Positivity (2025) 29:47
https://doi.org/10.1007/s11117-025-01139-7

Positivity

RESEARCH

On pairs of projections

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Received: 18 June 2025 / Accepted: 2 July 2025
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Abstract

Two projections P and Q on a Hilbert space $\mathcal{H}$ are called acute if $\|PQ\| < 1$. We utilize the von Neumann alternating projection theorem to prove that if P and Q are acute, then $P \wedge Q = 0$. Conversely, if $P \wedge Q = 0$ and PQ is a compact operator, then P and Q are acute. An example is presented to show that the assumption of compactness is necessary. Let $\mathcal{M}$ be a von Neumann algebra, $\mathcal{M}^{\mathrm{pr}}$ be the lattice of all projections in $\mathcal{M}$, and $P, Q \in \mathcal{M}^{\mathrm{pr}}$. A pair (P, Q) is called modular in $\mathcal{M}^{\mathrm{pr}}$ if $(R \vee P) \wedge Q = (R \wedge Q) \vee (P \wedge Q)$ for every $R \in \mathcal{M}^{\mathrm{pr}}$ with $R \leq Q$. We present several characterizations of modular pairs of projections in a von Neumann algebra. In particular, for a factor $\mathcal{M}$ of type I or III, we investigate certain modularity conditions.

Keywords Acute projections · Modular projections · Isoclinic projections

Mathematics Subject Classification 47A46 · 47A30 · 47B15 · 46L10

1 Introduction and preliminaries

Pairs of projections in a Hilbert space play a crucial role in the Quantum Hall Effect [2] and are the subject of study for a wide group of mathematicians as seen in [3, 4, 6, 13, 16, 24], and the references therein.

The examination of pairs of projections is a key point in problems of non-commutative integration theory as discussed in [7, 18, 20]. Modular pairs of projections may be utilized to determine when an isomorphism between projection lattices extends to an algebra isomorphism, see [9]. Halmos [13] defined the minimal angle between closed subspaces and showed that if two projections are acute, then their ranges are not orthogonal but still do not contain identical nonzero vectors.

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Published online: 12 July 2025

The algebra generated by two projections is of independent interest because it encapsulates the interplay between the ranges of the projections, with applications in operator algebras and spectral theory; see the works of Spitkovsky [22, 23].

Throughout this paper, let $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$, let $\mathcal{K}(\mathcal{H})$ be the $*$-ideal of all compact operators in $\mathcal{B}(\mathcal{H})$, and let I be the identity operator on $\mathcal{H}$. The cone of positive operators in $\mathcal{B}(\mathcal{H})$ is denoted by $\mathcal{B}(\mathcal{H})^+$. We consider the Löwner order $\leq$ on self-adjoint operators stating that $A \leq B$ whenever $B - A \in \mathcal{B}(\mathcal{H})^+$. If $A \in \mathcal{B}(\mathcal{H})$, then we denote the modulus of A by $|A| := (A^*A)^{1/2}$. The range and null space of an operator A are denoted by $\operatorname{ran}(A)$ and $\ker(A)$, respectively.

Recall that the strong operator topology on $\mathcal{B}(\mathcal{H})$ is defined by the family of seminorms $A \mapsto \|Ax\|$, $x \in \mathcal{H}$. Let $(s_n(X))_{n=1}^{\infty}$ be the sequence of the singular numbers of an operator $X \in \mathcal{B}(\mathcal{H})$. Then $X \in \mathcal{K}(\mathcal{H})$ if and only if $s_n(X) \to 0$ as $n \to \infty$; see [12, 21].

Let $\mathcal{M}$ be a von Neumann algebra of operators acting on a Hilbert space. Denote $\mathcal{M}^{\mathrm{pr}}$ as the lattice of all projections ($P = P^2 = P^*$) in $\mathcal{M}$. If $P, Q \in \mathcal{M}^{\mathrm{pr}}$, then $P^{\perp} = I - P \in \mathcal{M}^{\mathrm{pr}}$ and the operator $P \wedge Q$ is defined as the projection onto $\operatorname{ran}(P) \cap \operatorname{ran}(Q)$, while $P \vee Q = (P^{\perp} \wedge Q^{\perp})^{\perp}$ is the projection onto the closed linear span of $\operatorname{ran}(P) \cup \operatorname{ran}(Q)$. For $P, Q \in \mathcal{M}^{\mathrm{pr}}$, we write $P \sim Q$ (the *Murray–von Neumann equivalence*) if $P = U^*U$ and $Q = UU^*$ for some $U \in \mathcal{M}^{\mathrm{pr}}$.

We frequently utilize fundamental properties of projections, which are summarized in the following theorem:

Theorem 1.1 [19, Theorem 2.3.2] For $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$, the following conditions are equivalent: (i) $P \leq Q$, (ii) $PQ = P$, (iii) $QP = P$, (iv) $\operatorname{ran}(P) \subseteq \operatorname{ran}(Q)$, (v) $\|Px\| \leq \|Qx\|$ for all $x \in \mathcal{H}$, (vi) $Q - P \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$.

Readers are referred to [15] and [19] for any undefined notation and terminology.

In this paper, we explore acute projections and their connections with the Dixmier angle between their range spaces. We demonstrate that if PQ is a compact operator and $P \wedge Q = 0$, then P and Q are acute (Theorem 2.3). It is crucial to note that the compactness condition of PQ is essential (Example 2.6). For a factor $\mathcal{M}$ of type I or III, we study certain modularity conditions (Theorem 3.2 and Proposition 3.3). We investigate projections satisfying $PQP = \lambda P$ for some $0 < \lambda \leq 1$. In this context, we examine some pairs of isoclinic projections (Theorem 4.8).

2 Acute projections

In the theory of projections acting on Hilbert spaces, two projections P and Q on a Hilbert space are called *acute* if $\|PQ\| < 1$. This condition implies that their corresponding closed subspaces are not too closely aligned, in particular, there are no common eigenvector with eigenvalue 1. Consequently, if P and Q share a nontrivial closed subspace, they cannot be acute (see part (i) of Theorem 2.3).

A simple geometric example in the real Hilbert space $\mathbb{R}^2$ is given by two rank-1 projections P and Q onto lines separated by an angle $\theta > 0$. In this case, $\|PQ\| = \cos\theta < 1$.

Recall that the cosine of the *Dixmier angle* between subspaces of $\mathrm{ran}(P)$ and $\mathrm{ran}(Q)$ of two projections P and Q is defined as follows:

$$c_0(\mathrm{ran}(P), \mathrm{ran}(Q)) := \sup\{|\langle x, y\rangle| : x \in \mathrm{ran}(P), \|x\| \le 1, y \in \mathrm{ran}(Q), \|y\| \le 1\}.$$

It is a known fact that $c_0(\mathrm{ran}(P), \mathrm{ran}(Q)) = \|PQ\|$ as shown in [11, Lemma 10]. The projections $P - P \wedge Q$ and $Q - P \wedge Q$ are of particular interest because their ranges intersect trivially.

Proposition 2.1 *Given projections $P, Q \in \mathcal{B}(\mathcal{H})$, the Dixmier angle of the subspaces of $\mathrm{ran}(P) \ominus \mathrm{ran}(P \wedge Q)$ and $\mathrm{ran}(Q) \ominus \mathrm{ran}(P \wedge Q)$ is nonzero if and only if it holds that $\|PQ - P \wedge Q\| < 1$.*

Proof Set $R := P \wedge Q$ and let $P_1 := P - R$ and $Q_1 := Q - R$. The ranges of these operators are $\mathrm{ran}(P) \ominus \mathrm{ran}(R)$ and $\mathrm{ran}(Q) \ominus \mathrm{ran}(R)$, respectively. The operator $P_1 Q_1$ fulfills

$$\|P_1 Q_1\| = \|(P - R)(Q - R)\| = \|PQ - PR - RQ + R\| = \|PQ - R\| < 1,$$

since, by $R \le P$ and $R \le Q$, we have $PR = R$ and $RQ = R$. □

To prove the next result, we require the celebrated von Neumann alternating projection theorem [25]. For the reader's convenience, we provide an alternative proof of Theorem 2.2; see [14, Problem 122] for another approach.

Theorem 2.2 *(*von Neumann theorem*) If $P, Q \in \mathcal{B}(\mathcal{H})^{pr}$, then the sequence $\{(PQP)^n\}$ is decreasing with respect to the Löwner order and*

$$\text{so-}\lim_{n\to\infty} (PQP)^n = P \wedge Q. \tag{2.1}$$

Proof Since $0 \le PQP \le I$, we obtain

$$\begin{aligned} 0 \le (PQP)^n &= (PQP)^{(n-1)/2}(PQP)(PQP)^{(n-1)/2} \\ &\le (PQP)^{(n-1)/2} I (PQP)^{(n-1)/2} \\ &= (PQP)^{n-1} \end{aligned}$$

for all positive integers $n \ge 2$. It follows from [19, Theorem 4.1.1] that $\{(PQP)^n\}$ converges to a self-adjoint operator R in the strong operator topology. Since $\text{so-}\lim_{m\to\infty}(PQP)^{n+m} = \text{so-}\lim_{m\to\infty}(PQP)^m$, we get $(PQP)^n R = R$ for all n. By taking limits as $n \to \infty$, we get $R^2 = R$, which ensures that R is a projection. We shall show that $R = P \wedge Q$. To do this, we need to show that $R(\mathcal{H}) = \mathrm{ran}(P) \cap \mathrm{ran}(Q)$:

(1) Let $x \in \mathrm{ran}(P) \cap \mathrm{ran}(Q)$. Then, $x = Px = Qx$. Therefore, $x = \text{so-}\lim_{n\to\infty}(PQP)^n x = Rx \in R(\mathcal{H})$.

(2) Let $x \in R(\mathcal{H})$. It follows from $(PQP)^{n+1}x = (PQP)(PQP)^n x$ $(n \geq 1)$ that $x = PQPx \in \operatorname{ran}(P)$, since $Rx = x$. Therefore, $P(Qx) = x$. Hence

$$\langle Qx, Qx\rangle = \langle Qx, x\rangle = \langle Qx, Px\rangle = \langle PQx, x\rangle = \langle x, x\rangle.$$

Therefore,

$$\langle Qx - x, Qx - x\rangle = \langle Qx, Qx\rangle - 2\mathrm{Re}\langle Qx, x\rangle + \langle x, x\rangle = 0.$$

Thus, $x = Qx \in \operatorname{ran}(Q)$.

□

Now, we present one of our main results.

Theorem 2.3 *Let $P, Q \in \mathcal{B}(\mathcal{H})^{pr}$.*

(i) *If P and Q are acute, then $P \wedge Q = 0$.*
(ii) *If $P \wedge Q = 0$ and $PQ \in \mathcal{K}(\mathcal{H})$, then P and Q are acute.*

Proof (i). Since $\|PQ\| < 1$, we have $\|PQP\| < 1$. Therefore, the sequence $\{(PQP)^n\}$ converges to 0 in the norm topology, and hence, in the strong operator topology. It follows from Theorem 2.2, that $P \wedge Q = 0$.

(ii). If $P \wedge Q = 0$, then by employing (2.1), we arrive at so-$\lim_{n\to\infty}(PQP)^n = P \wedge Q = 0$.

Recall a "Basic lemma" of the theory of projection methods [8, pp. 18–19] (for a more general case see [5, Theorem 2]): *If Y is compact and $X_n \to X$strongly, then $X_nY \to XY$uniformly, that is, $\|X_nY - XY\| \to 0$ as $n \to \infty$.*

Next, we show that $PQ \in \mathcal{K}(\mathcal{H})$ if and only if $PQP \in \mathcal{K}(\mathcal{H})$: From $PQP = |QP|^2 = |(PQ)^*|^2$, we infer that

$$s_n(PQP) = s_n(|(PQ)^*|)^2 = s_n((PQ)^*)^2 = s_n(PQ)^2$$

for all $n \in \mathbb{N}$.

Now, for $X_n = (PQP)^{n-1}$ and $Y = PQP$ we obtain

$$(PQP)^n = (PQP)^{n-1}(PQP) \to (P \wedge Q)PQP = P \wedge Q = 0 \ \text{ uniformly as } \ n \to \infty.$$

Since PQP is self-adjoint, we conclude that

$$\|PQP\|^{2^n} = \|(PQP)^{2^n}\| \to 0 \ \text{ as } \ n \to \infty,$$

whence $\|PQ\|^2 = \|PQP\| < 1$. Thus, $\|PQ\| < 1$. □

Corollary 2.4 (i) If $\|P^\perp Q^\perp\| < 1$, then $P \vee Q = I$.
(ii) $PQP + (P - Q)^2 \leq P \vee Q$ for all $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$.

Proof (i). If $\|P^\perp Q^\perp\| < 1$, then $I = 0^\perp = (P^\perp \wedge Q^\perp)^\perp = P \vee Q$ by the De Morgan law.

(ii). For all $P, Q \in \mathcal{B}(\mathcal{H})^{\rm pr}$, we have

$$\begin{aligned}0 &\le P^\perp Q^\perp P^\perp - P^\perp \wedge Q^\perp = I - PQP + PQ + QP - P - Q - P^\perp \wedge Q^\perp \\ &= P \vee Q - PQP - (P-Q)^2\end{aligned}$$

by Theorem 2.2 and the De Morgan law.

□

Corollary 2.5 Let $P, Q \in \mathbb{M}_n^{\rm pr}$. Then, $\|PQ\| < 1$ if and only if $\operatorname{ran}(P) \cap \operatorname{ran}(Q) = \varnothing$.

Example 2.6 The condition "$PQ \in \mathcal{K}(\mathcal{H})$" is essential in item (ii) of Theorem 2.3. Choose a countable orthonormal basis in $\mathcal{H} = \ell_2$ and consider a sequence $(t_n)_{n=1}^\infty \subset (0,1)$ such that $t_n \nearrow 1$ as $n \to \infty$; for example, we can put $t_n = 1 - 2^{-n}$ for all $n \in \mathbb{N}$. Then $\sqrt{t_n} \nearrow 1$ as $n \to \infty$. Define infinite-dimensional projections in $\mathcal{B}(\mathcal{H})^{\rm pr}$ as

$$P = \operatorname{diag}(1, 0, 1, 0, \ldots, 1, 0, \ldots,), \qquad Q = R^{(t_1)} \oplus R^{(t_2)} \oplus \ldots \oplus R^{(t_n)} \oplus \ldots,$$

where

$$R^{(t)} = \begin{pmatrix} t & \sqrt{t-t^2} \\ \sqrt{t-t^2} & 1-t \end{pmatrix} \in \mathbb{M}_2^{\rm pr} \ \text{ for } \ 0 \le t \le 1.$$

Then

$$P \wedge Q = 0, \quad PQP = \operatorname{diag}(t_1, 0, t_2, 0, \ldots, t_n, 0, \ldots,),$$

$$|QP| = \sqrt{PQP} = \operatorname{diag}(\sqrt{t_1}, 0, \sqrt{t_2}, 0, \ldots, \sqrt{t_n}, 0, \ldots,) \ge \sqrt{t_1}P = \frac{1}{\sqrt{2}}P$$

and $QP \notin \mathcal{K}(\mathcal{H})$. Therefore, $PQ = (QP)^* \notin \mathcal{K}(\mathcal{H})$ and we have

$$\|PQ\| = \|QP\| = \||QP|\| = \sup_{n \in \mathbb{N}} \sqrt{t_n} = 1.$$

3 Modular projections

This section opens with a definition and its equivalent forms.

Definition 3.1 A pair (P, Q) is called *modular* in $\mathcal{M}^{\rm pr}$ if $(R \vee P) \wedge Q = (R \wedge Q) \vee (P \wedge Q) = R \vee (P \wedge Q)$ for every $R \in \mathcal{M}^{\rm pr}$ with $R \le Q$. For $P, Q \in \mathcal{B}(\mathcal{H})^{\rm pr}$ the following statements are equivalent [17, Remark 5]:

(α) (P, Q) is modular in $\mathcal{B}(\mathcal{H})^{\rm pr}$;
(β) $\|PQ - P \wedge Q\| < 1$;
(γ) the linear space $\operatorname{ran}(P) + \operatorname{ran}(Q)$ is closed.

The *Friedrichs angle* θ between closed subspaces $\operatorname{ran}(P)$ and $\operatorname{ran}(Q)$ is defined by

$$\cos\theta = \sup\left\{ |\langle x, y\rangle| : \|x\| = \|y\| = 1, \ {\scriptstyle x \in \operatorname{ran}(P) \ominus \operatorname{ran}(P \wedge Q) \atop y \in \operatorname{ran}(Q) \ominus \operatorname{ran}(P \wedge Q)} \right\}.$$

It is known that $\cos\theta = \|PQ - P\wedge Q\|$ [11, Lemma 10]. Thus, the pair (P, Q) is modular if and only if $\theta > 0$.

In Example 2.6, $\|PQ - P\wedge Q\| = 1$ indicating that the pair (P, Q) is not modular in $\mathcal{B}(\mathcal{H})^{\rm pr}$.

Let us now present another condition that is equivalent to the modularity of projections:

Since $PQP - P\wedge Q \geq 0$ for all $P, Q \in \mathcal{B}(\mathcal{H})^{\rm pr}$ and

$$\begin{aligned}\|PQ - P\wedge Q\| &= \|(PQ - P\wedge Q)^*\| = \|QP - P\wedge Q\| = \||QP - P\wedge Q|\|\\ &= \|\sqrt{(PQ - P\wedge Q)(QP - P\wedge Q)}\| = \|\sqrt{PQP - P\wedge Q}\|\\ &= \sqrt{\|PQP - P\wedge Q\|},\end{aligned}$$

we can say that

$$\text{“}A \text{ pair } (P, Q) \text{ is modular if and only if } \|PQP - P\wedge Q\| < 1\text{”}. \tag{3.1}$$

Let $\mathcal{M}$ be a factor of type I or III on a Hilbert space $\mathcal{H}$ (of course, if $\mathcal{M}$ is of type I, then $\mathcal{M} = \mathcal{B}(\mathcal{H})$). Then,

$$\text{“}a \text{ pair } (P, Q) \text{ is modular in } \mathcal{M}^{\rm pr} \text{ if and only if } \|PQP - P\wedge Q\| < 1,\text{”} \tag{3.2}$$

see Remark 5 and Corollary 4 in [17].

Let $E^X(B)$ be the spectral projection of a self-adjoint operator $X \in \mathcal{B}(\mathcal{H})$ relative to a Borel subset B of $\mathbb{R}$. Then $P\wedge Q = E^{PQP}(\{1\})$ for all $P, Q \in \mathcal{B}(\mathcal{H})^{\rm pr}$. To prove this equality, let $x \in \mathcal{H}$ such that $\langle PQPx, x\rangle = 1$. Then $\langle QPx, Px\rangle = 1$. It follows from the equality case in the Cauchy–Schwarz inequality that $QPx = Px$. Hence, $Px \in \operatorname{ran}(Q)$ and $\langle Px, Px\rangle = 1$. Therefore, $\langle Px, x\rangle = 1$. Again, by the Cauchy–Schwarz inequality, $x = Px \in \operatorname{ran}(P)$. Hence, $x \in (P\wedge Q)(\mathcal{H})$. The reverse statement evidently holds. Thus, we can assert that

$$\text{“}A \text{ pair } (P, Q) \text{ is modular in } \mathcal{M}^{\rm pr} \text{ if and only if } \left\|\int_{[0,1)} \lambda dE^{PQP}(\lambda)\right\| < 1,\text{"}$$

since

$$\|PQP - P\wedge Q\| = \left\|\int_{[0,1]} \lambda dE^{PQP}(\lambda) - \int_{\{1\}} \lambda dE^{PQP}(\lambda)\right\| = \int_{[0,1)} \lambda dE^{PQP}(\lambda).$$

Theorem 3.2 *Let $\mathcal{M}$ be a factor of type I or III on a Hilbert space $\mathcal{H}$. For $P, Q \in \mathcal{M}^{pr}$, the following conditions are equivalent:*

(i) *the pair $(P^\perp, Q^\perp)$ is modular in $\mathcal{M}^{pr}$;*

(ii) *there exists $0 \leq \alpha < 1$ such that $P\vee Q \leq (P - Q)^2 + PQP + \alpha I$;*

(iii) *there exists* $0 \le \alpha < 1$ *such that* $P \vee Q \le (P - Q)^2 + PQP + \alpha P^{\perp}$.

Proof (i)⇔(ii). By (3.2), which holds for a factor of type I or III, and the De Morgan law we have a chain of equivalences as follows:

"the pair $(P^{\perp}, Q^{\perp})$ is modular in $\mathcal{M}^{\mathrm{pr}}$" ⇔ "$\|P^{\perp}Q^{\perp} - P^{\perp} \wedge Q^{\perp}\| < 1$" ⇔ "$\|P^{\perp}Q^{\perp}P^{\perp} - P^{\perp} \wedge Q^{\perp}\| < 1$" ⇔ "there exists $0 \le \alpha < 1$ such that $P^{\perp}Q^{\perp}P^{\perp} - P^{\perp} \wedge Q^{\perp} \le \alpha I$" $(*)$ ⇔ "there exists $0 \le \alpha < 1$ such that $I - 2P - Q + P + PQ + QP - PQP - I + P \vee Q \le \alpha I$" ⇔ "there exists $0 \le \alpha < 1$ such that $-(P - Q)^2 - PQP + P \vee Q \le \alpha I$" ⇔ "there exists $0 \le \alpha < 1$ such that $P \vee Q \le (P - Q)^2 + PQP + \alpha I$".

Recall that $(P - Q)^2 + PQP \le P \vee Q$ for all $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$, see Corollary 2.4.

(i)⇒(iii). We multiply both sides of the inequality $P^{\perp}Q^{\perp}P^{\perp} - P^{\perp} \wedge Q^{\perp} \le \alpha I$, see $(*)$, by the projection $P^{\perp}$ from the left and the right. This yields $P^{\perp}Q^{\perp}P^{\perp} - P^{\perp} \wedge Q^{\perp} \le \alpha P^{\perp}$. By applying the De Morgan law, we can express this as "there exists $0 \le \alpha < 1$ such that $I - 2P - Q + P + PQ + QP - PQP - I + P \vee Q \le \alpha P^{\perp}$" ⇔ "there exists $0 \le \alpha < 1$ such that $-(P - Q)^2 - PQP + P \vee Q \le \alpha P^{\perp}$" ⇔ "there exists $0 \le \alpha < 1$ such that $P \vee Q \le (P - Q)^2 + PQP + \alpha P^{\perp}$".

The implication (iii)⇒(ii) is obvious. □

Similarly, it can be shown that the pair (P, Q) is modular in $\mathcal{M}^{\mathrm{pr}}$ if and only if there exists $0 \le \alpha < 1$ such that $PQP - P \wedge Q \le \alpha P$.

Proposition 3.3 *Let $\mathcal{M}$ be a factor of type I or III on a Hilbert space $\mathcal{H}$. Let $P, Q, P_1, Q_1 \in \mathcal{M}^{pr}$ be such that $PQ = P_1Q_1$. Then, the pair (P, Q) is modular in $\mathcal{M}^{pr}$ if and only if the pair (P_1, Q_1) is modular in $\mathcal{M}^{pr}$.*

Proof By von Neumann Theorem 2.2, we have

$$P_1 \wedge Q_1 = \text{so-}\lim_{n\to\infty}(P_1Q_1)^n = \text{so-}\lim_{n\to\infty}(PQ)^n = P \wedge Q.$$

Therefore, $P_1Q_1 - P_1 \wedge Q_1 = PQ - P \wedge Q$ and the assertion follows from Corollary 4 in [17] for a factor of type III and from Remark 5 in [17] for a factor of type I. □

Corollary 3.4 Let $\mathcal{M}$ be as in Proposition 3.3, let $P, Q \in \mathcal{M}^{\mathrm{pr}}$, and let $P_1 := P \vee Q - P$, $Q_1 := Q - P \wedge Q$. Then the following conditions are equivalent:

(i) the pair $(P^{\perp}, Q)$ is modular in $\mathcal{M}^{\mathrm{pr}}$;
(ii) the pair (P_1, Q_1) is modular in $\mathcal{M}^{\mathrm{pr}}$;
(iii) the pair (P_1, Q) is modular in $\mathcal{M}^{\mathrm{pr}}$.

Proof We have

$$P_1Q_1 = (P \vee Q - P)(Q - P \wedge Q) = Q - PQ = P^{\perp}Q$$

and

$$P_1Q = (P \vee Q - P)Q = Q - PQ = P^{\perp}Q.$$

Now, the assertions are concluded from Proposition 3.3 by considering $P^{\perp}$ instead of P. □

Analogously, by employing Proposition 3.3, for $P_2 := P \vee Q - P \wedge Q$ and $Q_2 := P - P \wedge Q$ and noting that $P_2 P = P - P \wedge Q = P_2 Q_2$, we have: *"The pair* (P_2, P) *is modular in* $\mathcal{M}^{\rm pr}$ *if and only if the pair* (P_2, Q_2) *is modular in* $\mathcal{M}^{\rm pr}$".

We conclude this section with general observations about products of projections of the form PQP, which will be employed in the discussions of the next section: In general, products $PQP \in P\mathcal{B}(\mathcal{H})P$ play an important role in the study of projections on Hilbert spaces. For example $\|PQP\|$ is equal to the square of the cosine of the Dixmier angle between subspaces of $\operatorname{ran}(P)$ and $\operatorname{ran}(Q)$. In quantum mechanics interpretation, the quantity PQP represents the probability of observing Q after a measurement has confirmed P.

It is shown in [1] that an operator $T \in \mathcal{B}(\mathcal{H})^+$ belongs to the set $\mathcal{D} := \{PQP : P, Q \in \mathcal{B}(\mathcal{H})^{\rm pr}\}$ if and only if $T \le I$ and $\dim \overline{\operatorname{ran}(T - T^2)} \le \dim \ker(T)$. The authors of [10] characterize the set $\mathcal{D}_S = \{(P, Q) : P, Q \in \mathcal{B}(\mathcal{H})^{\rm pr}, S = PQP\}$ and find all pairs $(P_0, Q_0) \in \mathcal{D}_S$ such that $\|P_0 - Q_0\| = \min\{\|P - Q\| : (P, Q) \in \mathcal{D}_S\}$.

It is easy to see that the projections of $P\mathcal{B}(\mathcal{H})P$ are exactly the operators PQP, where Q is a projection commuting with P. In fact, if $R = PTP \in P\mathcal{B}(\mathcal{H})P$ is a projection, then $Q := PRP = PR = RP$ is a projection commuting with P such that $R = PQP$ [11, Lemma 10].

If P and Q commute, then $\operatorname{ran}(P) + \operatorname{ran}(Q)$ is closed. Therefore, (P, Q) is modular as well as $PQP = PQ$ is a projection. Thus, one may claim that (P, Q) is modular if and only if PQP is a projection. However, this statement fails in general, as we show below:

If PQP is a projection, it follows from (2.1) that $P \wedge Q = \text{so-}\lim (PQP)^n = PQP$. Hence, $\|PQP - P \wedge Q\| = 0 < 1$ and from statement (3.1) we conclude that the pair (P, Q) is modular. However, if the pair (P, Q) is modular, then PQP may not be a projection. For example, let us consider the projections $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ acting on $\mathcal{H} = \mathbb{C}^2$. Then, $\operatorname{ran}(P) + \operatorname{ran}(Q) = \mathcal{H}$ is closed and so (P, Q) is modular. However, $PQP = \frac{1}{2}P$ is not a projection.

4 Isoclinic projections

We begin this section with the following lemma that we need to prove the next result.

Lemma 4.1 *Let $A \in \mathcal{B}(\mathcal{H})$ be an idempotent. Then, A is a projection if and only if $\|A\| \le 1$.*

Proof ($\Longrightarrow$) is evident. We merely prove ($\Longleftarrow$):

First proof. To reach a contradiction, assume two unit vectors $x \in \operatorname{ran}(A)$ and $y \in \ker(A)$ such that $\langle x, y\rangle \ne 0$. By replacing x with $\mathrm{i}x$, if necessary, we can assume that $\operatorname{Re}\langle x, y\rangle \ne 0$. Replacing x with $\frac{-2x}{\operatorname{Re}\langle x, y\rangle}$, we can assume $\operatorname{Re}\langle x, y\rangle = -2 < \frac{-1}{2}$. Note that the norm of x need not be equal to one in this proof.

Set $z := x + y$. We have $Az = Ax = x$, and

$$\|Az\|^2 = \|x\|^2 > \|x\|^2 + \|y\|^2 + 2\operatorname{Re}\langle x, y\rangle = \|z\|^2.$$

This implies $\|A\| > 1$, contradicting $\|A\| \leq 1$.

Hence, $\operatorname{ran}(A) \perp \ker(A)$, meaning that A is a projection.

Second proof. If $\|A\| \leq 1$, then

$$\begin{aligned} 0 \leq \|Ax - A^*Ax\|^2 &= \langle Ax - A^*Ax, Ax - A^*Ax\rangle \\ &= \|Ax\|^2 - \langle Ax, A^*Ax\rangle - \langle A^*Ax, Ax\rangle + \|A^*Ax\|^2 \\ &= \|Ax\|^2 - \langle A^2x, Ax\rangle - \langle Ax, A^2x\rangle + \|A^*Ax\|^2 \\ &= \|Ax\|^2 - \|Ax\|^2 - \|Ax\|^2 + \|A^*Ax\|^2 \\ &= \|A^*Ax\|^2 - \|Ax\|^2 \\ &\leq \|A^*\|^2\|Ax\|^2 - \|Ax\|^2 \\ &\leq \|Ax\|^2 - \|Ax\|^2 = 0. \end{aligned}$$

Thus $A = A^*A$ is self-adjoint and so A is a projection. □

Proposition 4.2 *Let $P, Q \in \mathcal{B}(\mathcal{H})^{pr}$. Then, $PQP = P$ (or $QPQ = P$) if and only if $P \leq Q$.*

Proof ($\Longleftarrow$) It follows from Theorem 1.1.

($\Longrightarrow$) *First proof.* If $PQP = P$, then $(PQ)^2 = PQP \cdot Q = PQ$ and the operator PQ is idempotent. Since $\|PQ\| \leq \|P\|\|Q\| \leq 1$ we conclude from Lemma 4.1 that $PQ \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$. Hence, $PQ = (PQ)^* = QP$. Therefore, $P = PQP = PQ = QP$ and $P \leq Q$.

If $QPQ = P$ we multiply both sides of this equality by Q from the left (resp., from the right), and obtain $QPQ = PQ = QP = P$ and $P \leq Q$.

Second proof. It follows from $PQP = P$ that $(PQP)^n = P$. From (2.1), we derive that $P \wedge Q = \text{so-}\lim_{n\to\infty}(PQP)^n = P$. Therefore, $P = P \wedge Q \leq Q$.

Third proof. Let's use the decomposition $\mathcal{H} = \operatorname{ran}(P) \oplus \ker(P)$. Then P and Q can be represented as

$$P = \begin{bmatrix} id_{\operatorname{ran}(P)} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \geq 0.$$

Employing the assumption $PQP = P$ and the matrix representations above, we find $A = id_{\operatorname{ran}(P)}$. Since $Q^2 = Q$, by examining the (1,1)-entries, we obtain $A^2 + XX^* = A$. This equality along with $A = id_{\operatorname{ran}(P)}$ leads to $XX^* = 0$. Hence $X = 0$. Consequently,

$$P = \begin{bmatrix} id_{\operatorname{ran}(P)} & 0 \\ 0 & 0 \end{bmatrix} \leq \begin{bmatrix} id_{\operatorname{ran}(P)} & 0 \\ 0 & B \end{bmatrix} = Q.$$

□

Recall that an operator $U \in \mathcal{B}(\mathcal{H})$ is called a *partial isometry* if U is isometric on $\ker(U)^\perp$. This is equivalent to any one of the following conditions: (i) $UU^*U = U$, (ii) $U^*U \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$, (iii) $UU^* \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$. (iv) U^* is a partial isometry; see [19, Theorem 2.3.3]. The next result can be stated as follows.

Theorem 4.3 *Let $P, Q \in \mathcal{B}(\mathcal{H})^{pr}$ and $PQP = \lambda P$ for some $0 < \lambda \leq 1$.*

(i) *If* $0 < \lambda < 1$, *then* $P \wedge Q = 0$.
(ii) *There exists* $R \in \mathcal{B}(\mathcal{H})^{pr}$ *such that* $P \sim R \leq Q$ *and* $RPR = \lambda R$.
(iii) *If* $P \sim Q$ *with a partial isometry* U, *then* $\lambda^{-1/2}U^2$ *is also a partial isometry.*

Proof (i). It is supported by Theorem 2.2.

(ii). The operator $V = \lambda^{-1/2}QP$ is a partial isometry because of the equality $V^*V = P$. Therefore, V^* is also a partial isometry. Let us set $R := VV^* \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$. Then, $R \sim P$ and

$$\lambda^{-1}QPQ = R.$$

Multiplying both sides of this equality by Q from the left and from the right, we obtain $\lambda^{-1}QPQ = R = QRQ$ and $R \leq Q$ according to Proposition 4.2. Now, we prove that $RPR = \lambda R$: We have $\lambda R = QPQ$ and

$$RPR = \lambda^{-2}Q(PQP)QPQ = \lambda^{-2}Q(\lambda P)QPQ = \lambda^{-1}Q(PQP)Q = QPQ.$$

(iii). Let us assume that $P = U^*U$ and $Q = UU^*$, where $U \in \mathcal{B}(\mathcal{H})$ is a partial isometry. We can rewrite the equality $PQP = \lambda P$ as

$$U^*UUU^*U^*U = \lambda U^*U.$$

Multiply both sides of this equality by U from the left and by U^* from the right, apply the equalities

$$UU^*U = U, \quad U^*UU^* = U^*,$$

and obtain $U^2U^{*2}U^2 = \lambda U^2$, which simplifies to $\lambda^{-3/2}U^2U^{*2}U^2 = \lambda^{-1/2}U^2$. Hence, $\lambda^{-1/2}U^2$ is a partial isometry.

□

Remark 4.4 The converse of Theorem 4.3(i) is not true in general. For example, let us consider the following projections in $\mathcal{B}(\mathbb{C}^4)$:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, we have $PQP = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and thus $PQP \neq \lambda P$ for any λ. Moreover, a straightforward verification shows that $\mathrm{ran}(P) \cap \mathrm{ran}(Q) = \varnothing$, and hence $P \wedge Q = 0$.

Definition 4.5 Projections $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ are called *isoclinic* if $PQP = \cos^2\theta\, P$ and $QPQ = \cos^2\theta\, Q$ for some angle $\theta \in (0, \pi/2)$. Then, we write $P \overset{\theta}{\approx} Q$.

A consequence of Theorem 4.3 is as follows.

Corollary 4.6 If $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ and $PQP = \lambda P$ for some $0 < \lambda < 1$ and $\mathrm{tr}(Q) \leq \mathrm{tr}(P) < +\infty$, then $QPQ = \lambda Q$, that is, $P \overset{\theta}{\approx} Q$ for the angle $\theta = \arccos(\sqrt{\lambda})$.

Proof In light of Theorem 4.3(ii), there exists $R \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ such that $P \sim R \leq Q$. Hence, $\mathrm{tr}(P) = \mathrm{tr}(R) \leq \mathrm{tr}(Q)$. It follows from the hypotheses that $\mathrm{tr}(Q) \leq \mathrm{tr}(P) < +\infty$. Hence, $\mathrm{tr}(R - Q) = 0$. Since the trace functional is faithful, $R = Q$. It follows from Theorem 4.3(ii) that $QPQ = \lambda Q$. □

If $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ with $P \overset{\theta}{\approx} Q$ for some angle $\theta \in (0, \pi/2)$, then the pair (P, Q) is modular in $\mathcal{B}(\mathcal{H})^{\mathrm{pr}}$. Proposition 4.6 ensures that $P \wedge Q = 0$ and

$$\cos^2\theta = \|PQP\| = \||QP|^2\| = \||QP|\|^2 = \|QP\|^2 = \|(QP)^*\|^2 = \|PQ\|^2.$$

Therefore, $\|PQ - P \wedge Q\| = \|PQ\| = \cos\theta \in (0, 1)$. In particular, the linear space $\mathrm{ran}(P) + \mathrm{ran}(Q)$ is closed.

The next result is derived from Theorem 4.3(i).

Corollary 4.7 [20, Chap. 2, Theorem 10.5] If $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ such that $P \overset{\theta}{\approx} Q$ for some angle $\theta \in (0, \pi/2)$, then $P \wedge Q = 0$.

Theorem 4.8 *Let $P, Q \in \mathcal{B}(\mathcal{H})^{pr}$ and $P \overset{\theta}{\approx} Q$ for some angle $\theta \in (0, \pi/2)$.*

(i) *If $P_1 := P \vee Q - P$ and $Q_1 := Q - P \wedge Q$, then $P_1 \overset{\pi/2-\theta}{\approx} Q_1$.*
(ii) *If $P^\perp \wedge Q^\perp = 0$, then $P^\perp \overset{\pi/2-\theta}{\approx} Q$ and $Q^\perp \overset{\pi/2-\theta}{\approx} P$.*

Proof We have $P \wedge Q = 0$ and $P \vee Q = \sin^{-2}\theta\,(P - Q)^2$, see [20, Chap. 2, Theorem 10.5(iii)]. Hence, $P_1 = \sin^{-2}\theta\,(P - Q)^2 - P$, $Q_1 = Q$ and

$$\begin{aligned}
Q_1P_1Q_1 &= Q(\sin^{-2}\theta(P + Q - PQ - QP) - P)Q \\
&= \sin^{-2}\theta(Q - QPQ) - QPQ = \sin^{-2}\theta(Q - \cos^2\theta Q) - \cos^2\theta Q \\
&= Q - \cos^2\theta Q = \cos^2(\pi/2 - \theta)Q_1,
\end{aligned}$$

$$\begin{aligned}
&P_1Q_1P_1 \\
&= (\sin^{-2}\theta(P + Q - PQ - QP) - P) - P)Q(\sin^{-2}\theta(P + Q - PQ - QP) - P) - P) \\
&= \sin^{-4}\theta(Q - 2QPQ + (QPQ)^2) - \sin^{-2}\theta(QP + PQ - QPQP - PQPQ) + PQP \\
&= \sin^{-4}\theta(Q - 2\cos^2\theta Q + \cos^4\theta Q) - \sin^{-2}\theta(QP + PQ - \cos^2\theta(QP + PQ) + \cos^2\theta P \\
&= \sin^{-4}\theta(1 - \cos^2\theta)^2 Q - \sin^{-2}\theta Q\sin^2\theta(QP + PQ) + \cos^2\theta P \\
&= Q - PQ - QP + \cos^2\theta P = (1 - \sin^2\theta)P + PQ - QP \\
&= \cos^2(\pi/2 - \theta)(\sin^{-2}\theta(P + Q - PQ - QP) - P) = \cos^2(\pi/2 - \theta)P_1.
\end{aligned}$$

(ii). By [20, Chap. 2, Theorem 10.5(iii)] and the equality $P \vee Q - P = P^{\perp} - P^{\perp} \wedge Q^{\perp}$, we have

$$\begin{aligned} &P^{\perp} Q P^{\perp} \\ &= (I-P)Q(I-P) = Q - PQ - QP + PQP = Q + \cos^2\theta P - PQ - QP \\ &= Q + P - PQ - QP - (1-\cos^2\theta)P = (Q-P)^2 - (1-\cos^2\theta)P \\ &= (1-\cos^2\theta)(P \vee Q - P) = (1-\cos^2\theta)(P^{\perp} - P^{\perp} \wedge Q^{\perp}) \\ &= \sin^2\theta P^{\perp} = \cos^2(\pi/2-\theta)P^{\perp} \end{aligned}$$

and $QP^{\perp}Q = Q - QPQ = Q - \cos^2\theta Q = \sin^2\theta Q = \cos^2(\pi/2-\theta)Q$. □

Acknowledgements The authors sincerely thank the referee for several valuable comments, which have improved the paper.

Author Contributions Both Bikchentaev and Moslehian wrote the main manuscript text and they edited and reviewed the manuscript.

Funding This work was performed under the development programme of the Volga Region Mathematical Center (agreement no. 075-02-2025-1725/1).

Data Availability No datasets were generated or analysed during the current study.

Declarations

Conflict of Interest On behalf of the authors, the corresponding author states that there is no conflict of interest.

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