# Width of the Gakhov Class Over the Dirichlet Space is Equal to 2 

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#### Abstract

Gakhov class $\mathcal{G}$ is formed by the holomorphic and locally univalent functions in the unit disk with no more than unique critical point of the conformal radius. Let $\mathcal{D}$ be the classical Dirichlet space, and let $P: f \mapsto F=f^{\prime \prime} / f^{\prime}$. We prove that the radius of the maximal ball in $P(\mathcal{G}) \cap \mathcal{D}$ with the center at $F=0$ is equal to 2 .


DOI: 10.1134/S1995080216040120
Keywords and phrases: Hyperbolic derivative, conformal radius, Bloch space, Dirichlet space, Gakhov class, Gakhov width.

## 1. INTRODUCTION

It was my good fortune in 2002 to work under the supervision of Professor D.H. Mushtari within the subject "Fundamental structures of mathematics and applied problems" which was carried out under the auspices of Tatarstan Academy of Sciences. Together with R.N. Gumerov and S.G. Haliullin we have prepared the report devoted to our study of linear functionals. Supervisor's review noted "the good balance of the report reflecting both functional-analytic nature of the studied objects, and the dynamics of the development of the investigations around the studying of the functionals behavior."

In this report the author developed the research of the hyperbolic derivatives of holomorphic functions (historically appeared as conformal radii of the hyperbolic domains). The concrete results concerned the study of the class $\mathcal{A}$ of the images of holomorphic functions with the unique extremum of the hyperbolic derivative under the pre-Schwarzian imbedding into the Bloch or Dirichlet space. The main tool was the Minkowski functional which allowed us to establish the series of topological properties of $\mathcal{A}$, to compute the width of $\mathcal{A}$, and to give an explicit description of the boundary part of $\mathcal{A}$ where such a width is attained.

Author's contribution to the above mentioned report has been reflected in the digest [1]; the expanded form of the Bloch case has been given in [2]. In the present note we reincarnate the results on the Dirichlet case announced in [1]. The author considers the work making here as the sacred duty and the cause to remember once again the unforgettable days of the fruitful experience of the job alongside the remarkable man and outstanding mathematician, Daniar Hamidovich Mushtari.

## 2. GAKHOV CLASSES

Let $H$ be the class of all holomorhic functions in $\mathbb{D}=\{\zeta \in \mathbb{C}:|\zeta|<1\}$, let $A=\{f \in H: f(0)=$ $\left.f^{\prime}(0)-1=0\right\}$, and let $H_{0}$ be the subclass of $f$ 's taking away from the class $A$ by the local univalence condition in the unit disk $\mathbb{D}: f^{\prime}(\zeta) \neq 0$ when $\zeta \in \mathbb{D}$. Hyperbolic derivative (conformal radius) of the function $f \in H_{0}$ is defined as

$$
\begin{equation*}
h_{f}(\zeta)=\left(1-|\zeta|^{2}\right)\left|f^{\prime}(\zeta)\right| . \tag{1}
\end{equation*}
$$

[^0]Systematic research of the extrema of the function (1) was begun by H.R. Haegi [3] as the development of the classical subjects of G. Polya and G. Szegö [4]; in the same, 1950, year the historical F.D. Gakhov's report has been sound at the conference on the complex function theory in Moscow (see [5], p. 649). This report became the starting point of the history of Kazan Seminar on the geometric function theory.

A surge of interest in the extrema of (1), i.e. the elements of the set

$$
\begin{equation*}
M_{f}=\left\{a \in \mathbb{D}: \nabla h_{f}(a)=0\right\}, \tag{2}
\end{equation*}
$$

came in the 1980-ies due to the connection between (2) and correctness of a number of problems in the complex function theory [6]-[8] and mathematical physics (see [9, 10] and bibliography there). One can considered the work [11] as a prolongation of the article [3]: the invariant form of the Haegi classification of the points in (2) turned out to be generic for the Nehari inequality [12]

$$
\begin{equation*}
\left|S_{f}(\zeta)\right| \leq 2 /\left(1-|\zeta|^{2}\right)^{2}, \zeta \in \mathbb{D}, \tag{3}
\end{equation*}
$$

as a condition for (2) to be a singleton or empty set [13]; $S_{f}(\zeta)=\left(f^{\prime \prime} / f^{\prime}\right)^{\prime}(\zeta)-\left(f^{\prime \prime} / f^{\prime}\right)^{2}(\zeta) / 2$ is the Schwarzian derivative of the function $f$. The study of this genericity was given in [14] and led to the new (in comparison with $[13,11]$ ) proof of the implication

$$
\begin{equation*}
(3) \Longrightarrow k_{f} \leq 1 \tag{4}
\end{equation*}
$$

where $k_{f}=\# M_{f}$ is the number of elements of (2) (see also [15]).
Later on the themes concentrating around the implication (4) have been caught up by $M$. Chuaqui et al (see, e.g., [16], and, especially, [17]) while within the walls of Kazan Seminar the new approach arose to treat the set of functions $f \in H_{0}$ with $k_{f} \leq 1$ as an object to study "im großen". The history of this approach is presented in [2].

Gakhov class (see, e.g., [17, 18, 2]) is defined as $\mathcal{G}=\left\{f \in H_{0}: k_{f} \leq 1\right\}$ and is equal to the union of its subclasses $\mathcal{G}_{-}=\left\{f \in H_{0}: k_{f}=0\right\}$ and $\mathcal{G}_{+}=\left\{f \in H_{0}: k_{f}=1\right\}$. The latter in turn breaks up in the disjoint union $\mathcal{G}_{+}=\mathcal{G}_{1} \bigsqcup \mathcal{G}_{s}$ of the subclasses $\mathcal{G}_{1}=\left\{f \in H_{0}: k_{f}=1, \gamma_{f}\left(M_{f}\right)=+1\right\}$ and $\mathcal{G}_{s}=\{f \in$ $\left.H_{0}: k_{f}=1, \gamma_{f}\left(M_{f}\right) \neq+1\right\}$. For the function $f \in H_{0}$ M.I. Kinder's mapping $\gamma_{f}: M_{f} \rightarrow\{-1,0,+1\}$ associates to an element $a \in M_{f}$ the index $\gamma_{f}(a)$ of the vector field $\nabla h_{f}(\zeta)$; depending on values of $\gamma_{f}(a)$ the points $a \in M_{f}$ may be of only three types: the case $\gamma_{f}(a)=+1$ corresponds to the local maximum, the case $\gamma_{f}(a)=-1$ defines the saddle, and in the case $\gamma_{f}(a)=0$ we have the semi-saddle [8].

As in [2] we consider the mapping

$$
\begin{equation*}
P: H_{0} \rightarrow H: f \mapsto F=f^{\prime \prime} / f^{\prime} \tag{5}
\end{equation*}
$$

which assigns to each function $f \in H_{0}$ its pre-Schwarzian $F=f^{\prime \prime} / f^{\prime}$ and is one-to-one due to $H_{0} \subset A$. In [2] the Gakhov class studied in the following setting [19, 20, 1].

Let $\mathcal{B}$ be the classical Bloch space consisting of all functions $F \in H$ with finite semi-norm $|F|_{\mathcal{B}}=$ $\sup _{\zeta \in \mathbb{D}} h_{f}(\zeta)$. It is well known that $\mathcal{B}$ is the Banach space relative to the norm $\|F\|_{\mathcal{B}}=|F(0)|+|F|_{\mathcal{B}}$, and that the small Bloch class is defined by $\mathcal{B}_{0}=\left\{F \in H: \lim _{\zeta \rightarrow \partial \mathbb{D}} h_{F}(\zeta)=0\right\}$. Since $P^{-1}(\mathcal{B}) \subset \mathcal{B}_{0}$ we have $P^{-1}(\mathcal{B}) \bigcap \mathcal{G}_{-}=P^{-1}(\mathcal{B}) \bigcap \mathcal{G}_{s}=\varnothing[2]$. Therefore $\mathcal{B} \bigcap P(\mathcal{G})=\mathcal{B} \bigcap P\left(\mathcal{G}_{+}\right)=\mathcal{B} \bigcap P\left(\mathcal{G}_{1}\right)$ which simplifies the study of the representation

$$
\begin{equation*}
\mathcal{A}=\mathcal{B} \bigcap P(\mathcal{G}) \tag{6}
\end{equation*}
$$

of the Gakhov class in the Bloch space. Restriction on the subspace $\tilde{\mathcal{B}}=\{F \in \mathcal{B}: F(0)=0\}$ where the Gakhov class is represented as $\tilde{\mathcal{A}}=\mathcal{A} \bigcap \tilde{\mathcal{B}}$ equalize pre-balls $B_{\varepsilon}(F)=\left\{G \in \mathcal{B}:|G-F|_{\mathcal{B}}<\varepsilon\right\}$ and balls $\mathbb{B}_{\varepsilon}(F)=\left\{G \in \mathcal{B}:\|G-F\|_{\mathcal{B}}<\varepsilon\right\}: \tilde{\mathbb{B}}_{\varepsilon}(F)=\mathbb{B}_{\varepsilon}(F) \bigcap \tilde{\mathcal{B}}=B_{\varepsilon}(F) \bigcap \tilde{\mathcal{B}}=\tilde{B}_{\varepsilon}(F)$. We will use also the sets $B=\left\{F \in H:|F|_{\mathcal{B}}<2\right\}, S=\left\{F \in H:|F|_{\mathcal{B}}=2\right\}, \bar{B}=B \bigcup S$, and $\mathbb{S}=\left\{F \in A:|F|_{\mathcal{B}}=1\right\}$.

Let's remark the following statement from [2]. For a function $F \in \mathbb{S}$ and a number $\xi \in(-1,1)$ we define the set

$$
\begin{equation*}
\mathcal{S}_{\xi}(F)=\left\{\frac{2 \bar{\varepsilon} \xi}{1-\xi^{2}}-2 \bar{\varepsilon}^{2} F \circ \Delta_{\xi \varepsilon}:|\varepsilon|=1\right\}, \tag{7}
\end{equation*}
$$

where $\Delta_{\omega}(\zeta)=(\zeta-\omega) /(1-\bar{\omega} \zeta),|\omega|<1$, and the set

$$
\begin{equation*}
\mathcal{S}=\bigcup_{F \in \mathcal{S}, \xi \in(-1,1)} \mathcal{S}_{\xi}(F) . \tag{8}
\end{equation*}
$$

Proposition 1. The following relations hold:

$$
\begin{gather*}
\bar{B} \subset \mathcal{A},  \tag{9}\\
\bar{B} \bigcap F r_{\mathcal{B}} \mathcal{A}=\mathcal{S} . \tag{10}
\end{gather*}
$$

Defining the (Gakhov) width of the class $\mathcal{A}$ as the least upper bound

$$
\begin{equation*}
\Gamma(\mathcal{A})=\sup \left\{r>0: \overline{B_{r}(0)} \subset \mathcal{A}\right\} \tag{11}
\end{equation*}
$$

we can rewrite Proposition 1 as
Proposition 2. Gakhov width of the class (6) is equal to $\Gamma(\mathcal{A})=2$.
In the present note we give an analogue of this assertion and some consequences for the classical Dirichlet space $\mathcal{D}$ with the Banach norm $\|F\|_{\mathcal{D}}=|F(0)|+|F|_{\mathcal{D}}$ where

$$
|F|_{\mathcal{D}}=\left\{\frac{1}{\pi} \iint_{\mathbb{D}}\left|F^{\prime}(\zeta)\right|^{2} d x d y\right\}^{1 / 2}<+\infty
$$

and $\zeta=x+i y$. We introduce the notations $D=\left\{F \in H:|F|_{\mathcal{D}}<2\right\}, T=\left\{F \in H:|F|_{\mathcal{D}}=2\right\}, \bar{D}=$ $D \bigcup T$, and collect the required facts on the classes $\mathcal{B}$ and $\mathcal{D}$ in the following

Lemma 1. ([4, 21, 22]) The inclusion $D \subset \mathcal{B}_{0}$ is valid with estimate

$$
\begin{equation*}
|F|_{\mathcal{B}} \leq|F|_{\mathcal{D}} . \tag{12}
\end{equation*}
$$

Equality in (12) is attained only on the functions of the form $F(\zeta)=a+b \Delta_{\omega}(\zeta)$ where $a \in \mathbb{C}$, $b=|F|_{\mathcal{D}}$, and $\omega \in \mathbb{D}$ is the (unique) maximum of the function $h_{F}$ (on $\overline{\mathbb{D}}$ ).

It remains to introduce the analogues of the sets (7) and (8):

$$
\mathcal{I}_{\xi}:=\mathcal{S}_{\xi}(i d)=\left\{\frac{2 \bar{\varepsilon} \xi}{1-\xi^{2}}-2 \bar{\varepsilon}^{2} \Delta_{\xi \varepsilon}:|\varepsilon|=1\right\}, \quad \mathcal{T}=\bigcup_{\xi \in(-1,1)} \mathcal{T}_{\xi} .
$$

We define the Gakhov width of the class $\mathcal{A} \bigcap \mathcal{D}$ by analogy with (11):

$$
\Gamma(\mathcal{A} \bigcap \mathcal{D})=\sup \left\{r>0: \overline{D_{r}(0)} \subset \mathcal{A} \bigcap \mathcal{D}\right\}
$$

where $\overline{D_{r}(0)}=\left\{F \in \mathcal{D}:|F|_{\mathcal{D}} \leq r\right\}$.

## 3. GAKHOV WIDTH OVER DIRICHLET SPACE

Now we prove the main result of this note:
Theorem 1. There holds the equality $\Gamma(\mathcal{A} \bigcap \mathcal{D})=2$. In more detail, the following relations take place: the inclusion

$$
\begin{equation*}
\bar{D} \subset \mathcal{A} \bigcap \mathcal{D}, \tag{13}
\end{equation*}
$$

and the equality

$$
\begin{equation*}
\bar{D} \bigcap F r_{\mathcal{B}} \mathcal{A}=\mathcal{T} . \tag{14}
\end{equation*}
$$

Proof. The inclusion (9) is completed to (13) by the inclusion

$$
\begin{equation*}
\bar{D} \subset \bar{B} \tag{15}
\end{equation*}
$$

being the consequence of the inequality (12). Let's prove (14).

Suppose $G \in \bar{D} \bigcap F r_{\mathcal{B}} \mathcal{A}$. The relations (15) and (10) lead to

$$
\begin{equation*}
G \in \bar{B} \tag{16}
\end{equation*}
$$

and to $G \in \mathcal{S}$. According to (7) and (8) the latter means the existence of the function $\tilde{G} \in \mathbb{S}$ and the number $\xi \in(-1,1)$ such that

$$
\begin{equation*}
G=\frac{2 \bar{\varepsilon} \xi}{1-\xi^{2}}-2 \bar{\varepsilon}^{2} \tilde{G} \circ \Delta_{\xi \varepsilon} . \tag{17}
\end{equation*}
$$

The direct calculations with $\tilde{G} \in \mathbb{S}$ give

$$
\begin{equation*}
h_{G}(\varepsilon \xi)=2, \tag{18}
\end{equation*}
$$

whence, by virtue of (16), we have

$$
\begin{equation*}
|G|_{\mathcal{B}}=2 . \tag{19}
\end{equation*}
$$

Then, the relations $G \in \bar{D}$, (12) for $F=G$, and (19) establish the equality $|G|_{\mathcal{B}}=|G|_{\mathcal{D}}$; due to the Lemma 1 it is possible only for

$$
\begin{equation*}
G=a+b \Delta_{\omega} \tag{20}
\end{equation*}
$$

where $a, b \in \mathbb{C}, \omega \in \mathbb{D}$. From (19) we have $|b|=|G|_{\mathcal{B}}=2$, i.e. $b=2 \eta$ for some $\eta,|\eta|=1$; furthermore, the estimate $h_{f}(\zeta)=2 h_{\Delta_{\omega}}(\zeta) \leq 2$ holds in $\mathbb{D}$ where the equality is attained only for $\zeta=\omega$. Due to (18) and $G \in \mathcal{B}_{0}$ it follows that $\omega=\varepsilon \xi$. In view of the latter we equate the representations (17) and (20) for the function $G$, and in the resulting identity we make the change of variables $\Lambda_{\omega}(\zeta)=w$ :

$$
\begin{equation*}
a+2 \eta w=\frac{2 \bar{\varepsilon} \xi}{1-\xi^{2}}-2 \bar{\varepsilon}^{2} \tilde{G}(w), \quad w \in \mathbb{D} \tag{21}
\end{equation*}
$$

In the following transform we use the belonging $\tilde{G} \in \mathbb{S}$. Substituting $w=0$ in (21) we have $a=2 \bar{\varepsilon} \xi /\left(1-\xi^{2}\right)$. Subtracting the latter from (21), differentiating the remainder and substituting $w=0$ once again we get $\eta=-2 \bar{\varepsilon}^{2}$. Thus, $\tilde{G}(w)=w$, and $G \in \mathcal{T}_{\xi}$.

So, the inclusion $\subset$ in (14) is established. Let's prove the opposite inclusion.
We start with $\mathcal{T} \subset \mathcal{S}$. By (10) it means that if $G \in \mathcal{T}$, then $G \in F r_{\mathcal{B}} \mathcal{A}$; moreover, the direct calculation shows that $|G|_{\mathcal{D}}=2$. As a result we obtain $G \in \bar{D} \bigcap F r_{\mathcal{B}} \mathcal{A}$ which implies the desired inclusion $\supset$ in (14).

Corollary 1. The following equality takes place:

$$
\begin{equation*}
\bar{D} \bigcap F r_{\mathcal{D}}(\mathcal{A} \bigcap \mathcal{D})=\mathcal{T} \tag{22}
\end{equation*}
$$

Proof. The inclusion $\subset$ in (22) follows from the inclusion $\operatorname{Fr}_{\mathcal{D}}(\mathcal{A} \bigcap \mathcal{D}) \subset \mathcal{D} \bigcap F r_{\mathcal{B}} \mathcal{A}$ which is established in [23], Sect. 2.1. Now we turn to the inclusion $\supset$.

Let $D \in \mathcal{T}$. As it was done at the end of the proof of Theorem 1 we get $|G|_{\mathcal{D}}=2$, so, $G \in \bar{D}$. Obviously, it remains to prove that

$$
\begin{equation*}
G \in F r_{\mathcal{D}}(\mathcal{A} \bigcap \mathcal{D}) \tag{23}
\end{equation*}
$$

The bijectivity of the mapping (5) implies the existence of a function $g \in H_{0}$ such that $g^{\prime \prime} / g^{\prime}=G$. The idea of the proof of the inclusion (23) is based on the study of the dynamics of the critical points of the hyperbolic derivatives $h_{g_{r}}$ for the level lines $g_{r}(\zeta)=g(r \zeta) / r$ satisfying the condition $g_{r}^{\prime \prime} / g_{r}^{\prime}=G_{r}$ where $G_{r}(\zeta)=r G(r \zeta)$, and on the estimate of the norm $\left\|G_{r}-G\right\|_{\mathcal{D}}$ when $r \rightarrow 1$.

The use of the representation for $G_{r}$ on the base of $G \in \mathcal{T}$ allows us to conclude that $\left|G_{r}\right|_{\mathcal{D}} \leq 2$ when $0 \leq r \leq 1$, and $2<\left|G_{r}\right|_{\mathcal{D}}<\infty$ when $1<r<1 /|\xi|$. It follows from here at once that for $0 \leq r \leq 1$ we have $G_{r} \in \mathcal{A} \bigcap \mathcal{D}$. In particular, for $r=1$ the Schwarzian derivative is equal to $S_{g}(\varepsilon \xi)=-2 \bar{\varepsilon}^{2} /(1-$ $\left.|\varepsilon \xi|^{2}\right)^{2}$, whence $\left|S_{g}(\varepsilon \xi)\right|=2 /\left(1-|\varepsilon \xi|^{2}\right)^{2}$, i.e. the critical point $\zeta=\varepsilon \xi$ is parabolic for the function $h_{g}$ (see, e.g., [11]).

Since $G=G_{1} \in \mathcal{A} \bigcap \mathcal{D}$, the point $\zeta=\varepsilon \xi$ is the unique critical point of $h_{g}$, and since $G \in \mathcal{B}_{0}$ by Lemma 1, then $\zeta=\varepsilon \xi$ is the (parabolic) maximum point of $h_{g}$. As is well known (see, for example, [18]), it means that, when $r$ increasing, the maximum $\zeta=\varepsilon \xi$ for $r=1$ bifurcates into two maxima and one saddle for $r>1$. Therefore, for $1<r<1 /|\xi|$ we have $G_{r} \in \mathcal{D} \backslash \mathcal{A}$.

The estimate $\left|\left|G_{r}-G \|_{\mathcal{D}} \leq M(\xi)\right| r-1\right|$ with the explicitly representable constant $M(\xi), \xi \in$ $(-1,1)$, and $0 \leq r \leq(1+1 /|\xi|) / 2$ is established by virtue of an estimate of sup-norm of the difference $G_{r}-G$ in $\mathbb{D}$, and implies that when $r \rightarrow 1$, the convergence $G_{r} \rightarrow G$ holds in the norm of $\mathcal{D}$. According to the proved above it follows that the function $G$ belongs to both $C l_{\mathcal{D}}(\mathcal{A} \bigcap \mathcal{D})$ and $C l_{\mathcal{D}}(\mathcal{D} \backslash \mathcal{A})$, i.e. we obtain (23).

Corollary 2. There holds the following equality:

$$
\bar{D} \bigcap F r_{\tilde{\mathcal{B}}} \tilde{\mathcal{A}}=\mathcal{T}_{0}
$$

Proof. We remind that the class $\mathcal{T}_{0}$ reduces to the family of the functions $F(\zeta)=2 \eta \zeta$ where $|\eta|=1$. It is evident that $\mathcal{T}_{0} \subset \bar{D}$. In order to verify the inclusion $\mathcal{T}_{0} \subset F r_{\tilde{\mathcal{B}}} \tilde{\mathcal{A}}$ it is enough to analyze the dynamics of the sets $M_{f_{r}}$ for the level lines $f_{r}$ defining by the equation $f_{r}^{\prime \prime} / f_{r}^{\prime}=2 \eta r^{2} \zeta$ with $r$ varies near 1 . An outcome of such an analysis is analogous to one in the proof of Corollary 1 for $M_{g_{r}}$. So, this way leads to the inclusion $\supset$. Let's prove the inclusion $\subset$.

First of all we note that $D \subset B \subset \operatorname{Int}_{\mathcal{B}} \mathcal{A}$ by Lemma 1 above and Theorem 2 from [2], and $\tilde{\mathcal{B}} \bigcap \operatorname{Int}_{\mathcal{B}} \mathcal{A}=\operatorname{Int}_{\tilde{\mathcal{B}}} \tilde{\mathcal{A}}$ by Theorem 6 from [2]. Therefore, $D \bigcap \operatorname{Fr}_{\tilde{\mathcal{B}}} \tilde{\mathcal{A}} \subset \operatorname{Int}_{\tilde{\mathcal{B}}} \tilde{\mathcal{A}} \bigcap F r_{\tilde{\mathcal{B}}} \tilde{\mathcal{A}}=\varnothing$, and, consequently, $\bar{D} \bigcap F r_{\tilde{\mathcal{B}}} \tilde{\mathcal{A}}=T \bigcap F r_{\tilde{\mathcal{B}}} \tilde{\mathcal{A}}$.

Now we fix an arbitrary function $G \in T \bigcap F r_{\tilde{\mathcal{B}}} \tilde{\mathcal{A}}$, and suppose $G \notin \mathcal{T}_{0}$. Then $|G|_{\mathcal{B}}<|G|_{\mathcal{D}}$ by Lemma 1. Let's consider the pre-ball $\tilde{B}_{\varepsilon}(G)=\left\{F \in \tilde{H}:|F-G|_{\mathcal{B}}<\varepsilon\right\}$ where $\varepsilon=\left(|G|_{\mathcal{D}}-|G|_{\mathcal{B}}\right) / 2(>$ $0)$. We have $|G|_{\mathcal{B}}=|G|_{\mathcal{D}}-2 \varepsilon=2-2 \varepsilon<2-\varepsilon$ by $G \in T$, and the condition $F \in \tilde{B}_{\varepsilon}(G)$ implies the following chain of inequalities:

$$
|F|_{\mathcal{B}} \leq|G|_{\mathcal{B}}+|F-G|_{\mathcal{B}} \leq|G|_{\mathcal{B}}+|F-G|_{\mathcal{D}}<2-\varepsilon+\varepsilon=2,
$$

whence $F \in \mathcal{A}$ due to Proposition 1. By the arbitrariness of $F \in \tilde{B}_{\varepsilon}(G)$ it follows from $\tilde{B}_{\varepsilon}(F)=\tilde{\mathbb{B}}_{\varepsilon}(F)$ that $G \in \tilde{\mathcal{B}} \bigcap \operatorname{Int}_{\mathcal{B}} \mathcal{A}=I n t_{\tilde{\mathcal{B}}} \tilde{\mathcal{A}}$, which is in contradiction with $G \in \operatorname{Fr}_{\tilde{\mathcal{B}}} \tilde{\mathcal{A}}$. Thus $G \in \mathcal{T}_{0}$.

Remark 1. Assertion of Corollary 2 may be given by considering the trace on $\tilde{\mathcal{B}}$ of the inequality (14), and by using Theorem 7 from [2].

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