Iskander Sh. Kalimullin

COMPUTABLE EMBEDDINGS OF CLASSES OF STRUCTURES UNDER ENUMERATION AND TURING OPERATORS

Institute of Mathematics and Mechanics, Kazan (Volga region) Federal University, Kremlyovskaya 35, Kazan, 420008, Russia

Abstract. In the paper we study the differences and partial characterizations of the Turing and enumeration computable embeddings of classes of structures

1. Introduction

In the papers [1] and [2] the following two notions were introduced as a computability analog of Borel embedding.

Definition 1. Let K_0, K_1 be classes of structures in finite languages (for each class the language is the same).
1) We say that K_0 is *computably embeddable via an e-operator* into K_1 (and write $K_0 \leq_c K_1$) iff there are a function $f : K_0 \to K_1$ and an e-operator Φ such that $D(f(\mathcal{A})) = \Phi(D(\mathcal{A}))$ for any $\mathcal{A} \in K_0$ and and for any $\mathcal{A}_1, \mathcal{A}_2 \in K_0$

$$\mathcal{A}_0 \cong \mathcal{A}_1 \iff f(\mathcal{A}_0) \cong f(\mathcal{A}_1).$$

2) We say that K_0 is *computably embeddable via an Turing operator* into K_1 (and write $K_0 \leq_{tc} K_1$) iff there are a function $f : K_0 \to K_1$ and a Turing operator φ_e such that $\chi_{D(f(\mathcal{A}))} = \varphi_e^{D(\mathcal{A})}$ for any $\mathcal{A} \in K_0$ and and for any $\mathcal{A}_1, \mathcal{A}_2 \in K_0$

$$\mathcal{A}_0 \cong \mathcal{A}_1 \iff f(\mathcal{A}_0) \cong f(\mathcal{A}_1).$$

It follows from the next proposition then $K_0 \leq_c K_1$ implies $K_0 \leq_{tc} K_1$.

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Proposition 2. *$K_0 \leq_{tc} K_2$ iff there are a function $f : K_0 \to K_1$ and an integer $e \in \omega$ such that $D(f(\mathcal{A})) = W_e^{D(\mathcal{A})}$ for any $\mathcal{A} \in K_0$ and and for any $\mathcal{A}_1, \mathcal{A}_2 \in K_0$*

$$\mathcal{A}_0 \cong \mathcal{A}_1 \iff f(\mathcal{A}_0) \cong f(\mathcal{A}_1).$$

Proof. ($\Longrightarrow$) Obvious.

($\Longleftarrow$) Without loss of generality we can assume that $card(W_{e,s+1}^X - W_{e,s})^X \leq 1$ for all s and X. We denote via $T(a)$ the atomic sentence $a = a$ for each $a \in \omega$.

Suppose that $\mathcal{A} \in K_0$ is given. Define

$$S = \{s \in \omega : (\exists a)[T(a) \in W_{e,s+1}^{D(\mathcal{A})} - W_{e,s}^{D(\mathcal{A})}]\}$$

and $T(a_s) \in W_{e,s+1}^{D(\mathcal{A})} - W_{e,s}^{D(\mathcal{A})}$ for each $s \in S$. Let $\mathcal{S}_{\mathcal{A}}$ be the structure with universe S such that $\mathcal{S}_{\mathcal{A}} \cong \Phi(\mathcal{A})$ via the isomorphism $s \mapsto a_s$.

It is easy to see that there is an index i such that $D(\mathcal{S}_{\mathcal{A}}) = \varphi_i^{D(\mathcal{A})}$ for each $\mathcal{A} \in K_0$. $\square$

Corollary 3. *If $K_0 \leq_c K_1$ then $K_0 \leq_{tc} K_1$.*

To see that the reverse implication is not true we can note that for the case when $K_0 = \{\mathcal{X} : \mathcal{X} \cong \mathcal{A}\}$ and $K_1 = \{\mathcal{X} : \mathcal{X} \cong \mathcal{B}\}$ the embedding $K_0 \leq_{tc} K_1$ is equivalent to the Medvedev reducibility (the uniform Turing reducibility) $\mathcal{B} \leq_{uT} \mathcal{A}$, and the embedding $K_0 \leq_c K_1$ is equivalent to $\mathcal{B} \leq_{ue} \mathcal{A}$ (the uniform enumeration reducibility). It follows from [3] that there are structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{B} \leq_{uT}$ and $\mathcal{B} \not\leq_{ue} \mathcal{B}$. But such structures $\mathcal{A}$ and $\mathcal{B}$ should not have a computable presentation, while the embeddings $\leq_c$ and $\leq_{tc}$ are more interesting for the classes of computable and even finite structures.

2. Embedding of classes with finitely many isomorphic types

The following two theorem give a full descriptions of c- and tc-embeddings of classes with finitely many isomorphic types. These will give an easy example of a pair of classes of computable structures for which the c- and tc-embeddings differ.

Theorem 4. *Let a class of finite structures K_0 contains only finitely many different isomorphic types. Then $K_0 \leq_c K_1$ iff there is a function f from K_0 into a subclass of K_1 containing only computable structures such that for any $\mathcal{A}, \mathcal{B} \in K_0$*

$$f(\mathcal{A}) \cong f(\mathcal{B}) \implies \mathcal{A} \cong \mathcal{B}, \textit{ and}$$

$$\mathcal{A} \text{ is embeddable into } \mathcal{B} \implies f(\mathcal{A}) \subseteq f(\mathcal{B}).$$

Theorem 5. *Let a class of finite structures K_0 contains only finitely many different isomorphic types. Then $K_0 \le_{tc} K_1$ iff there is a function f from K_0 into a subclass of K_1 containing only computable structures such that for any $\mathcal{A}, \mathcal{B} \in K_0$*

$$\mathcal{A} \cong \mathcal{B} \iff f(\mathcal{A}) \cong f(\mathcal{B}) \iff f(\mathcal{A}) = f(\mathcal{B}), \text{ and}$$

$$\mathcal{A} \text{ is embeddable into } \mathcal{B} \implies Th_\exists(f(\mathcal{A})) \subseteq Th_\exists(f(\mathcal{B})).$$

Corollary 6. *Let a class of finite structures K_0 contains only finitely many different isomorphic types. Then for all classes of finite structures K_1 we have $K_0 \le_c K_1 \iff K_0 \le_{tc} K_1$.*

Corollary 7. *There are classes K_0 and K_1 such that $K_0 \le_{tc} K_1$ and $K_0 \not\le_c K_1$.*

Proof of Corollary 7. Let K_0 consists from the empty linear ordering and all one-element orderings, and let K_1 consists from all linear orderings isomorphic either to ω, or to ω^*. By Theorem 4 $K_0 \not\le_c K_1$. Since all infinite linear orderings have one existensional theory we have $K_0 \le_{tc} K_1$ by Theorem 5. $\square$

Proof of Theorem 4. ($\Longrightarrow$) Let $K_0 \le_c K_1$ via an e-operator Φ. Since K_0 contains only finitely many different isomorphic types there is a finite collection I_0 of structures from K_0 such that any structure from K_0 has an isomorphic copy in I_0 and such that from any $\mathcal{A}_0, \mathcal{A}_1 \in K_0$ and any $\mathcal{C}_0, \mathcal{C}_1 \in I_0$

$$\mathcal{A}_0 \cong \mathcal{C}_0 \;\&\; \mathcal{A}_1 \cong \mathcal{C}_1 \;\&\; \mathcal{A}_0 \text{ is embeddable into } \mathcal{A}_1 \implies \mathcal{C}_0 \subseteq \mathcal{C}_1.$$

For any $\mathcal{A} \in K_0$ we define $f(\mathcal{A})$ as the structure from K_1 such that $D(f(\mathcal{A})) = \Phi(D(\mathcal{C}_\mathcal{A}))$, where $\mathcal{C}_\mathcal{A} \in I_0$ and $\mathcal{C}_\mathcal{A} \cong \mathcal{A}$.

($\Longleftarrow$) Suppose that such function $f : K_0 \to K_1$ exists. We define an e-operator Φ via the c.e. set of all axioms $\langle \varphi, D(\mathcal{A}) \rangle$, where $\mathcal{A} \in K_0$ and $\varphi \in D(f(\mathcal{A}))$. Then $K_0 \le_c K_1$ via the e-operator Φ. $\square$

Proof of Theorem 5. ($\Longrightarrow$) Let $K_0 \le_{tc} K_1$ via a Turing operator φ_e and let I_0 be as in the proof of Theorem 4. For any $\mathcal{A} \in K_0$ we define $f(\mathcal{A})$ as the structure from K_1 such that $D(f(\mathcal{A})) = \varphi_e^{D(\mathcal{C}_\mathcal{A})}$, where $\mathcal{C}_\mathcal{A} \in I_0$ and $\mathcal{C}_\mathcal{A} \cong \mathcal{A}$. Then the implications $\mathcal{A} \cong \mathcal{B} \iff f(\mathcal{A}) \cong f(\mathcal{B}) \iff f(\mathcal{A}) = f(\mathcal{B})$ are obvious. Suppose that $\mathcal{A} \in K_0$ is embeddable into $\mathcal{B} \in K_0$ and $f(\mathcal{A}) \models \theta$ for some existensional sentence θ. Let $\mathcal{F}$ be a finite substructure of $f(\mathcal{A})$ such that $\mathcal{F} \models \theta$ and let n be such integer that $\varphi_e^{D(\mathcal{C}_\mathcal{A})\restriction n}(\psi)\downarrow= 1$ for all $\psi \in D(\mathcal{F})$. Since $\mathcal{A}$ is embeddable into $\mathcal{B}$ there is a structure $\mathcal{B}' \cong \mathcal{B}$ such that $\psi \in D(\mathcal{C}_\mathcal{A}) \iff \psi \in D(\mathcal{B}')$ for all atomic sentences with code $< n$. Then $\varphi_e^{D(\mathcal{B}')}(\psi) = 1$ for all $\psi \in D(\mathcal{F})$ so that $\mathcal{F} \subseteq \mathcal{B}'$ and hence $f(\mathcal{B}) \models \theta$.

($\Longleftarrow$) Let $f : K_0 \to K_1$ be such function. For any $\mathcal{C} \in I_0$ and any finite consistent set of atomic sentences Δ in the language of class K_1 we denote via $\mathcal{E}_\Delta^\mathcal{C}$ a structure isomorphic to $f(\mathcal{C})$ such that $\Delta \subseteq D(\mathcal{E}_\Delta^\mathcal{C})$.

Since $f(\mathcal{C})$ is always a computable structure, for each $\mathcal{C} \in I_0$ the correspondence $\Delta \mapsto \mathcal{E}_\Delta^\mathcal{C}$ can be chosen partial computable in the sense that knowing a canonical index of a finite set Δ we can effectively determine membership of any atomic sentence ψ in $D(\mathcal{E}_\Delta^\mathcal{C})$ if $\mathcal{E}_\Delta^\mathcal{C}$ exists, and the last condition is c.e.

Let $\mathcal{A} \in K_0$ be given. Let $s_0 < s_1 < s_2 < \dots$ be all integers $s \in \omega$ such that for some $\mathcal{A}' \in K_0$ we have $D(\mathcal{A}) \restriction s = D(\mathcal{A}')$. Then for any $n \in \omega$ we denote via $\mathcal{C}_n$ the structure from the finite collection I_0 such that for some $\mathcal{A}' \cong \mathcal{C}_n$ we have $D(\mathcal{A}) \restriction s_n = D(\mathcal{A}')$. Note that $\mathcal{C}_n \subseteq \mathcal{C}_{n+1}$ for each $n \in \omega$.

Now we inductively construct a sequence $\{\Delta_n\}_{n\in\omega}$ of finite consistent sets of atomic sentences in the language of class K_1:
$\Delta_0 = \emptyset$,
$\Delta_{n+1} = D(\mathcal{E}_{\Delta_m}^{\mathcal{C}_{n+1}}) \restriction n$, where $m \leq n$ is the least integer such that $\mathcal{C}_k = \mathcal{C}_{n+1}$ for all k, $m < k \leq n$.
Note that $\mathcal{E}_{\Delta_m}^{\mathcal{C}_{n+1}}$ always exists since $\mathcal{C}_m \subseteq \mathcal{C}_{n+1}$ and hence $Th_\exists(f(\mathcal{C}_m)) \subseteq Th_\exists(f(\mathcal{C}_{n+1}))$. Moreover, $\bigcup_n \Delta_n = D(\mathcal{B})$ for some $\mathcal{B} \cong f(\mathcal{A})$. It remains to note that $D(\mathcal{B}) = W_e^{D(\mathcal{A})}$ for some e and apply Proposition 2.$\square$

3. Embedding of classes of finite structures

The following two theorem give a full descriptions of c- and tc-embeddings of the class FLO of all finite linear orderings. The third theorem gives an example of a pair of classes for which the c- and tc-embeddings differ.

Theorem 8. *$FLO \leq_c K$ iff there is a computable Friedberg numbering $\{\mathcal{X}_n\}_{n\in\omega}$ of a subclass of K such that $\mathcal{X}_n \subseteq \mathcal{X}_{n+1}$ for each n.*

Theorem 9. *$FLO \leq_{tc} K$ iff there is a computable Friedberg numbering $\{\mathcal{X}_n\}_{n\in\omega}$ of a subclass of K such that $Th_\exists(\mathcal{X}_n) \subseteq Th_\exists(\mathcal{X}_{n+1})$ for each n.*

Theorem 10. *There is a class K of undirected finite graphs with such that*
a) there is a computable Friedberg numbering $\{\mathcal{X}_n\}_{n\in\omega}$ of the class K

such that each graph $\mathcal{X}_n$, $n \in \omega$, is embeddable into the graph $\mathcal{X}_{n+1}$, and b) there is no computable Friedberg numbering $\{\mathcal{Y}_n\}_{n\in\omega}$ of a subclass of K such that $\mathcal{Y}_n \subseteq \mathcal{Y}_{n+1}$ for each n.

Corollary 11. *There is a class K of undirected finite graphs such that $FLO \leq_{tc} K$ and $FLO \nleq_c K$.*

Proof of Theorem 8. ($\Longrightarrow$) Suppose that $FLO \leq_c K$ via an e-operator Φ. Let $\mathcal{L}_n$ be the standard linear ordering of natural numbers $< n$. Then for each $n \in \omega$ $\Phi(D(\mathcal{L}_n)) = D(\mathcal{X}_n)$ for some $\mathcal{X}_n \in K$. It is easy to check that $\{\mathcal{X}_n\}_{n\in\omega}$ is the computable Friedberg numbering such that $\mathcal{X}_n \subseteq \mathcal{X}_{n+1}$ for each n.

($\Longleftarrow$) Let there exists such computable Friedberg numbering $\{\mathcal{X}_n\}_{n\in\omega}$. We define an e-operator Φ via the c.e. set of all axioms $\langle \varphi, D(\mathcal{A})\rangle$, where $\mathcal{A}$ is a linear ordering with n elements and $\varphi \in D(\mathcal{X}_n)$, $n \in \omega$. Then $FLO \leq_c K$ via the e-operator Φ. $\Box$

Proof of Theorem 9. ($\Longrightarrow$) Let $FLO \leq_{tc} K$ via a Turing operator φ_e and let $\mathcal{L}_n$ be the standard linear ordering of natural numbers $< n$. For any $n \in \omega$ we define $\mathcal{X}_n$ as the structure from K such that $D(\mathcal{X}_n) = \varphi_e^{D(\mathcal{L}_n)}$. It is easy to see that $\{\mathcal{X}_n\}_{n\in\omega}$ is the computable Friedberg numbering.

We prove that $Th_\exists(\mathcal{X}_n) \subseteq Th_\exists(\mathcal{X}_{n+1})$ for each n. Let $\mathcal{X}_n \models \theta$ for some existensional sentence θ. Let $\mathcal{F}$ be a finite substructure of $\mathcal{X}_n$ such that $\mathcal{F} \models \theta$ and let k be such integer that $\varphi_e^{D(\mathcal{L}_n)\restriction k}(\psi) \downarrow = 1$ for all $\psi \in D(\mathcal{F})$. We choose a linear odrering $\mathcal{L}'_{n+1}$ with $n+1$ elements such that $\psi \in D(\mathcal{L}_n) \iff \psi \in D(\mathcal{L}'_{n+1})$ for all atomic sentences with code $< k$. Then $\varphi_e^{D(\mathcal{L}'_{n+1})}(\psi) = 1$ for all $\psi \in D(\mathcal{F})$ and hence $\mathcal{X}_{n+1} \models \theta$.

($\Longleftarrow$) Let there exists such computable Friedberg numbering $\{\mathcal{X}_n\}_{n\in\omega}$. For any $n \in \omega$ and any finite consistent set of atomic sentences Δ in the language of class K we denote via $\mathcal{E}^n_\Delta$ a structure isomorphic to $\mathcal{X}_n$ such that $\Delta \subseteq D(\mathcal{E}^n_\Delta)$. As in the proof of Theorem 5 we can choose a partially computable correspondence $(n, \Delta) \mapsto \mathcal{E}^n_\Delta$.

Let a finite linear ordering $\mathcal{L}$ be given. Let $s_0 < s_1 < s_2 < \dots$ be all integers $s \in \omega$ such that $D(\mathcal{L}) \restriction s$ is a diagram of some linear ordering. We denote via $c(n)$ the number of elements in the linear ordering with the diagram $D(\mathcal{L}) \restriction s_n$. Note that $c(n) \leq c(n+1)$ for each $n \in \omega$.

We inductively construct a sequence $\{\Delta_n\}_{n\in\omega}$ of finite consistent sets of atomic sentences in the language of class K:
$\Delta_0 = \emptyset$,
$\Delta_{n+1} = D(\mathcal{E}^{c(n+1)}_{\Delta_m}) \restriction n$, where $m \leq s$ is the least integer such that $c(k) = c(n+1)$ for all k, $m < k \leq n$.

Note that $\mathcal{E}^{c(n+1)}_{\Delta_m}$ exists since $Th_\exists(\mathcal{X}_{c(m)}) \subseteq Th_\exists(\mathcal{X}_{c(n+1)})$. Moreover, $\bigcup_s \Delta_s = D(\mathcal{B})$ for some $\mathcal{B} \cong \mathcal{X}_{card(\mathcal{L})}$. It remains to note that $D(\mathcal{B}) = W_e^{D(\mathcal{L})}$ for some e and apply Proposition 2.□

Proof of Corollary 11. Let the language of undirected graphs contain one binary predicate R ($R(a,b)$ means that vertices a and b are connected by an edge). For each $m \geq 4$ define the following finite undirected graphs:
$\mathcal{A}_m$ is the graph with vertices $a_1, \ldots, a_{m+1}$ and with edges $\{a_1, a_2\}$, $\{a_2, a_3\}$, $\{a_3, a_4\}, \ldots \{a_{m-1}, a_m\}, \{a_m, a_1\}, \{a_m, a_{m+1}\}$;
$\mathcal{B}_m$ is the graph with vertices $a_1, \ldots, a_{m+2}$ and with edges $\{a_1, a_2\}$, $\{a_2, a_3\}$, $\{a_3, a_4\}, \ldots \{a_{m-1}, a_m\}, \{a_m, a_1\}, \{a_m, a_{m+1}\}, \{a_{m+1}, a_{m+2}\}$;
$\mathcal{C}_m$ is the graph with vertices $a_1, \ldots, a_{m+2}$ and with edges $\{a_1, a_2\}$, $\{a_2, a_3\}$, $\{a_3, a_4\}, \ldots \{a_{m-1}, a_m\}, \{a_m, a_1\}, \{a_m, a_{m+1}\}, \{a_{m+1}, a_{m+2}\}, \{a_m, a_{m+2}\}$;
$\mathcal{D}_m$ is the graph with vertices $a_1, \ldots, a_{m+4}$ and with edges $\{a_1, a_2\}$, $\{a_2, a_3\}, \{a_3, a_4\}, \ldots \{a_{m-1}, a_m\}, \{a_m, a_1\}, \{a_m, a_{m+1}\}, \{a_{m+1}, a_{m+2}\}, \{a_m, a_{m+2}\}$, $\{a_m, a_{m+3}\}$, $\{a_{m+3}, a_{m+4}\}$.

Then for each $i, j \in \omega$ we set $\mathcal{F}_{i,j} = \mathcal{C}_{\langle i,j\rangle+4}$ and $\mathcal{G}_{i,j} = \mathcal{D}_{\langle i,j\rangle+4}$ if there are integers $x_1, \ldots, x_{\langle i,j\rangle+6}$ such that
1) $\langle i, R(x_k, x_{k+1})\rangle \in W_i$ for each k, $1 \leq k \leq \langle i,j\rangle + 3$, and
2) $\langle i, R(x_1, x_{\langle i,j\rangle+4})\rangle \in W_i$;
3) $\langle i, R(x_{\langle i,j\rangle+4}, x_{\langle i,j\rangle+5})\rangle \in W_i$;
4) $\langle i+1, R(x_{\langle i,j\rangle+5}, x_{\langle i,j\rangle+6})\rangle \in W_i$ and $\langle i+1, \neg R(x_{\langle i,j\rangle+4}, x_{\langle i,j\rangle+6})\rangle \in W_i$.
Otherwise we set $\mathcal{F}_{i,j} = \mathcal{A}_{\langle i,j\rangle+4}$ and $\mathcal{G}_{i,j} = \mathcal{B}_{\langle i,j\rangle+4}$.

Let finite undirected graph $\mathcal{H}_n$, $n \in \omega$, be the disjoint and disconnected union of all graphs $\mathcal{F}_{i,n}$, $i \leq n$, and all graphs $\mathcal{G}_{i,j}$, $i \leq j < n$. It is easy to see that for all n the graph $\mathcal{H}_n$ is finite and $\mathcal{H}_n$ is embeddable into $\mathcal{H}_{n+1}$ since each $\mathcal{A}_m$ is embeddable into $\mathcal{B}_m$ and each $\mathcal{C}_m$ is embeddable into $\mathcal{D}_m$. Moreover, there is a computable Friedberg numbering of finite undirected graphs $\{\mathcal{X}_n\}_{n\in\omega}$ such that $\mathcal{X}_n \cong \mathcal{H}_n$ for each n since the conditions 1)-4) are Σ_1 and since each $\mathcal{A}_m$ is embeddable into $\mathcal{C}_m$ and each $\mathcal{B}_m$ is embeddable into $\mathcal{D}_m$.

Let K be the class containing all graphs isomorphic to $\mathcal{H}_n$ for some $n \in \omega$. Then K satisfies the condition a) of the theorem. Suppose that there is a computable Friedberg numbering $\{\mathcal{Y}_n\}_{n\in\omega}$ of a subclass of K such that $\mathcal{Y}_n \subseteq \mathcal{Y}_{n+1}$ for each n. Then there is a c.e. set W_i of pairs $\langle n, \varphi\rangle$, where $n \in \omega$, φ is either an atomic sentence, or its negation, such that $D(\mathcal{Y}_n) = \{\varphi : \langle n, \varphi\rangle \in W_i\}$.

Since $\mathcal{Y}_n \subsetneq \mathcal{Y}_{n+1}$ for each n we have $\mathcal{Y}_i \cong \mathcal{H}_j$ for some $j \geq i$ and $\mathcal{Y}_{i+1} \cong \mathcal{H}_{j'}$ for some $j' > j$. Then $\mathcal{Y}_i$ contains a subgraph $\mathcal{Y}'_i$ isomorphic to $\mathcal{F}_{i,j}$ and $\mathcal{Y}_{i+1}$ contains a subgraph $\mathcal{Y}'_{i+1}$ isomorphic to $\mathcal{G}_{i,j}$ such that $\mathcal{Y}'_i \subseteq \mathcal{Y}'_{i+1}$.

If $\mathcal{F}_{i,j} = \mathcal{A}_{\langle i,j\rangle+4}$ and $\mathcal{G}_{i,j} = \mathcal{B}_{\langle i,j\rangle+4}$ then obviously there are integers $x_1, \ldots, x_{\langle i,j\rangle+5} \in Supp(\mathcal{Y}'_i)$ and $x_{\langle i,j\rangle+6} \in Supp(\mathcal{Y}'_{i+1})$ satisfying the conditions 1)-4), and hence $\mathcal{F}_{i,j} = \mathcal{C}_{\langle i,j\rangle+4}$, $\mathcal{G}_{i,j} = \mathcal{D}_{\langle i,j\rangle+4}$, contradiction.

Thus, $\mathcal{Y}'_i \cong \mathcal{F}_{i,j} = \mathcal{C}_{\langle i,j\rangle+4}$ and $\mathcal{Y}'_{i+1} \cong \mathcal{G}_{i,j} = \mathcal{D}_{\langle i,j\rangle+4}$. Hence there are integers $x_1, \ldots, x_{\langle i,j\rangle+6}$ satisfying the conditions 1)-4). Then $x_1, \ldots, x_{\langle i,j\rangle+5} \in Supp(\mathcal{Y}'_i)$ and $x_{\langle i,j\rangle+6} \in Supp(\mathcal{Y}'_{i+1})$. Since $\mathcal{Y}'_i \cong \mathcal{C}_{\langle i,j\rangle+4}$, there is an integer $y \in Supp(\mathcal{Y}'_i)$ such that the atomic sentences $R(x_{\langle i,j\rangle+4}, y)$, $R(x_{\langle i,j\rangle+5}, y)$ belong to $D(\mathcal{Y}'_i) \subseteq D(\mathcal{Y}'_{i+1})$. By the condition 4) we have $R(x_{\langle i,j\rangle+5}, x_{\langle i,j\rangle+6}) \in D(\mathcal{Y}'_{i+1})$ and $\neg R(x_{\langle i,j\rangle+4}, x_{\langle i,j\rangle+6}) \in D(\mathcal{Y}'_{i+1})$ so that $y \neq x_{\langle i,j\rangle+6}$, and we get a contradiction with $\mathcal{Y}'_{i+1} \cong \mathcal{D}_{\langle i,j\rangle+4}$. $\square$

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