *N* on Ω. The set $FP(N,\Omega)$ can be empty. Let *V* be a subset of $\Omega \setminus FP(N)$ closed under operation *N*. A dissimilarity function *D* is called *non-contradictive* or *consistent* on *V* if for *x* in *V* it is fulfilled:

$$D\big(x, N(x)\big) = 1. \qquad \text{(consistency)}$$

This property means that the dissimilarity between opposite elements in maximal.

A dissimilarity function *D* is called *co-symmetric* on *V* if for all *x*,*y* in *V* it is fulfilled:

$$D\big(N(x), N(y)\big) = D(x, y). \qquad \text{(co-symmetry)}$$

This property means symmetry of dissimilarity functions with respect to opposite elements (complements, is we talk about sets).

Theorem 1 [3,8]. Let Ω be a set with a reflection operation *N*, and *V* be a subset of $\Omega \setminus FP(N)$ closed under operation *N*. If *D* is a co-symmetric and consistent dissimilarity function on *V*, then the function:

$$A(x,y) \ = \ D\big(x, N(y)\big) - \ D(x,y), \tag{1}$$

is a correlation function on *V*.

Originally in Theorem 1, an association measure (correlation function) was constructed using similarity functions but replacing similarity functions by complementary dissimilarity functions we obtain (1).

4 Dissimilarity and Correlation Functions on Circular Scales

Let $(c_1, \dots, c_n)$ be a sequence of the grades of the *n*-point circular scale. We will suppose here that *n* is even, i.e., $n = 2m$ for some positive integer *m*. Instead of names of the grades of the circular scale, we will use the indexes $I_{2m} = (1, \dots, 2m)$ of these grades. Generally, what point of the scale will obtain the index 1 is not important because we will calculate differences between them, that are invariant under rotation of indexing when the circular order of indexes coincide with the circular order of the points on the scale. For example, for the 8-point compass rose shown on Fig. 1, with the grades (**NO, O, SO, S, SW, W, NW, N**), the indexing $I_8 = (1, 2, \dots, 7, 8)$ can correspond to the points **NO, O, …, NW, N,** respectively. In the examples below, we will use this indexing of the points of the circular scale. For simplicity of interpretation of operations, we write this indexing of 8-point compass rose as follows:

$$1(\mathbf{NO}), 2(\mathbf{O}), 3(\mathbf{SO}), 4(\mathbf{S}), 5(\mathbf{SW}), 6(\mathbf{W}), 7(\mathbf{NW}), 8(\mathbf{N}). \tag{2}$$

But if one assigns the index 1 to **O**, then **SO** will have index 2, and so on, and the final index 8 will be assigned to **NO**. Below we will operate with the indexes of a circular scale, for this reason, the set $I_{2m} = (1, \ldots, 2m)$ sometimes also will be referred to as a circular scale.

Let us introduce a negation *N* on the circular scale with indexes $I_{2m} = (1, \ldots, 2m)$ as follows:

$$N(k) = \begin{cases} k+m, & if\ k \le m \\ k-m, & if\ k > m \end{cases}. \quad (3)$$

It is important to note that the set $I_{2m} = (1, \ldots, 2m)$ has an even number of grades. For this reason, the negation (3) has not fixed points, and we can apply Theorem 1 for the set $V = \Omega$ and for all *x*, y in Ω.

The negation (3) maps any point of the circular scale into the "opposite" point of the scale. For example, for circular scale from Fig. 1 with indexing (2) we have $m = 4$, and we obtain for $N(\mathbf{NO})$: $N(1) = 1 + 4 = 5$, i.e., $N(\mathbf{NO}) = \mathbf{SW},$ and we obtain for $N(\mathbf{W})$: $N(6) = 6 - 4 = 2$, i.e., $N(\mathbf{W}) = \mathbf{O}$.

Let us introduce the distance on the circular scale $I_{2m} = (1, \ldots, 2m)$ as follows:

$$d(k,j) = min\{|k-j|, 2m - |k-j|\}. \quad (4)$$

The distance on the circular scale equals to the minimum of "directional" distances between 2 points *k* and *j* calculated in clockwise and counterclockwise directions. For example, to calculate the distance $d(1,7)$ between two points $1(\mathbf{NO})$ and $7(\mathbf{NW})$ we calculate, first, the directional distance in the clockwise direction starting from 1: $|k-j| = |1-7| = 6$. Since the sum of distances in clockwise and counterclockwise directions equal to $2m = 8$, we calculate the distance in the counterclockwise direction as follows: $2m - |k-j| = 8 - 6 = 2$. The minimal distance between $1(\mathbf{NO})$ and $7(\mathbf{NW})$ will be equal to $d(1,7) = min\{|1-7|, 8-|1-7|\} = min\{6,2\} = 2$.

Since the distance (4) has the maximal value $M = m$, we obtain the dissimilarity function on a circular scale as follows:

$$D(k,j) = \frac{1}{m} d(k,j) = \frac{1}{m}[min\{|k-j|, 2m - |k-j|\}]. \quad (5)$$

It is easy to see that *D* is symmetric and irreflexive. Also, we can prove the fulfillment of the following properties of dissimilarity function (5).

Proposition 1. The dissimilarity function (5) is consistent, i.e., for all $k = 1, \ldots, 2m$ it is fulfilled:

$$D\big(k, N(k)\big) = 1. \quad (6)$$

Proposition 2. The dissimilarity function (5) is co-symmetric, i.e., for all $k = 1, \ldots, 2m$ it is fulfilled:

$$D\big(N(k), N(j)\big) = D(k,j). \quad (7)$$

From the definition of dissimilarity functions it follows that a dissimilarity function *D* is co-symmetric if the corresponding distance *d* is co-symmetric, i.e., for all $k, j = 1, \ldots, 2m$ it is fulfilled:

$$d\big(N(k), N(j)\big) = d(k, j). \tag{8}$$

From Proposition 1, Proposition 2, (6)-(8), and Theorem 1 it follows that the following function will be a correlation function on a circular scale:

$$A(k, j) = D\big(k, N(j)\big) - D(k, j) = \tag{9}$$

$$= \frac{1}{m}[min\{|k - N(j)|, 2m - |k - N(j)|\} - min\{|k - j|, 2m - |k - j|\}]). \tag{10}$$

As we can see from (9) the *correlation between k and j is positive if* distance $D(k, j)$ is less than $D\big(k, N(j)\big)$, i.e. *k is closer to j than to its negation* $N(j)$. In inverse case, *when k is closer to the negation* $N(j)$ *than to j, the correlation between k and j is negative.*

Example 1. Consider 8-point Compass Rose from Fig. 1 with indexing (2). We have $m = 4, 2m = 8$. Compare correlation values for different pairs of points from this circular scale. For example, grades $1(\mathbf{NO})$ and $4(\mathbf{S})$ have indexes $k = 1$ and $j = 4$, respectively. From (3) and $m = 4$ we have $N(4) = 4 + m = 8$, i.e., $N(\mathbf{S}) = \mathbf{N}$. Using (9) and (10) calculate correlation $A(\mathbf{NO}, \mathbf{S})$ step by step:

$$A(1,4) = D\big(1, N(4)\big) - D(1,4),$$

$$D\big(1, N(4)\big) = D(1,8) = \frac{1}{4}[min\{|1 - 8|, 8 - |1 - 8|\}] = \frac{1}{4}[min\{7,1\}] = \frac{1}{4}.$$

$$D(1,4) = \frac{1}{4}[min\{|1 - 4|, 8 - |1 - 4|\}] = \frac{1}{4}[min\{3,5\}] = \frac{3}{4}.$$

$$A(1,4) = \frac{1}{4} - \frac{3}{4} = -\frac{2}{4} = -\frac{1}{2}, \text{ i.e., } A(\mathbf{NO}, \mathbf{S}) = -0.5.$$

As we can see from the example, the correlation between **NO** and **S** is negative because **NO** is closer to the negation of **S**, $N(\mathbf{S}) = \mathbf{N}$, than to **S**, see Fig. 1. Table 1 presents correlation values between all directions (points) of 8-Point Compass Rose.

Table 1. Correlations of directions in 8-point Compass Rose, see Fig. 1.

		1	2	3	4	5	6	7	8
		NO	**O**	**SO**	**S**	**SW**	**W**	**NW**	**N**
1	**NO**	1	0.5	0	-0.5	-1	-0.5	0	0.5
2	**O**	0.5	1	0.5	0	-0.5	-1	-0.5	0
3	**SO**	0	0.5	1	0.5	0	-0.5	-1	-0.5
4	**S**	-0.5	0	0.5	1	0.5	0	-0.5	-1
5	**SW**	-1	-0.5	0	0.5	1	0.5	0	-0.5
6	**W**	**-0.5**	**-1**	**-0.5**	0	**0.5**	**1**	**0.5**	0
7	**NW**	0	-0.5	-1	-0.5	0	0.5	1	0.5
8	**N**	0.5	0	-0.5	-1	-0.5	0	0.5	1

Another interpretation of the obtained result can be the following. Suppose that the circular scale from Fig. 1 is used for measuring the direction of the movement of some vehicles. Then, comparing the movement of one vehicle in the direction **NO** with the directions of the movement of other vehicles we can say that the directions **N** and **O** are “near” to **NO** and positively correlated with **NO**, but the directions **W, SW** and **S** are “opposite” to **NO** and negatively correlated with it. In such a case, the correlation function can measure the sign and the strength of the relationship between directions. In our example, we obtained a negative correlation between **NO** and **S** because they move in “opposite” directions.

As a simple example, suppose that some vehicle moves in direction **W**. Then its movement is positively correlated with the movements of other vehicles moving in the “similar” directions **NW, W** and **SW** and it is negatively correlated with the movements of vehicles moving in the “opposite” directions **NO, O** and **SO**, (see bold font numbers in Table 1).

5 Correlation of Measurements in Circular Scales

Using the correlations of points of a circular scale, one can calculate the correlation between sets of measurements in circular scales. If these measurements are ordered in time, we can say about the time series of circular values. Consider two *n*-tuples $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ such that x_i, y_i for all $i = 1, \dots, n$ belong to the same circular scale $I_{2m} = (1, \dots, 2m)$. As an example, one can consider the directions of movements of two vehicles x_i, y_i at different moments of time $i = 1, \dots, n$. If these measurements have been done during the movements of these vehicles at the same moments of time and ordered in time, these *n*-tuples will give time series of circular measurements or trajectories. It can be proposed several methods of correlation between sets of circular measurements or trajectories.

Since we can calculate correlations between *n*-tuples element-by-element, then the total correlation between them can be calculated as an average correlation:

$$A(x, y) = \frac{1}{n}\sum_{i=1}^{n} A(x_i, y_i).$$

It is clear that the correlation $A(x, y)$ takes values in the interval [-1,1].

If we talk about the movements of two vehicles *x* and *y* then $A(x, y)$ will be negative if they generally move in opposite directions, and this correlation will be positive if they generally move in the “similar” direction.

In an analysis of circular time correlations of events, the correlation between two sets of measurements can be negative, for example, if the events from one set usually happen in the morning and events from another set happen usually in the evening. For example, John is usually jogging in the morning, and Bob is jogging in the evenings, then these events for John and Bob will be negatively correlated. For example, a high positive correlation of the hours of visiting the store by some groups of customers can give rise to an analysis of the demands of these groups of customers. The time correlations can be useful in the analysis of the weather conditions or contamination levels during the day or the year, in the analysis of the behavior of animals or birds, etc.

6 Relative and Dynamic Circular Scales

The usage of the same scale, for example, for analysis of the correlation between the movements of different agents (vehicles, people, animals) can be useful in many tasks, but in some situations, it is more convenient to use for different agents different scales. For example, if we will calculate the correlation between the movements of two ships moving to the North on the opposite sides of the Earth, then in the circular scale of one ship, these two ships will move in opposite directions towards each other and can be considered as negatively correlated. From another point of view, we can consider that their movements as positively correlated because they move in the same direction.

A similar situation appears when we want to calculate the correlation between the movement of two vehicles in the city when we consider their movements as positively correlated if both move to the Center of the city. But if they move from the North and from the South of the city, they will move in opposite directions. Hence if we will use one circular scale for both vehicles, then their movements will be negatively correlated.

To consider such situations, one can introduce relative circular scales, and for each vehicle to measure the angle between the "goal of the movement" (Center of the city) and the real direction of the movement of the vehicle. To measure such angles, one can use the angular circular scale $I_{360} = (1, \dots, 360)$ where direction 360 (or equally 0) will correspond to the direction to the goal of the movement. In this case, vehicles can move in opposite directions, one from the North and another from the South of the city, but the correlation between their movements will be positive if both are moving to the common goal, to the Center of the city.

Another situation appears when the goals of vehicles are different and dynamically changed. Such situations appear for "predator and prey" or "leader and follower" agents. In these cases, the goals of movements for different agents can be different and will depend on the movements of other agents. Hence the correlation between movements of different agents can be calculated in a different manner.

7 Conclusion and Future Work

The paper introduced a new correlation function on a circular scale. It does not require that the circular scale should be angular as it is usually required in previous works on correlations on circular scales. Another advantage of the proposed method is that it gives a possibility to measure the correlation between two points of the scale and does not require the sets of measurements of circular variables. In future, the proposed approach will be extended on circular scales with an odd number of points. Also, the comparison of the proposed circular correlation coefficient with the existing circular correlations will be made.

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