ISSN 0037–4466, Siberian Mathematical Journal, 2026, Vol. 67, No. 1, pp. 10–17.

HYPONORMAL MEASURABLE OPERATORS AFFILIATED WITH A SEMIFINITE VON NEUMANN ALGEBRA. IV

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Abstract—Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$ of operators. For a normal operator A in $\mathcal{M}$, a condition on a τ-integrable operator B is found under which the operator $A+B$ is normal. For an operator whose square is τ-integrable, equivalent conditions for its normality are established in terms of trace inequalities. For an operator in $\mathcal{M}$, a criterion for hyponormality is found in terms of trace inequalities. It is shown that, given an arbitrary natural n, the power $(PQ)^n$ of the product of projections P and Q in $\mathcal{M}$ is hyponormal if and only if $PQ = QP$. Operator inequalities are obtained for powers of hyponormal contractions. It is shown that every natural power of a hyponormal partial isometry is a hyponormal partial isometry with the same initial space.

DOI: 10.1134/S0037446626010027

Keywords: Hilbert space, von Neumann algebra, normal trace, measurable operator, hyponormal operator

1. Introduction

Hyponormal bounded operators on a Hilbert space have been studied by many authors (see, for instance, [1–10] and the references therein). In the context of semifinite von Neumann algebras, the author published the papers [11–19] devoted to properties of (unbounded) τ-measurable hyponormal operators (see also [20]). Let $\mathcal{M}$ be a von Neumann algebra of operators acting in a Hilbert space $\mathcal{H}$, let $\mathcal{M}^{\mathrm{pr}}$ be the lattice of projections in $\mathcal{M}$, and let τ be a faithful normal semifinite trace on $\mathcal{M}$. We list the main results of the paper; some of them are new even in the case $\mathcal{M} = \mathcal{B}(\mathcal{H})$ with $\tau = \mathrm{tr}$.

- Let $A \in \mathcal{M}$ be normal and let $B \in L_1(\mathcal{M},\tau) \cap L_2(\mathcal{M},\tau)$. If the operator $A+B$ is hyponormal, then $A+B$ is normal (Theorem 1).
- For an operator $A \in L_2(\mathcal{M},\tau)$, the following conditions are equivalent (see Theorem 2):
  (i) A is normal;
  (ii) $\tau(PA^*AP) \geq \tau(PAA^*P)$ for all $P \in \mathcal{M}^{\mathrm{pr}}$;
  (iii) $\tau(PA^*AP) \leq \tau(PAA^*P)$ for all $P \in \mathcal{M}^{\mathrm{pr}}$.
- $A \in \mathcal{M}$ is hyponormal if and only if $\tau(PA^*AP) \geq \tau(PAA^*P)$ for all $P \in \mathcal{M}^{\mathrm{pr}}$ with $\tau(P) < +\infty$.
- If $A \in \mathcal{B}(\mathcal{H})_1$ is hyponormal, then $(A^{*n}A^n)^{1/4} \geq (A^*)^{n-2}A^{n-2}$ for all $n \in \mathbb{N}$ with $3 \leq n$ (Theorem 4).
- Given $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ and $n \in \mathbb{N}$, the operator $(PQ)^n$ is hyponormal if and only if $PQ = QP$ (Theorem 5).
- If a partial isometry $U \in \mathcal{B}(\mathcal{H})$ is hyponormal, then U^n is also a hyponormal partial isometry and $U^{*n}U^n = U^*U$ for each $n \in \mathbb{N}$ (Corollary 5).

Apparently, some of our assertions extend to locally measurable operators from [21].

2. Definitions and Notation

Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$, let $\mathcal{M}^{\mathrm{pr}}$ be the lattice of projections ($P = P^2 = P^*$) in $\mathcal{M}$, let I be the unit of $\mathcal{M}$, and let $P^{\perp} = I - P$ for $P \in \mathcal{M}^{\mathrm{pr}}$. Let $\mathcal{M}^+$ be the cone of positive elements in $\mathcal{M}$, let $\|\cdot\|$ be the C^*-norm on $\mathcal{M}$, and let $\mathcal{M}_1 = \{X \in \mathcal{M} : \|X\| \leq 1\}$ be the unit ball of $\mathcal{M}$.

Given $P, Q \in \mathcal{M}^{\mathrm{pr}}$, we write $P \sim Q$ (Murray–von Neumann equivalence) if $P = U^*U$ and $Q = UU^*$ for some $U \in \mathcal{M}$.

For $(P_n)_{n=1}^{\infty} \subset \mathcal{M}^{\mathrm{pr}}$, the infimum $\bigwedge_{n=1}^{\infty} P_n \in \mathcal{M}^{\mathrm{pr}}$ is defined by $\big(\bigwedge_{n=1}^{\infty} P_n\big)\mathcal{H} = \bigcap_{n=1}^{\infty} P_n\mathcal{H}$.

Original article submitted July 27, 2025; revised September 2, 2025; accepted September 26, 2025.

An operator $V \in \mathcal{M}$ is an *isometry* if $V^*V = I$. An operator $V \in \mathcal{M}$ is a *coisometry* if V^* is an isometry.

A mapping $\varphi : \mathcal{M}^+ \to [0, +\infty]$ is called a *trace* if $\varphi(X+Y) = \varphi(X) + \varphi(Y)$, $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{M}^+$ and $\lambda \geq 0$ (with $0 \cdot (+\infty) \equiv 0$), and $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{M}$. A trace φ is called

- *faithful* if $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$, $X \neq 0$;
- *normal* if $X_i \nearrow X$ $(X_i, X \in \mathcal{M}^+)$ implies $\varphi(X) = \sup \varphi(X_i)$;
- *semifinite* if $\varphi(X) = \sup\{\varphi(Y) : Y \in \mathcal{M}^+,\ Y \leq X,\ \varphi(Y) < +\infty\}$ for all $X \in \mathcal{M}^+$ (see [22, Chapter V, Section 2; 23, Chapter 1, Section 1.15]).

An operator on $\mathcal{H}$ (not necessarily bounded or densely defined) is *affiliated with the von Neumann algebra* $\mathcal{M}$ if it commutes with every unitary operator from the commutant $\mathcal{M}'$ of $\mathcal{M}$.

Further on, τ denotes a faithful normal semifinite trace on $\mathcal{M}$, and

$$\mathcal{M}^{\mathrm{pr}}_\tau = \{P \in \mathcal{M}^{\mathrm{pr}} : \tau(P) < +\infty\}.$$

A closed operator X affiliated with $\mathcal{M}$ whose domain $\mathcal{D}(X)$ is dense in $\mathcal{H}$ is called τ-*measurable* if for every $\varepsilon > 0$ there exists $P \in \mathcal{M}^{\mathrm{pr}}$ such that $P\mathcal{H} \subset \mathcal{D}(X)$ and $\tau(P^\perp) < \varepsilon$. The set $S(\mathcal{M}, \tau)$ of all τ-measurable operators is a $*$-algebra with respect to taking the adjoint, scalar multiplication, and the operations of strong addition and multiplication obtained by taking closures of the usual operations (see [24, Chapter IX; 23, Chapter 2, Section 2.3]). For a family $\mathcal{L} \subset S(\mathcal{M}, \tau)$, we denote by $\mathcal{L}^+$ and $\mathcal{L}^{\mathrm{h}}$ the positive and Hermitian parts of $\mathcal{L}$, respectively. The partial order on $S(\mathcal{M}, \tau)^{\mathrm{h}}$ induced by the proper cone $S(\mathcal{M}, \tau)^+$ is denoted by $\leq$. If $X \in S(\mathcal{M}, \tau)$ and $X = U|X|$ is the polar decomposition of X, then $U \in \mathcal{M}$ and $|X| = \sqrt{X^*X} \in S(\mathcal{M}, \tau)^+$.

An operator $A \in S(\mathcal{M}, \tau)$ is called *hyponormal* if $A^*A \geq AA^*$. It is called *cohyponormal* if the operator A^* is hyponormal.

By $\mu(t; X)$ we denote the *singular value function* of an operator $X \in S(\mathcal{M}, \tau)$, i.e., the nonincreasing right-continuous function $\mu(\cdot; X) : (0, +\infty) \to [0, +\infty)$ defined by

$$\mu(t; X) = \inf\{\|XP\| : P \in \mathcal{M}^{\mathrm{pr}},\ \tau(P^\perp) \leq t\}, \quad t > 0.$$

Lemma 1 [25]. *Let $X, Y \in S(\mathcal{M}, \tau)$ and let $A, B \in \mathcal{M}$. Then*

(i) $\mu(t; X) = \mu(t; |X|) = \mu(t; X^*)$ *for all* $t > 0$*;*
(ii) *if* $|X| \leq |Y|$*, then* $\mu(t; X) \leq \mu(t; Y)$ *for all* $t > 0$*;*
(iii) $\mu(t; AXB) \leq \|A\|\,\|B\|\,\mu(t; X)$ *for all* $t > 0$*;*
(iv) $\mu(t; f(|X|)) = f(\mu(t; X))$ *for all continuous functions* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $f(0) = 0$ *and all* $t > 0$.

The set

$$S_0(\mathcal{M}, \tau) = \big\{X \in S(\mathcal{M}, \tau) : \mu(\infty; X) := \lim_{t\to\infty} \mu(t; X) = 0\big\}$$

of τ-compact operators is an ideal in $S(\mathcal{M}, \tau)$.

Let m be the Lebesgue measure on $\mathbb{R}$. The noncommutative Lebesgue L_p-space $(1 \leq p < +\infty)$ associated with $(\mathcal{M}, \tau)$ can be defined by

$$L_p(\mathcal{M}, \tau) = \big\{X \in S(\mathcal{M}, \tau) : \mu(\,\cdot\,; X) \in L_p(\mathbb{R}^+, m)\big\}$$

with the norm $\|X\|_p = \|\mu(\,\cdot\,; X)\|_p$, $X \in L_p(\mathcal{M}, \tau)$. We denote by the same symbol τ the extension of the trace τ to the entire Banach space $L_1(\mathcal{M}, \tau)$.

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ is the $*$-algebra of all bounded linear operators on $\mathcal{H}$ and $\tau = \mathrm{tr}$ is the canonical trace, then $S(\mathcal{M}, \tau)$ coincides with $\mathcal{B}(\mathcal{H})$, and $S_0(\mathcal{M}, \tau)$ coincides with the ideal of compact operators $\mathcal{S}(\mathcal{H})$. The following holds:

$$\mu(t; X) = \sum_{n=1}^{\infty} s_n(X)\chi_{[n-1,n)}(t), \quad t > 0,$$

where $\{s_n(X)\}_{n=1}^\infty$ is the sequence of s-numbers of a compact operator X, and χ_A denotes the indicator function of a set $A \subset \mathbb{R}$ (see [26, Chapter II]). Then the space $L_p(\mathcal{M}, \tau)$ is the Schatten–von Neumann ideal $\mathcal{S}_p(\mathcal{H})$, $1 \leq p < +\infty$.

3. Main Results

Lemma 2 [24, Chapter IX, Theorem 2.13]. *If $X \in \mathcal{M}$ and $Y \in L_1(\mathcal{M}, \tau)$, then $XY, YX \in L_1(\mathcal{M}, \tau)$.*

Lemma 3 [27, Theorem 17]. *If $X, Y \in S(\mathcal{M}, \tau)^+$ and $XY, YX \in L_1(\mathcal{M}, \tau)$, then $\tau(XY) = \tau(YX)$.*

Theorem 1. *Let an operator $A \in \mathcal{M}$ be normal and let $B \in L_1(\mathcal{M}, \tau) \cap L_2(\mathcal{M}, \tau)$. If the operator $T := A + B$ is hyponormal, then T is normal.*

PROOF. Since $T^*T \geq TT^*$ and $A^*A = AA^*$, we have

$$A^*B + B^*A + B^*B \geq AB^* + BA^* + BB^*.$$

Therefore,

$$D := A^*B + B^*A + B^*B - AB^* - BA^* - BB^* \geq 0.$$

The terms A^*B, B^*A, AB^*, and BA^* belong to $L_1(\mathcal{M}, \tau)$ by Lemma 2, and $B^*B, BB^* \in L_1(\mathcal{M}, \tau)$ by the definition of the space $L_2(\mathcal{M}, \tau)$. Hence $D \in L_1(\mathcal{M}, \tau)^+$. By Lemma 3 we have

$$\tau(A^*B) = \tau(BA^*), \quad \tau(B^*A) = \tau(AB^*), \quad \tau(B^*B) = \tau(BB^*),$$

and therefore, by linearity of the extension of τ to $L_1(\mathcal{M}, \tau)$, we get $\tau(D) = 0$. Since this extension is faithful on the cone $L_1(\mathcal{M}, \tau)^+$, we conclude that $D = 0$. Thus $T^*T = TT^*$, and the operator T is normal. $\square$

Passing to adjoint operators, we obtain the following.

Corollary 1. *Let an operator $A \in \mathcal{M}$ be normal and let $B \in L_1(\mathcal{M}, \tau) \cap L_2(\mathcal{M}, \tau)$. If the operator $T := A + B$ is cohyponormal, then T is normal.*

Corollary 2 [28]. *Let an operator $A \in \mathcal{B}(\mathcal{H})$ be normal and let $B \in \mathcal{S}_2(\mathcal{H})$. If the operator $T := A + B$ is hyponormal* (*or cohyponormal*), *then T is normal.*

PROOF. The assertion follows from the inclusion $\mathcal{S}_2(\mathcal{H}) \subset \mathcal{S}_1(\mathcal{H})$.

Theorem 2. *For an operator $A \in L_2(\mathcal{M}, \tau)$, the following conditions are equivalent:*
(i) *A is normal;*
(ii) *$\tau(PA^*AP) \geq \tau(PAA^*P)$ for all $P \in \mathcal{M}^{\mathrm{pr}}$;*
(iii) *$\tau(PA^*AP) \leq \tau(PAA^*P)$ for all $P \in \mathcal{M}^{\mathrm{pr}}$.*

PROOF. (i)⇒(ii): Every normal operator $X \in S(\mathcal{M}, \tau)$ is hyponormal. Hence, from the inequality $A^*A \geq AA^*$ we get

$$PA^*AP \geq PAA^*P \quad \text{for all } P \in \mathcal{M}^{\mathrm{pr}}.$$

Therefore, by monotonicity on the cone $L_1(\mathcal{M}, \tau)^+$ of the extension of τ to $L_1(\mathcal{M}, \tau)$, we obtain (ii).

(ii)⇒(i): Assume that (ii) holds, but A is not normal. Since $L_2(\mathcal{M}, \tau) \subset S_0(\mathcal{M}, \tau)$, the operator A is not hyponormal by [11, Theorem 2.2]. Therefore, in the Jordan decomposition

$$X := A^*A - AA^* = X_+ - X_-, \tag{1}$$

with $X_+, X_- \in L_1(\mathcal{M}, \tau)^+$ and $X_+X_- = 0$, we have $X_- \neq 0$. Multiplying both sides of equality (1) on the left and on the right by the projection $P = \mathrm{supp}(X_-)$, we get

$$PA^*AP - PAA^*P = -X_-. \tag{2}$$

Consequently, $\tau(PA^*AP) - \tau(PAA^*P) = -\tau(X_-) < 0$ since the extension of τ to $L_1(\mathcal{M}, \tau)$ is faithful on the cone $L_1(\mathcal{M}, \tau)^+$. This contradicts (ii).

Since an operator $X \in S(\mathcal{M}, \tau)$ is normal if and only if X^* is normal, we obtain (i)⇔(iii). $\square$

Since the trace τ on the algebra $\mathcal{M}$ is semifinite, there exists a nonzero subprojection Q of the projection P such that $Q \in \mathcal{M}_\tau^{\text{pr}}$ and $QX_-Q \neq 0$ (see (2)). Therefore, each of the conditions

- $\tau(PA^*AP) \geq \tau(PAA^*P)$ for all $P \in \mathcal{M}_\tau^{\text{pr}}$;
- $\tau(PA^*AP) \leq \tau(PAA^*P)$ for all $P \in \mathcal{M}_\tau^{\text{pr}}$

implies that the operator $A \in L_2(\mathcal{M}, \tau)$ is normal.

An operator $T \in S(\mathcal{M}, \tau)$ is hyponormal (cohyponormal) if and only if $\mu(t; TP) \geq \mu(t; T^*P)$ (respectively, $\mu(t; T^*P) \geq \mu(t; TP)$) for all $t > 0$ and all $P \in \mathcal{M}_\tau^{\text{pr}}$ (see [17, Theorem 6]). Hence, by Lemma 1(i), (iv), we conclude the following:

An operator $T \in S(\mathcal{M}, \tau)$ is hyponormal (cohyponormal) if and only if $\mu(t; PT^*TP) \geq \mu(t; PTT^*P)$ (respectively, $\mu(t; PTT^*P) \geq \mu(t; PT^*TP)$) for all $t > 0$ and all $P \in \mathcal{M}_\tau^{\text{pr}}$.

Arguing in the same way as in the proof of Theorem 2, we verify the following statement:

For an operator $A \in \mathcal{M}$, the following conditions are equivalent:
(i) *A is hyponormal;*
(ii) *$\tau(PA^*AP) \geq \tau(PAA^*P)$ for all $P \in \mathcal{M}_\tau^{\text{pr}}$.*

Theorem 3. *Let an operator $A \in \mathcal{M}$ be an isometry, let $B \in S_0(\mathcal{M}, \tau)$, and let $T := A + B$. If $TT^* \geq I$ and T is hyponormal, then T is normal.*

Proof. By assumption, $P := AA^* \in \mathcal{M}^{\text{pr}}$. Since $T^*T \geq TT^* \geq I$, we have

$$A^*B + B^*A + B^*B \geq AB^* - BA^* - BB^* - P^\perp \geq 0. \tag{3}$$

In particular, $P^\perp \in S_0(\mathcal{M}, \tau)$. If $X, Y \in S(\mathcal{M}, \tau)^+$, $Y \neq 0$, and $X \geq \mu(\infty; X)I$, then there exists $s > 0$ such that $\mu(s; X) < \mu(s; X + Y)$ (see [29, Proposition 2.2]). The operator

$$X := AB^* - BA^* - BB^* - P^\perp$$

belongs to $S_0(\mathcal{M}, \tau)^+$ and satisfies $X \geq \mu(\infty; X)I = 0 \cdot I = 0$. Put

$$X + Y := A^*B + B^*A + B^*B$$

(see the left-hand side of inequality (3)). Assume that T is not normal, i.e., $Y \neq 0$. For an arbitrary $Z \in S(\mathcal{M}, \tau)^+$, we have

$$\mu(t; I + Z) = 1 + \mu(t; Z) \quad \text{for all } t > 0$$

(see the proof of [30, Theorem 7]). By Lemma 1(i), (iv), we obtain

$$\mu(t; T^*T) = \mu(t; |T|^2) = \mu(t; |T|)^2 = \mu(t; T)^2 = \mu(t; T^*)^2 = \mu(t; |T^*|)^2 = \mu(t; |T^*|^2) = \mu(t; TT^*) \tag{4}$$

for all $t > 0$. Now, for $s > 0$ we have

$$\mu(s; T^*T) = \mu(s; I + X + Y) = 1 + \mu(s; X + Y) > 1 + \mu(s; X) = \mu(s; I + X) = \mu(s; TT^*).$$

This contradicts (4). Therefore, $Y = 0$, and the operator T is normal. $\square$

Passing to adjoint operators, we obtain the following.

Corollary 3. *Let an operator $A \in \mathcal{M}$ be a coisometry, let $B \in S_0(\mathcal{M}, \tau)$, and let $T := A + B$. If $TT^* \geq I$ and T is cohyponormal, then T is normal.*

Lemma 4. *The function $f(t) = t^p$ is operator monotone on the half-line $[0, +\infty)$ for $0 < p \leq 1$.*

Lemma 5 [31]. *If $A \in \mathcal{B}(\mathcal{H})^+$ and $B \in \mathcal{B}(\mathcal{H})_1$, then $Bf(A)B^* \leq f(BAB^*)$ for every function f that is operator monotone on $[0, +\infty)$ and satisfies $f(0) \leq 0$.*

Theorem 4. *Let an operator $A \in \mathcal{B}(\mathcal{H})_1$ be hyponormal. Then $(A^{*n}A^n)^{1/4} \geq (A^*)^{n-2}A^{n-2}$ for all $3 \leq n \in \mathbb{N}$.*

Proof. Multiplying both sides of the inequality $A^*A \geq AA^*$ on the left by A^* and on the right by A, we obtain

$$A^{*2}A^2 \geq (A^*A)^2. \tag{5}$$

Then $(A^{*2}A^2)^{1/2} \geq A^*A$ by Lemma 4. Multiplying both sides of the latter inequality on the left by A^* and on the right by A, by Lemmas 4, 5, and inequality (5) we get

$$(A^{*3}A^3)^{1/2} \geq A^*(A^{*2}A^2)^{1/2}A \geq A^{*2}A^2 \geq (A^*A)^2,$$

and hence $(A^{*3}A^3)^{1/4} \geq A^*A$. Multiplying both sides of the latter inequality on the left by A^* and on the right by A, by Lemmas 4 and 5 we obtain $(A^{*4}A^4)^{1/4} \geq A^*(A^{*3}A^3)^{1/4}A \geq A^{*2}A^2$; hence, in the same way we derive

$$(A^{*5}A^5)^{1/4} \geq A^*(A^{*4}A^4)^{1/4}A \geq A^{*3}A^3.$$

Continuing this process, we get $(A^{*n}A^n)^{1/4} \geq (A^*)^{n-2}A^{n-2}$, i.e., $|A^n|^{1/2} \geq |A^{n-2}|^2$ for all $3 \leq n \in \mathbb{N}$. ☐

Corollary 4. *Let an operator $A \in \mathcal{M}_1$ be hyponormal and let $A^n \in S_0(\mathcal{M},\tau)$ for some $2 \leq n \in \mathbb{N}$. Then A belongs to $S_0(\mathcal{M},\tau)$ and is normal.*

Proof. Without loss of generality, we may assume that n is odd (if n is even, consider the operator $A^{n+1} = A \cdot A^n \in S_0(\mathcal{M},\tau)$). For every $t > 0$, by Lemma 1(i), (ii), (iv), we have

$$\mu(t; A^n)^{1/2} = \mu(t; A^{*n}A^n)^{1/4} = \mu\big(t; (A^{*n}A^n)^{1/4}\big) \geq \mu\big(t; (A^*)^{n-2}A^{n-2}\big) = \mu\big(t; |A^{n-2}|^2\big) = \mu(t; A^{n-2})^2.$$

On the other hand, for every $t > 0$, by Lemma 1(iii),

$$\mu(t; A^n) = \mu\big(t; A \cdot A^{n-2} \cdot A\big) \leq \|A\|^2\, \mu(t; A^{n-2}) \leq \mu(t; A^{n-2}),$$

and hence $\mu(t; A^{n-2}) \geq \mu(t; A^n) \geq \mu(t; A^{n-2})^4$, $t > 0$. Thus $A^{n-2} \in S_0(\mathcal{M},\tau)$. Continuing the process of reducing the exponent of A, we obtain $A \in S_0(\mathcal{M},\tau)$. By [11, Theorem 2.2], the operator A is normal. In particular, $\mu(t; A^k) = \mu(t; A)^k$ for all $t > 0$ and all $k \in \mathbb{N}$ by [11, Theorem 3.1(ii)]. ☐

Recall that in [16, Theorem 3.3(i)] it was established that if an operator $A \in S(\mathcal{M},\tau)$ is paranormal and $A^n \in S_0(\mathcal{M},\tau)$ for some $n \in \mathbb{N}$, then $A \in S_0(\mathcal{M},\tau)$.

Theorem 5. *Let $P, Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$, let $n \in \mathbb{N}$, and let $A := (PQ)^n$. Then the following are equivalent:*
(i) *A is hyponormal;*
(ii) *$PQ = QP$ (hence $A = P \wedge Q \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$).*

Proof. (i)⇒(ii): We have $(QP)^n(PQ)^n \geq (PQ)^n(QP)^n$, that is,

$$(QPQ)^{2n-1} \geq (PQP)^{2n-1}.$$

Applying Lemma 4 with $p = (2n-1)^{-1} \in (0,1]$, we obtain $QPQ \geq PQP$. Multiplying both sides of this inequality on the left and on the right by the projection P, we get $PQPQP = (PQP)^2 \geq PQP$. Since $PQP \in \mathcal{B}(\mathcal{H})^+ \cap \mathcal{B}(\mathcal{H})_1$, we obtain $PQP \geq (PQP)^2$. Consequently,

$$PQP = (PQP)^2 \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}.$$

By von Neumann's theorem [32, Problem 122] (a new proof is given in [33, Theorem 2.2]), the sequence $(PQP)^k$ is nonincreasing and converges in the strong operator topology of $\mathcal{B}(\mathcal{H})$ to the projection $P \wedge Q$. Thus $PQP = P \wedge Q$, and the operator $V := QP$ is a partial isometry. Then $V^* = (QP)^* = PQ$ is also a partial isometry [32, Corollary 2 to Problem 127], and $VV^* = QPQ \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$. Applying von Neumann's theorem once again, we obtain $QPQ = P \wedge Q$. Hence $PQP = QPQ = P \wedge Q$. Since $\|PQ - P \wedge Q\| = \sqrt{\|PQP - P \wedge Q\|} = 0$ (see [33, p. 6]), we get $PQ = P \wedge Q$ and $A = (PQ)^n = P \wedge Q$. ☐

Lemma 6. *Let $A \in \mathcal{B}(\mathcal{H})^+$, let $P \in \mathcal{B}(\mathcal{H})^{\text{pr}}$, and let $I \geq A \geq P$. Then*
(i) *$AP = PA$;*
(ii) *if $AP = A$, then $A = P$.*

Proof. (i): Since $P^\perp \geq I - A \geq 0$; according to [34, Chapter 2, 2.17] the operators $P^\perp \in \mathcal{B}(\mathcal{H})^{\text{pr}}$ and $I - A$ commute. Hence $AP = PA$.

(ii): We have $B := A - P \geq 0$ and $A = P + B$. Multiplying both sides of the latter equality on the left by the projection $P^\perp$, we get $BP^\perp = AP^\perp = 0$. Multiplying both sides of the inequality $I \geq P + B$ on the left and on the right by the projection P, we obtain $P \geq P + PBP$. Since $PBP \geq 0$, we have $PBP = |B^{1/2}P|^2 = 0$. Consequently, $B^{1/2}P = 0$ and $BP = B^{1/2} \cdot B^{1/2}P = 0$. Thus $B = BP + BP^\perp = 0 + 0 = 0$, and hence $A = P$. $\square$

Theorem 6. *Let $P, Q \in \mathcal{M}^{\text{pr}}$. Then*
(i) *$\mu(t; PQ^\perp)^2 = \mu(t; P - PQP) \leq \mu(t; P - Q)$ for all $t > 0$;*
(ii) *if $P \sim Q$ and $V \in \mathcal{M}$ with $V^*V = P$ and $VV^* = Q$, then*

$$\mu(t; P - V) = \mu(t; Q - V) \leq \mu(t; I - V) \quad \text{for all } t > 0;$$

(iii) *if a partial isometry $U \in \mathcal{B}(\mathcal{H})$ is hyponormal, then*

$$\cdots \geq U^{*n}U^n \geq \cdots \geq U^{*2}U^2 \geq U^*U \geq UU^* \geq U^2U^{*2} \geq \cdots \geq U^nU^{*n} \geq \cdots .$$

Proof. (i): For every $t > 0$, by Lemma 1(i), (iii), (iv), we have

$$\begin{aligned}\mu(t; PQ^\perp)^2 &= \mu(t; Q^\perp P)^2 = \mu(t; PQ^\perp P) = \mu(t; P - PQP) = \mu(t; P(P - Q)P) \\ &\leq \|P\|^2 \mu(t; P - Q) = \mu(t; P - Q).\end{aligned}$$

(ii): If $P \sim Q$ with $V \in \mathcal{M}$, then V is a partial isometry and $V = VV^*V$ (see [32, Corollary 3 to Problem 127]). Therefore, by Lemma 1(i), (iii), for every $t > 0$ we have

$$\begin{aligned}\mu(t; P - V) &= \mu(t; V^*V - VV^*V) = \mu(t; (V^* - VV^*)V) \\ &\leq \|V\|\, \mu(t; V^* - VV^*) = \mu(t; V^* - Q) = \mu(t; Q - V).\end{aligned}$$

In the same way, we obtain $\mu(t; Q - V) \leq \mu(t; P - V)$ for every $t > 0$. By Lemma 1(i), (iii), for every $t > 0$ we have

$$\mu(t; Q - V) = \mu(t; VV^* - V) = \mu(t; V(V^* - I)) \leq \|V\|\, \mu(t; V^* - I) = \mu(t; I - V).$$

(iii): Multiplying both sides of the inequality $U^*U \geq UU^*$ on the left by U and on the right by U^*, and taking into account the equality $U = UU^*U$, we obtain $UU^* \geq U^2U^{*2}$. Multiplying all parts of the chain $U^*U \geq UU^* \geq U^2U^{*2}$ on the left by U and on the right by U^*, and again using $U = UU^*U$, we get $UU^* \geq U^2U^{*2} \geq U^3U^{*3}$. Continuing this process, we obtain

$$U^*U \geq UU^* \geq U^2U^{*2} \geq \cdots \geq U^nU^{*n} \geq U^{n+1}U^{*(n+1)} \geq \cdots .$$

Multiplying both sides of the inequality $U^*U \geq UU^*$ on the left by U^* and on the right by U, and taking into account the equality $U = UU^*U$, we obtain $U^{*2}U^2 \geq U^*U$. Multiplying all parts of the chain $U^{*2}U^2 \geq U^*U \geq UU^*$ on the left by U^* and on the right by U, and again using $U = UU^*U$, we get $U^{*3}U^3 \geq U^{*2}U^2 \geq U^*U$. Continuing this process, we obtain

$$\cdots \geq U^{*(n+1)}U^{n+1} \geq U^{*n}U^n \geq \cdots \geq U^{*2}U^2 \geq U^*U \geq UU^*. \quad \square$$

Corollary 5. *If a partial isometry $U \in \mathcal{B}(\mathcal{H})$ is hyponormal, then U^n is also a hyponormal partial isometry and $U^{*n}U^n = U^*U$ for every $n \in \mathbb{N}$.*

Proof. Since $I \geq U^{*n}U^n \geq U^*U \geq UU^* \geq U^nU^{*n} \geq 0$, the operator U^n is hyponormal for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and put $A := U^{*n}U^n$, $P := U^*U$. Then $I \geq A \geq P$ and

$$AP = U^{*n}U^n \cdot U^*U = U^{*n}U^{n-1} \cdot UU^*U = U^{*n}U^n = A$$

by the equality $UU^*U = U$. Now, by Lemma 6(ii), we obtain $U^{*n}U^n = U^*U = P$ for every $n \in \mathbb{N}$. Consequently, U^n is also a partial isometry; hence, U^{*n} is a partial isometry for every $n \in \mathbb{N}$.

The sequence of projections $(Q_n)_{n=1}^{\infty} = (U^nU^{*n})_{n=1}^{\infty}$ is decreasing and, according to Vigier's theorem (see [35, Theorem 4.1.1] or [36, Chapter 1, Theorem 4.5]), it converges in the strong operator topology of $\mathcal{B}(\mathcal{H})$ to some nonnegative operator in $\mathcal{B}(\mathcal{H})_1$. Since the lattice $\mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ is monotone complete, the limit operator is the projection $\bigwedge_{n=1}^{\infty} Q_n$. $\square$

FUNDING

The research was carried out in the framework of the Development Program of the Scientific Educational Mathematical Center of the Volga Federal District (Agreement 075–02–2025–1725/1).

CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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