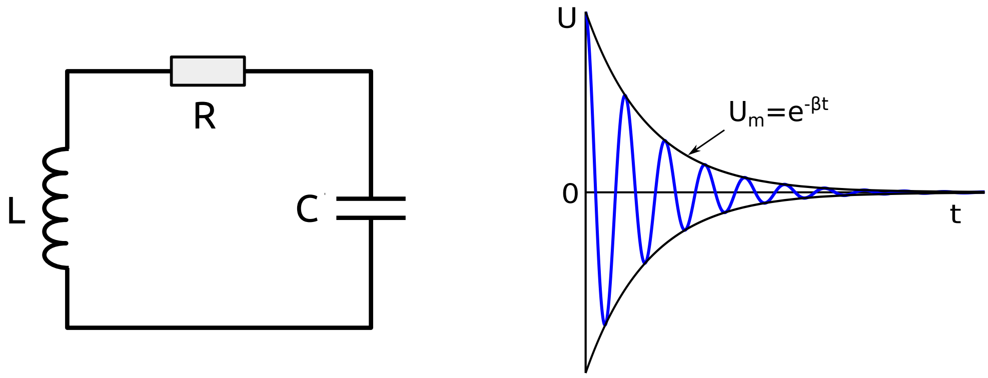
Investigation of free electrical oscillations in an oscillatory circuit

Oscillatory circuit is an electrical circuit consisting of a coil (inductive element) L and a capacitor C (Fig. 1a). Changes of the current I in the coil and charge q in the capacitor are free electrical oscillations. A real circuit has energy losses, mainly due to joule (ohmic) heating in the coil and connecting wires, which have a resistance R. For this reason, free oscillations are decaying.



(a)

(b)

Figure 1

The Kirchhoff's second rule for a closed electric circuit containing the inductane coil L, resistor R, and capacitor C, which are connected in a sequential manner, predicts that

$$U_L + U_R + U_C = 0\,, \tag{1}$$

where $U_L = L\ddot{q}$, $U_R = R\dot{q}$ and $U_C = \frac{1}{C}q$ are the voltage drops on the coil L, ohmic resistance R, and capacitor C. Thus we can derive differential equations for the free electrical oscillations of the charge q and voltage at the capacitor $U_C \equiv U$:

$$\ddot{q} + 2\beta\dot{q} + \omega_0^2 q = 0 \text{ and } \ddot{U} + 2\beta\dot{U} + \omega_0^2 U = 0\,. \tag{2}$$

The value $\beta = R/2L$ is called the damping coefficient, $\tau = 1/\beta$ is the relaxation time, and $\omega_0 = 1/\sqrt{LC}$ is the natural frequency of the oscillatory circuit.

The general solution of Eqs. (2) is actually the equation of decaying oscillations. For example, for the voltage U it is written as (Fig. 1b)

$$U(t) = U_m e^{-\beta t}\cos(\omega t)\,. \tag{3}$$

Let us investigate the solution (3) at different proportions of β^2 and ω_0^2.

1. $\beta^2 \ll \omega_0^2$ – weak damping. Free decaying oscillations take place in the circuit. The voltage at the capacitor is changed according to a periodic law with a decreasing amplitude (Fig. 1b). The oscillations frequency is given by the formula

$$\omega = \sqrt{\omega_0^2 - \beta^2}\,. \tag{4}$$

To characterize damping oscillations, a parameter λ (logarithmic decrement) is introduced in addition to the factor β. It is defined as a natural (Napierian) logarithm of the ratio of two consecutive voltage amplitudes $U_0(t)$ and $U_0(t+T)$, separated by one full oscillation period T:

$$\lambda = \ln \frac{U_0(t)}{U_0(t+T)} . \tag{5}$$

The value of λ can also be calculated as follows:

$$\lambda = \frac{1}{N} \ln \frac{U_0(t)}{U_0(t+NT)} , \ \lambda = \frac{1}{N_e} \text{ and } \lambda = \frac{\ln 2}{N_{0.5}} , \tag{6}$$

where $U_0(t)$ and $U_0(t+NT)$ are the amplitudes separated by N oscillation periods, and N_e and $N_{0.5}$ are the numbers of oscillations after which the amplitude decreases e times ($e = 2,71828\ldots$) or twice, respectively.

The quality of an oscillatory circuit is characterized by the Q-factor which is defined as

$$Q = \pi/\lambda . \tag{7}$$

The damping parameters β, τ, λ and Q relate to each other by the following expressions:

$$\lambda = \pi/Q = \beta T = T/\tau \text{ or } Q = \pi/\lambda = \pi/\beta T = \pi\tau/T . \tag{8}$$

Thus, the slower is the oscillation damping, the higher is the Q-factor of the circuit. In the weak damping regime the oscillations occur at a frequency close to the resonance frequency, i.e., $\omega \approx \omega_0 = 1/\sqrt{LC}$ or $T = 2\pi/\omega \approx 2\pi\sqrt{LC}$ and, hence,

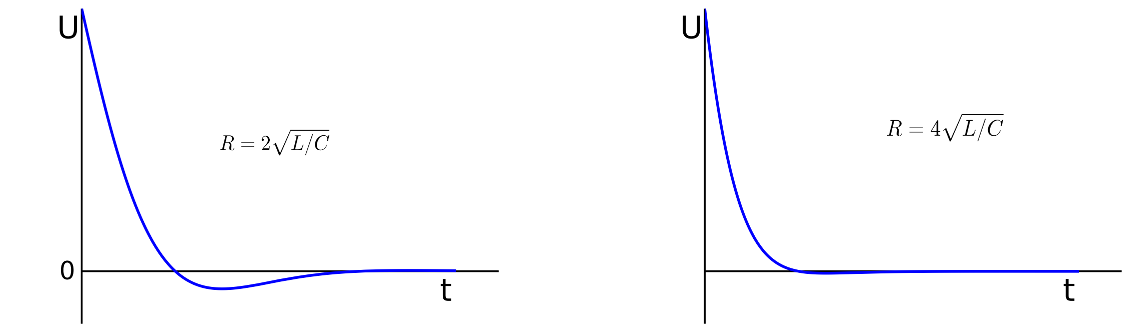
$$Q \approx \frac{1}{R}\sqrt{\frac{L}{C}} . \tag{9}$$

2. $\beta^2 \approx \omega_0^2$ – strong damping. An aperiodic process of changing the charge and voltage at the capacitor and the current in the circuit takes place instead of oscillations. Figures 2a and 2b show the example plots of the aperiodic voltage change $U(t)$ at the capacitor. If only the capacitor discharge process occurs in the circuit, then the curve $U(t)$ has the shape presented in Fig. 2b. The minimal active resistance of the circuit, with which the aperiodic process is observed, is called the critical resistance R_{cr}. The value of the critical resistance can be found from the equation $\beta = \omega_0$:

$$R_{\text{cr}} = 2\sqrt{L/C} \tag{10}$$

Task. Determination of the logarithmic damping coefficient of oscillatons and the Q-factor of a circuit.

1. Connect the generator of rectangular pulses to the contacts 1, 2 (Fig. 3). Set the generator frequency to 300 Hz. Make a short contact between the output contacts of the cable, attached to the oscilloscope, and match the signal line with the middle horizontal line of the display using the "↕" handwheel. Then reconnect the cable to the contacts 4 and 5. By turning the handwheels "stability" and "level" on the oscilloscope, get a stable image of decaying oscillations.



(a) (b)

Figure 2

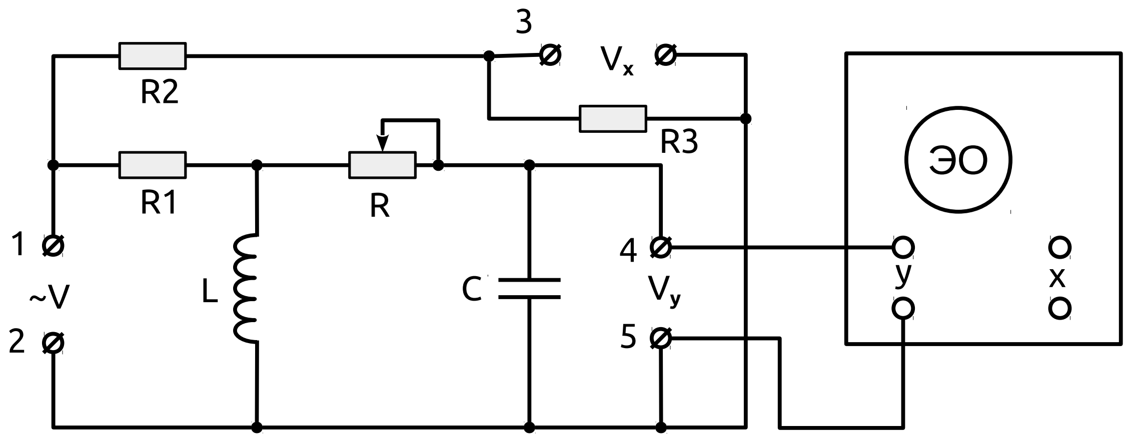


Figure 3

Set the resistance R to the minimal value. In case of problems, call the engineer or teacher, or check the following procedure:

Switch 1 of the oscilloscope set to the position "∼"; switch 2, to "AUTO"; switch 3, to "+" or "−"; switch 4, to "EXT." Get a stable sinusoidal pattern containing several oscilaltions by regulating the input amplification "Y" of the oscilloscope (selector VOLT/UNIT) and the sweep period (selector TIME/UNIT).

2. Measure the amplitudes of two oscillations separated by k periods: a_n and a_{n+k} in millimeters (n and k are integers). For exact measurement of the amplitude, the oscillation being measured should by positioned on the central vertical gauge using the horizontal position handwheel ("↔"). Calculate the logarithmic damping decrement and the Q-factor of the circuit as $\lambda = \frac{1}{k} \ln \left(\frac{a_n}{a_{n+k}} \right)$ and $Q = \frac{\pi}{\lambda}$. Repeat the measurement with different n and k (no less than 3 combinations!) and find the average Q_{av}. Calculate the resistance R using the expression $R = \frac{1}{Q_{\text{av}}} \sqrt{\frac{L}{C}}$, assuming that $L = 10$ mH and $C = 33$ nF.

3. Repeat the measurements of step 2 for three values of the resistance R distributed over

3/4 of the whole regulation range starting from the smallest value.

4. Build the plots of dependencies $Q = Q(R)$ and $\lambda = \lambda(R)$. Explain the results.

Questions

1. Free electrical oscillations. Equation of decaying oscillations and its solution.
2. Characteristics of decaying oscillations: damping factor, damping decrement and logarithmic damping decrement, relaxation time.
3. Q-factor of an oscillatory circuit, its physical meaning, equations.
4. Particular cases of free oscillations:
   - small resistance of the circuit,
   - large resistance of the circuit,
   - critical case, critical resistance.
5. Graphical determination of the damping parameters.