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ORDINARY DIFFERENTIAL EQUATIONS AND THEIR SYSTEMS

Учебно-методическое пособие на английском языке
с использованием СКМ Maple

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«КАЗАНСКИЙ (ПРИВОЛЖСКИЙ) ФЕДЕРАЛЬНЫЙ УНИВЕРСИТЕТ»

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В данном учебном пособии рассматриваются обыкновенные дифференциальные уравнения и их системы. В каждой главе пособия содержатся необходимые теоретические сведения (основные теоремы, определения, формулы, вычислительные схемы и т.д.), подробно разобранные примеры и задания для самостоятельного решения. Приведены примеры решения задач, приводящих к дифференциальным уравнениям. Пособие содержит рекомендации по решению обыкновенных дифференциальных уравнений и систем в СКМ Maple, англо-русский и русско-английский словарь.

Предназначено для студентов, обучающихся по направлению подготовки Педагогическое образование (математика, информатика, ИТ) на билингвальной основе, и школьников физико-математического класса, обучающихся в полилингвальных школах.

Оглавление

Introduction

A differential equation is an equation that relates independent variables, an unknown function, and derivatives (or differentials) of this function. The highest order of the derivative included in the differential equation is called the order of the differential equation. Differential equations are divided into two groups depending on the number of independent variables included in them. If the variable is one, the equation is called ordinary, if the variables are two or more, the equation is called partial differential equation.

This tutorial covers ordinary differential equations and their systems. The tutorial includes the following chapters:

1. Ordinary differential equations of the first order;
2. Higher order ordinary differential equations;
3. Systems of differential equations;
4. Problems leading to differential equations;
5. Solution of ordinary differential equations and systems in SCM Maple;

Each chapter of the tutorial contains the necessary theoretical information (basic theorems, definitions, formulas, methods of solution, etc.), examples and tasks for independent work. The tutorial contains examples of solving problems leading to differential equations, recommendations for solving ordinary differential equations and systems in SCM Maple, English-Russian and Russian-English dictionary.

Глава 1

Ordinary differential equations of the first order

Basic definitions and terms

A differential equation of first order is the relation of the independent variable x, the unknown function $y = y(x)$, and its first derivative y', i.e.

$$F\left(x, y, y'\right) = 0. \tag{1.1}$$

Also, the differential equation of the first order (1.1) can be written as

$$y' = f(x, y), \tag{1.2}$$

if it can be solved with respect to the derivative.

A differential equation of the first order can be written using differentials x and y , i.e.

$$P(x, y)dx + Q(x, y)dy = 0. \tag{1.3}$$

A solution of a differential equation is a function $y = \varphi(x)$, substitution of which into the equation, converts it to the identity with respect to the variable x.

For example, the function $y = 3x^2$ is a solution of the differential equation of the first order $y' = 6x$. Indeed, after substituting the function $y = 3x^2$ into the differential equation, we obtain the identity: $6x = 6x$.

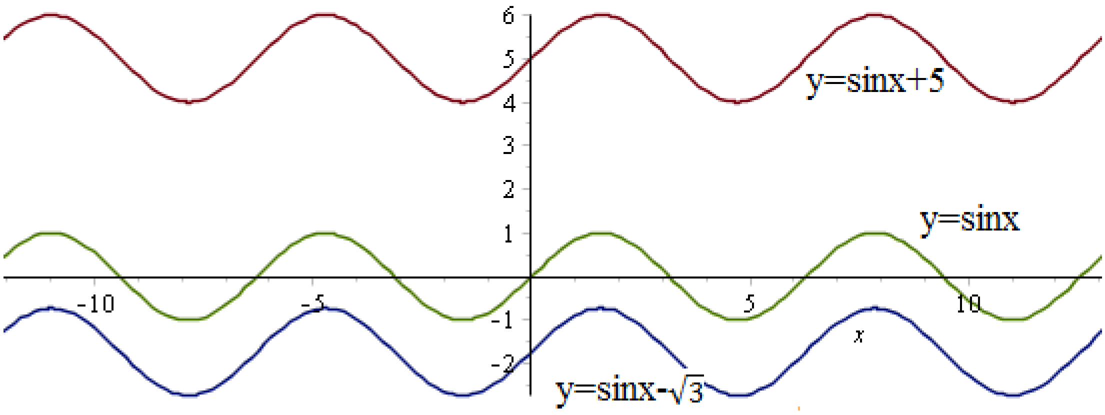
A solution of a differential equation given in implicit form $\Phi(x, y) = 0$ is called **an integral** of the differential equation.

The graph of a solution is called **an integral curve**. The process of finding solutions is called **integration** of the differential equation.

In fact, in the integration process we find the whole class of solutions

$$y = \varphi(x, C).$$

For example, it is easy to guess, that the function $y = \sin x$ is a solution of the differential equation of the first order $y' = \cos x$, as long as the functions $y = \sin x + 5$, $y = \sin x - \sqrt{3}$, and all functions $y = \sin x + C$, where C is an arbitrary constant.



The general solution of a differential equation of the first order is a function $y = \varphi(x, C)$, where C is a constant, such that:

1) it satisfies the differential equation for any values of the constant C;

2) whatever the initial condition $y(x_0) = y_0$, we can find such value $C = C_0$, that the function $y = \varphi(x, C_0)$ satisfies this initial condition.

A particular solution of the differential equation (1.2) is a solution obtained from the general solution at a certain value of C.

The relation in the form

$$\Phi(x, y, C) = 0 \text{ or } \psi(x, y) = C,$$

implicitly determining the general solution, is called **the general integral** of the differential equation of the first order.

The relation obtained from the general integral at a certain value of C is called **the particular integral** of the differential equation.

The Cauchy problem is the problem of finding the solution $y = y(x)$ of the equation (1.2), satisfying **the initial condition**

$$y(x_0) = y_0. \tag{1.4}$$

Geometrically this means that we need to find an integral curve passing through the point $M_0(x_0, y_0)$ on the plane xOy.

Theorem (of existence and uniqueness of solution of the Cauchy problem)

If a function $f(x,y)$ is defined in some closed rectangle $\overline{D}=\{|x-x_0|\leq a, |y-y_0|\leq b\}$ and satisfies two conditions:

1) it is continuous and, therefore, bounded, i.e. there is number $M>0$, such that $|f(x,y)|\leq M$;

2) it satisfies a Lipschitz condition in the variable y, i.e. there exists a constant $N>0$ such that for all points $(x,\overline{y})$ and $(x,\overline{\overline{y}})$ from $\overline{D}$ we have the inequality, $|f(x,\overline{y})-f(x,\overline{\overline{y}})|\leq N|\overline{y}-\overline{\overline{y}}|$, then the equation (1.2) has a unique solution $y=y(x)$, satisfying the initial condition (1.4), defined and continuously differentiable on segment the $[x_0-h, x_0+h]$ where $h=\min\{a, b/M\}$.

Tasks for independent work

Check that the given functions are solutions of the corresponding differential equations.

1. $y'\cdot x-2y=0, y=x^2$;
2. $y'-y=2e^x, y=(x-1)\cdot e^x$;
3. $y'=3x^2y, y=e^{x^3}$;
4. $xy'+y=\cos x, y=\frac{\sin x}{x}$;
5. $\left(1-x^2\right)y'+xy=2x, y=2+C\sqrt{1-x^2}$;
6. $y'+2y=e^x, y=Ce^{-2x}+\frac{1}{3}e^x$;
7. $yy'=x, y=\sqrt{x^2+C}$;
8. $y''-(k_1+k_2)y'+k_1k_2y=0, y=C_1e^{k_1x}+C_2e^{k_2x}$ (k_1, k_2 – constants);
9. $y''+2py'+p^2y=0, \quad y=e^{-px}(C_1+C_2x)$ (p – constant);
10. $y''+2py'+\left(p^2+q^2\right)y=0, y=e^{-px}\left(C_1\cos qx+C_2\sin qx\right)$ (p, q – const.).

Separated variables equations and separable equations

An equation of the form

$$P(x)dx+Q(y)dy=0$$

is called **a separated variables equation**.

The general integral of this equation is $\int P(x)dx+\int Q(y)dy=C$.

An equation of the form

$$P(x,y)dx+Q(x,y)dy=0$$

is called **a separable equation**, if both the functions $P(x,y)$ and $Q(x,y)$ admit representation as a product of two factors, each of which only depends on one variable:

$$P_1(x)\cdot P_2(y)dx+Q_1(x)\cdot Q_2(y)dy=0.$$

Dividing by the product $Q_1(x)\cdot P_2(y)$ we convert it to the following separated variables equation

$$\frac{P_1(x)}{Q_1(x)}dx+\frac{Q_2(y)}{P_2(y)}dy=0.$$

The general integral of this equation is $\int\frac{P_1(x)}{Q_1(x)}dx+\int\frac{Q_2(y)}{P_2(y)}dy=C.$

Remark. Note that because of division by expressions which contain variables, there may be a loss of solutions that turn these expressions to zero. Therefore, we should consider the existence of such solutions of the differential equation.

Example. Solve the Cauchy problem $xydx+\left(1+y^2\right)\sqrt{1+x^2}dy=0$, $y(\sqrt{8})=1$.

Solution. Divide both parts of the equation by $y\cdot\sqrt{1+x^2}\neq 0$ to separate the variables:

$$\frac{x}{\sqrt{1+x^2}}dx+\frac{1+y^2}{y}dy=0.$$

By integrating this equation, we obtain

$$\int\frac{x}{\sqrt{1+x^2}}dx+\int\left(\frac{1}{y}+y\right)dy=C,$$

or

$$\frac{1}{2}\int\frac{dx\left(1+x^2\right)}{\sqrt{1+x^2}}+\int\frac{dy}{y}+\int ydy=C.$$

From here we obtain the general integral of this differential equation:

$$\sqrt{1+x^2}+\ln|y|+\frac{y^2}{2}-C=0.$$

By separating the variables, we assumed that $y\cdot\sqrt{1+x^2}\neq 0$, which could lead to the loss of the solution $y=0$. After substituting it into the original equation, make sure that $y=0$ is a (singular) solution. However, note that it cannot be obtained from the general solution for any particular value of the arbitrary constant C.

From the condition $y(\sqrt{8})=1$ we obtain

$$\sqrt{1+8}+\ln 1+\tfrac{1}{2}-C=0,\ \text{i.e. } C=\tfrac{7}{2}.$$

The desired particular solution is determined implicitly (the particular integral)

$$\sqrt{1+x^2}+\ln|y|+\frac{y^2}{2}-\frac{7}{2}=0.$$

Answer: $\sqrt{1+x^2}+\ln|y|+\frac{y^2}{2}-\frac{7}{2}=0,\ y=0.$

A differential equation of the form

$$y'=\varphi(x)\cdot\psi(y)$$

is also called **a separable equation**.

Let us represent the derivative y' as the ratio of differentials $y'=\frac{dy}{dx}$.

By separating variables, we obtain

$$\frac{dy}{\psi(y)}=\varphi(x)dx.$$

The general integral is $\int\frac{dy}{\psi(y)}=\int\varphi(x)dx+C$.

Remark. Division by $\psi(y)$ can also lead to the loss of particular solutions when $\psi(y)=0$.

Example. Find the general solution of the equation $y'=\frac{1-y}{x}$.

Solution. We represent the derivative as the ratio of differentials: $\frac{dy}{dx}=\frac{1-y}{x}$. By multiplying both the parts of equality by dx and dividing by $1-y\neq0$, we obtain the separated variables equation:

$$\frac{dy}{1-y}=\frac{dx}{x}.$$

By integrating we obtain

$$\int\frac{dy}{1-y}=\int\frac{dx}{x}+C \text{ where } -\ln|1-y|=\ln|x|+C.$$

Take an arbitrary constant C as $\ln|c|: -\ln|1-y|=\ln|x|+\ln|c|$.

$$\ln|1-y|=-\ln|cx|;\ \ln|1-y|=\ln|cx|^{-1};$$

$y=1-\frac{c}{x}$ – the general solution of equarion.

After division by $1-y$ the solution $y=1$ could be lost. By substituting it into the original equation, we make sure that $y=1$ is a solution. However, note that it can be obtained from the general solution when $C=0$.

Answer: $y=1-\frac{C}{x}$.

A differential equation of the form $y'=f(ax+by+c)$, where a, b and c are constants, is converted to a separable equation by replacing $z=ax+by+c$. By solving the equation with respect to z, we find the unknown function y from the equality $z=ax+by+c$.

Example. Solve the equation $y'=(2x+3y+1)^2$.

Solution. Make the replacement $z=z(x)=2x+3y+1$, then $y=\frac{1}{3}(-2x+z-1)$. Therefore $y'=-\frac{2}{3}+\frac{1}{3}z'$. Substitute it into the original equation:

$$-\frac{2}{3}+\frac{1}{3}z'=z^2, \text{ where } \frac{dz}{dx}=3\left(z^2+2\right).$$

By separating variables, we obtain

$$\frac{dz}{z^2+2}=3dx.$$

By integrating the last equation and making the reverse substitution, we obtain

$$\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{z}{\sqrt{2}}=3x+C, \text{ where } \frac{1}{\sqrt{2}}\operatorname{arctg}\frac{2x+3y+1}{\sqrt{2}}=3x+C.$$

There is no loss of solutions because the expression z^2+2 at any values is not equal to zero.

Answer: $\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{2x+3y+1}{\sqrt{2}}=3x+C$.

Tasks for independent work

Solve the differential equations:

1. $\left(1+y^2\right)dx+\left(1+x^2\right)dy=0$;
2. $x\sqrt{1+y^2}+yy'\sqrt{1+x^2}=0$;
3. $e^{-y}\left(1+y'\right)=1$;
4. $2x\sqrt{1-y^2}=y'\left(1+x^2\right)$;
5. $\left(1+y^2\right)dx+xydy=0$;
6. $\left(1+y^2\right)dx=xdy$;

7. $e^y\left(1+x^2\right)dy-2x\left(1+e^y\right)dx=0$;
8. $e^x\sin^3 y+\left(1+e^{2x}\right)\cos y\cdot y'=0$;
9. $y'=\sin(x-y)$;
10. $y^2\sin x dx+\cos^2 x\ln y dy=0$.

Homogeneous differential equations

A function $f(x,y)$ is called a **homogeneous function of order** k, if the identity $f(tx,ty)\equiv t^k f(x,y)$ is true.

For example, the function $f(x,y)=x^2+xy$ is a second-order homogeneous function because $f(tx,ty)=(tx)^2+(tx)\cdot(ty)=t^2\left(x^2+xy\right)=t^2\cdot f(x,y))$.

A differential equation of the form

$$P(x,y)dx+Q(x,y)dy=0$$

is called **homogeneous** if both the functions $P(x,y)$ and $Q(x,y)$ are homogeneous functions of the same order.

A differential equation of the form

$$y'=f(x,y)$$

is homogeneous if $f(x,y)$ is a homogeneous function of zero order.

A homogeneous equation is transformed into a separable equation by substituting

$$y=ux \text{ or } u=\frac{y}{x}.$$

Here $y'=u'x+u, dy=udx+xdu$.

Example 1. Solve the equation $xdy=(x+y)dx$.

Solution. This equation is homogeneous. Let $y=ux$. Then $dy=udx+xdu$. By substituting this expression into the equation, we obtain

$$x(udx+xdu)=(x+ux)dx \text{ or } xdu=dx.$$

Solve the obtained separable equation $du=\frac{dx}{x}$, $u=\ln|x|+C$.

Returning to the variable y, we obtain the general solution of the equation

$$y=x(\ln|x|+C).$$

In addition, there is a solution $x=0$, which was lost by dividing by x.

Answer: $y = x(\ln|x| + C)$.

Example 2. Solve the Cauchy problem $y' = \frac{2xy}{x^2+y^2}$, $y(0) = -1$.

Solution. The function $f(x, y) = \frac{2xy}{x^2+y^2}$ is a homogeneous function of zero order, so this differential equation is homogeneous. Make the replacement $y = ux$, then $y' = u'x + u$. The original equation is written as

$$u'x + u = \frac{2u}{1+u^2}, \text{ where } x\frac{du}{dx} = \frac{u-u^3}{1+u^2}.$$

By separating the variables, we obtain

$$\frac{\left(1+u^2\right)du}{u\left(1-u^2\right)} = \frac{dx}{x}.$$

By converting the fraction on the left-hand side of the last equation, we have

$$\left(\frac{1}{u} + \frac{2u}{1-u^2}\right)du = \frac{dx}{x}.$$

Then $\int\left(\frac{1}{u} + \frac{2u}{1-u^2}\right)du = \int\frac{dx}{x}$ or $\ln|u| - \ln\left|1-u^2\right| = \ln|x| + C$. By taking the constant C as $\ln|C|$, we obtain

$$\frac{u}{1-u^2} = Cx.$$

By substituting $u = \frac{y}{x}$, we finally obtain

$$\frac{xy}{x^2-y^2} = Cx \text{ or } Cy = x^2 - y^2.$$

In the process of solving the equation we divide by x, u, $1-u^2$. It is easy to see that $x = 0$ is not a solution of the original equation, while $u = 0$ and $u = \pm 1$ are solutions of equation $u'x + u = \frac{2u}{1+u^2}$. Therefore, the original equation has other solutions, $y = 0$ and $y = \pm x$. Note that the solutions $y = \pm x$ are included in the family of solutions $Cy = x^2 - y^2$ (they are obtained by putting $C = 0$), and the solution $y = 0$ is not included in this set (but it is obtained from the first form of the general solution) by putting $C = 0$.

By substituting $x = 0$, $y = -1$, we obtain the solution of the Cauchy problem $y = x^2 - y^2$.

Answer: $y = x^2 - y^2$.

An equation of the form $y' = f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right)$, provided that $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$, is converted to homogeneous by the replacement $x = \xi + x_0$, $y = \eta + y_0$, where ξ, η are new variables, and (x_0, y_0) is the solution of the system $\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{cases}$.

If $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$, then $a_1x + b_1y = k(a_2x + b_2y)$, therefore, the equation has the form $y' = F(a_1x + b_1y)$ and it is converted to a separable equation by the replacement $z = a_1x + b_1y$ (or $z = a_1x + b_1y + c_1$).

Example. Solve the equation $(x + y - 2)dx + (x - y + 4)dy = 0$.

Solution. Consider the system of linear algebraic equations $\begin{cases} x + y - 2 = 0 \\ x - y + 4 = 0 \end{cases}$.

There is a unique solution of the system: $x_0 = -1$, $y_0 = 3$. Make a replacement $x = \xi - 1$, $y = \eta + 3$, then, $dx = d\xi$, $dy = d\eta$. Then the equation is converted to

$$(\xi + \eta)d\xi + (\xi - \eta)d\eta = 0.$$

This equation is homogeneous. Let $\eta = u\xi$, then we obtain

$$(\xi + \xi u)d\xi + (\xi - \xi u)(\xi du + ud\xi) = 0.$$

It follows $\left(1 + 2u - u^2\right) d\xi + \xi(1 - u)du = 0$. Separate the variables

$$\frac{d\xi}{\xi} + \frac{1 - u}{1 + 2u - u^2} du = 0.$$

By integrating, we obtain

$$\ln|\xi| + \tfrac{1}{2}\ln\left|1 + 2u - u^2\right| = \ln C \text{ or } \xi^2\left(1 + 2u - u^2\right) = C.$$

Returning to variables x and y, we obtain

$$(x + 1)^2\left(1 + 2\tfrac{y-3}{x+1} - \tfrac{(y-3)^2}{(x+1)^2}\right) = C \text{ or } x^2 + 2yx - y^2 - 4x + 8y = C.$$

Answer: $x^2 + 2yx - y^2 - 4x + 8y = C$.

Some equations can be converted to homogeneous by the replacemen $y = z^\alpha$. The number α is usually unknown. To find it, it is necessary to make a substitution $y = z^\alpha$ in the equation . By requiring the equation to be homogeneous, we obtain the number α, if it is possible. If it is impossible, the equation is not converted to homogeneous with this method.

Example. Solve the equation $\left(x^2y^2-1\right)dy+2xy^3dx=0$.

Solution. Let us make a substitution $y=z^\alpha$, $dy=\alpha z^{\alpha-1}dz$ where α is an undefined number which we will choose later. By substituting the expressions for y and dy into the equation, we obtain $\left(x^2z^{2\alpha}-1\right)\alpha z^{\alpha-1}dz+2xz^{3\alpha}dx=0$.

The obtained equation is homogeneous if the function $x^2z^{2\alpha}-1$ is homogeneous, i.e. if $2+2\alpha=0$ or $\alpha=-1$. By replacing α by -1, we obtain

$$-\left(x^2z^{-2}-1\right)z^{-2}dz+2xz^{-3}dx=0 \text{ or } \left(z^2-x^2\right)dz+2xzdx=0.$$

This equation is homogeneous. Let $z=ux$, then $dz=udx+xdu$. Next we write this equation in the form

$$\left(u^2-1\right)(udx+xdu)+2udx=0, \text{ where } u\left(u^2+1\right)dx+x\left(u^2-1\right)du=0.$$

By separating the variables, we obtain

$$\frac{dx}{x}+\frac{u^2-1}{u\left(u^2+1\right)}du=0.$$

By integrating, we obtain

$$\ln|x|+\ln\left|u^2+1\right|-\ln|u|=\ln C, \text{ or } \frac{x\left(u^2+1\right)}{u}=C.$$

By substituting u by $\frac{1}{xy}$ (because $z=\frac{1}{y}$, $u=\frac{z}{x}$)), obtain the general integral of the equation

$$1+x^2y^2=Cy.$$

Answer: $1+x^2y^2=Cy$.

Tasks for independent work

Solve the following differential equations:

1. $xy'=y(\ln y-\ln x)$;
2. $(4x-3y)dx+(2y-3x)dy=0$;
3. $x^2dy=\left(y^2-xy+x^2\right)dx$;
4. $2x^2y'=x^2+y^2$;
5. $(3y-7x+7)dx-(3x-7y-3)dy=0$;
6. $2x+3y-5+(3x+2y-5)y'=0$;

7. $(x+y)dx+(x-y-2)dy=0$;
8. $2xy'\left(x-y^2\right)+y^3=0$;
9. $y\left(1+\sqrt{x^2y^4+1}\right)dx+2xdy=0$;
10. $4y^6+x^3=6xy^5y'$.

Linear differential equation of the first order. The Bernoulli equations

A linear differential equation of the first order is an equation, which is an equation, linear relative to the unknown function $y(x)$ and its derivative y', i.e. an equation of the form

$$y'+p(x)\cdot y=q(x), \tag{1.5}$$

where $p(x)$ и $q(x)$ are continuous functions in the variable x.

If $q(x)\equiv 0$, the equation is called **a linear homogeneous equation.**

Remark. Some equations become linear if we change the roles of the desired function and the independent variable.

Example. Consider the equation $y=\left(2x+y^3\right)y'$, where y is a function of x. It is not linear with to y. Taking into account that $y'=\frac{1}{x'}$, we obtain the equation

$$y=\left(2x+y^3\right)\cdot\frac{1}{x'} \text{ or } x'-\frac{2x}{y}=y^2,$$

which is linear respect to the variable x, i.e. here $x=x(y)$ is the desired function and y is an independent variable.

A **linear homogeneous equation** of the form

$$y'+p(x)\cdot y=0 \tag{1.6}$$

is a **separable equation**.

Indeed, by separating the variables, we obtain $\frac{dy}{y}+p(x)dx=0$ (where $y\neq 0$).

Next we find

$$\ln|y|+\int p(x)dx=\ln C,\quad C>0$$

(for convenience the constant C is represented as $\ln C$);

$$|y|\cdot e^{\int p(x)dx}=C;\ y=C\cdot e^{-\int p(x)dx},\ C\neq 0.$$

In the process of solving the solution $y=0$ was lost. It can be obtained when $C=0$. Therefore, the general solution of the linear homogeneous equation (1.6) has the form

$$y=C\cdot e^{-\int p(x)dx},\ \forall C.$$

The general solution of a linear inhomogeneous equation can be found by the method of variation of an arbitrary constant (Lagrange method) and by the method of substitution (Bernoulli method).

1. The method of variation of an arbitrary constant (Lagrange method)

The solution of the equation (1.5) is sought in the same form as the solution of the corresponding homogeneous equation (1.6), but C is considered not as a constant, but as an unknown function of x. Thus, the equation (1.5) has the following solution:

$$y=C(x)\cdot e^{-\int p(x)dx}. \tag{1.7}$$

Then $y'=C'(x)e^{-\int p(x)dx}+C(x)\left(-p(x)e^{-\int p(x)dx}\right)$.

By substituting y and y' into the equation (1.5), we obtain

$$C'(x)=q(x)e^{\int p(x)dx}.$$

Therefore, $C(x)=\int q(x)e^{\int p(x)dx}dx+C$, $C-const$. By substituting it into the (1.7), we obtain the general solution of the nonhomogeneous linear differential equation

$$y=e^{-\int p(x)dx}\left(\int q(x)e^{\int p(x)dx}dx+C\right). \tag{1.8}$$

Remark. The formula (1.8) is difficult to remember, so by solving specific equations we need to perform calculations according to this scheme independently.

Example. Find the solution of the equation $y'+2xy=2xe^{-x^2}$.

Solution. This equation is linear. We use the method of variation of an arbitrary constant. First, we solve the corresponding homogeneous equation

$$y'+2xy=0,$$

which is a separable equation. Its general solution is

$$y = Ce^{-x^2}.$$

The general solution of the nonhomogeneous equation is found in the form

$$y = C(x)e^{-x^2},$$

where $C(x)$ is an unknown function of x. By substituting y into the original equation, we obtain

$$C'(x)e^{-x^2} - 2xC(x)e^{-x^2} + 2xC(x)e^{-x^2} = 2xe^{-x^2} \text{ or } C'(x) = 2x.$$

Therefore $C(x) = x^2 + C$. Thus, we obtain a general solution of the original equation:

$$y = \left(x^2 + C\right) e^{-x^2}.$$

Answer: $y = \left(x^2 + C\right) e^{-x^2}$.

2. The method of substitution (Bernoulli method)

The solution of equation (1.5) is sought in the form $y = u(x) \cdot v(x)$ where $u(x)$ and $v(x)$ are unknown functions. Then $y' = u' \cdot v + u \cdot v'$. By substituting it into the equation (1.5), we obtain

$$u' \cdot v + u \cdot v' + p(x) \cdot u \cdot v = q(x),$$

or, after transformations,

$$u' \cdot v + u\left(v' + p(x) \cdot v\right) = q(x).$$

We choose the function $v(x)$ such that $u\left(v' + p(x) \cdot v\right) = 0$.

But u cannot be equal to zero, because in this case both y and y' will be equal to zero, and this cannot be true for nonzero $q(x)$. Therefore,

$$v' + p(x) \cdot v = 0;$$

$$\frac{dv}{v} = -p(x)dx; \int \frac{dv}{v} = -\int p(x)dx;$$

$$\ln|v| = -\int p(x)dx; v = e^{-\int p(x)dx}.$$

The function $v(x)$ can be chosen arbitrary, the constant of integration is chosen to be equal to zero. The remaining parts of the equation are also separable equations:

$$u' \cdot v = q(x); u' \cdot e^{-\int p(x)dx} = q(x);$$

$$u' = q(x) \cdot e^{\int p(x)dx}; u = \int q(x) \cdot e^{\int p(x)dx} dx + C.$$

The general solution is

$$y = \left(\int q(x) \cdot e^{\int p(x)dx} dx + C \right) \cdot e^{-\int p(x)dx}.$$

Example. Solve the Cauchy problem

$$y' = \frac{1}{x\cos y + \sin 2y}, \quad y(-2) = 0.$$

Solution. This equation is linear if we consider x as a function of the variable y:

$$x' - x\cos y = \sin 2y. \tag{1.9}$$

The solution of the equation (1.9) is sought with the method of substitution, i.e. in the form $y = uv$. Then $x' = u'v + uv'$. By substituting the expressions for x and x' into the equation (1.9), we obtain

$$u'v + uv' - uv\cos y = \sin 2y,$$

or

$$u'v + u\left(v' - v\cos y\right) = \sin 2y. \tag{1.10}$$

The function $v = v(y)$ is sought as a particular solution of equation

$$v' - v\cos y = 0,$$

for example,

$$v = e^{\sin y}.$$

Then from the equation (1.10) we have

$$u'e^{\sin y} = \sin 2y, \quad \text{or} \quad u' = \sin 2y e^{-\sin y}.$$

By integrating, we obtain

$$u = \int \sin 2y e^{-\sin y} dy = 2\left(-\sin y e^{-\sin y} - e^{-\sin y}\right) + C.$$

Therefore, the general solution of the equation (1.9), as long as of the original equation is

$$x = uv = \left(2\left(-\sin y e^{-\sin y} - e^{-\sin y}\right) + C\right) e^{\sin y} = Ce^{\sin y} - 2(1 + \sin y).$$

According to the the initial condition, we obtain the equation to find C:

$$-2 = Ce^{\sin 0} - 2(1 + \sin 0)$$

where $C = 0$; therefore, the solution of the Cauchy problem is the function

$$x = -2(1 + \sin y).$$

Answer: $x = -2(1 + \sin y)$.

Bernoulli equation is an equation of the form

$$y' + p(x) \cdot y = q(x) \cdot y^n, \ n \neq 0, \ n \neq 1. \tag{1.11}$$

Divide both the parts of the equation by y^n. We obtain the equation

$$\frac{y'}{y^n} + \frac{p(x)}{y^{n-1}} = q(x).$$

Since $\left(\frac{1}{y^{n-1}}\right)' = (1-n)\frac{y'}{y^n}$, the substitution $z = y^{1-n}$ converts this equation to a linear relative to z:

$$\frac{z'}{1-n} + p(x)z = q(x). \tag{1.12}$$

For $n > 0$ the function $y = 0$ is a solution of the equation (1.11), and for $n < 0$ it is not.

Remark. Bernoulli equation can also be integrated with the method of variation of the constant, as well as the linear equation, and by substitution $y = u(x) \cdot v(x)$.

Example. Solve the equation

$$y' + \frac{y}{x} = y^2 \frac{\ln x}{x}.$$

Solution. It is Bernoulli equation. Multiply both the sides of the equation by y^{-2}:

$$y^{-2}y' + \frac{y^{-1}}{x} = \frac{\ln x}{x}.$$

We make a substitution $z = y^{1-2} = y^{-1}$, then $z' = -y^2 y'$. After substitution the last transform into the linear equation we have the equation

$$z' - \frac{z}{x} = -\frac{\ln x}{x},$$

the general solution of which is

$$z = 1 + \ln x + Cx.$$

Thus, we obtain the general solution of the original equation:

$$y = \frac{1}{1 + Cx + \ln x}.$$

Answer: $y = \frac{1}{1+Cx+\ln x}$.

Tasks for independent work

Solve the linear differential equations and, if necessary, the Cauchy problem:

1. $\left(2x - y^2\right) y' = 2y$;
2. $y' - ye^x = 2xe^x$;
3. $y' + 2xy = e^{-x^2}$;
4. $xy' - 2y = x^3 \cos x$;
5. $y' x \ln x - y = 3x^3 (\ln x)^2$;
6. $y' + y \cos x = \cos x,\ \ y(0) = 1$;
7. $\left(e^{-\frac{y^2}{2}} - xy\right) dy - dx = 0$;
8. $y' - y \operatorname{tg} x = \frac{1}{\cos^3 x},\ \ y(0) = 0$;
9. $y' \cos x - y \sin x = 2x,\ \ y(0) = 0$;
10. $y' + xe^x y = e^{(1-x)e^x}$.

Solve the Bernoulli equations

11. $\left(x^3 + e^y\right) y' = 3x^2$;
12. $2y' \sin x + y \cos x = y^3 \sin^2 x$;

13. $y' + 2xy = y^2 e^{x^2}$;
14. $2y' \ln x + \frac{y}{x} = y^{-1} \cos x$;
15. $\left(x^2 + y^2 + 1\right) dy + xydx = 0$.

Exact Differential Equations. Integrating factor

An equation of the form

$$P(x, y)dx + Q(x, y)dy = 0 \tag{1.13}$$

is called an **exact differential equation**, if there is a function $u(x, y)$, the total differential of which is equal to the left-hand side of the equation:

$$du(x, y) = P(x, y)dx + Q(x, y)dy.$$

Theorem (Cauchy-Riemann condition)

If in a simply-connected domain D the functions $P(x, y)$, $Q(x, y)$ and their partial derivatives are continuous, then equation (1.13) is an exact differential equation when

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

The relation $u(x, y) = C$ is the general integral of the exact differential equation, and it can obtained from the system:

$$\begin{cases} \frac{\partial u}{\partial x} = P(x, y); \\ \frac{\partial u}{\partial y} = Q(x, y). \end{cases}$$

Algorithm for solving an exact differential equation

1. Check the validity of the condition $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

2. Write down the system $\begin{cases} \frac{\partial u}{\partial x} = P(x, y); \\ \frac{\partial u}{\partial y} = Q(x, y). \end{cases}$

3. Integrate the first equation of the system over the variable x. Write an unknown function$\varphi(y)$ instead of the constant C:

$$u(x, y) = \int P(x, y)dx + \varphi(y).$$

4. Differentiate the function $u(x,y)$ with respect to the variable y and the result substitute into the second equation of the system:

$$\frac{\partial}{\partial y}\left(\int P(x,y)dx+\varphi(y)\right)=Q(x,y).$$

We obtain an expression for $\varphi(y)$:

$$\varphi'(y)=Q(x,y)-\frac{\partial}{\partial y}\left(\int P(x,y)dx\right).$$

5. By integrating the last expression, obtain $\varphi(y)$ and substitute it into the equality

$$u(x,y)=\int P(x,y)dx+\varphi(y).$$

6. Write down the general solution

$$u(x,y)=C.$$

Remark. In step 3, you can integrate the second equation of the system, instead of integrating the first one. After you need to find an unknown function $\psi(x)$.

Example. Solve the equation $\left(2xy+3y^2\right)dx+\left(x^2+6xy-3y^2\right)dy=0.$

Solution. In this case,

$$P(x,y)=2xy+3y^2,\quad Q(x,y)=x^2+6xy-3y^2,$$

$$\frac{\partial P}{\partial y}=2x+6y,\quad \frac{\partial Q}{\partial x}=2x+6y.$$

Thus, $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$, i.e. the equation is an exact differential equation. By definition, the left-hand side of the equation is the total differential of some function $u(x,y)$. For the function $u(x,y)$ we have

$$\begin{cases}\frac{\partial u}{\partial x}=2xy+3y^2;\\ \frac{\partial u}{\partial y}=x^2+6xy-3y^2.\end{cases}$$

From the first equation we obtain $u(x,y)=x^2y+3xy^2+\varphi(y)$.

To determine the function $\varphi(y)$, differentiate the last equation over the y and substitute the result into the second equation of the system:

$$\frac{\partial u}{\partial y} = x^2 + 6xy + \varphi'(y) = x^2 + 6xy - 3y^2, \text{ where } \varphi'(y) = -3y^2.$$

From here we obtain $\varphi(y) = -y^3 + C$. Therefore,

$$u(x, y) = x^2 y + 3xy^2 - y^3.$$

The general integral of equation is

$$x^2 y + 3xy^2 - y^3 = C.$$

Answer: $x^2 y + 3xy^2 - y^3 = C$.

Method of integrable combinations

In some cases, it is possible to solve or simplify the equation by selecting a group of members, which is a total differential or an expression easily leading to a total differential, by multiplying or dividing by some function. You can use the relation

$$ydx + xdy = d(xy), \quad ydy = \frac{1}{2}d\left(y^2\right), \quad xdx + ydy = \frac{1}{2}d\left(x^2 + y^2\right),$$

$$ydx - xdy = y^2 d\left(\frac{x}{y}\right) = -x^2 d\left(\frac{y}{x}\right), \quad \frac{dx}{x} = d(\ln x).$$

Example. Solve the equation $\left(xy + y^4\right)dx + \left(x^2 - xy^3\right)dy = 0$.

Solution. Group members to obtain total differentials:

$$x(ydx + xdy) + y^3(ydx - xdy) = 0,$$

$$xd(xy) + y^5 d\left(\frac{x}{y}\right) = 0.$$

Dividing by x and substituting $xy = u$, $\frac{x}{y} = v$, we obtain the equation

$$du + \frac{u^2}{v^3}dv = 0,$$

which is solved easily.

Let the equation (1.13) is not an exact differential equation. **An integrating factor** for equation (1.13) is a function $\mu(x, y)$, after multiplying by which this equation is converted to an exact differential equation.

Consider two cases when equation (1.13) has an integrating factor that depends either only on x or only on y.

1. $\mu = \mu(x)$. Then

$$\frac{d\ln\mu}{dx} = \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q},$$

and such an integrating factor exists if the right-hand side of the equation either depends only on x or is a constant.

2. $\mu = \mu(y)$. Then

$$\frac{d\ln\mu}{dy} = \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{-P},$$

and the right-hand side of the equation either depends only on y or is a constant.

Example. Solve the equation $\left(1 - x^2 y\right) dx + x^2(y - x)dy = 0$.

Solution. In this equation,

$$P(x, y) = \left(1 - x^2 y\right), \quad Q(x, y) = x^2(y - x)$$

and

$$\frac{\partial P}{\partial y} = -x^2, \quad \frac{\partial Q}{\partial x} = 2xy - 3x^2.$$

$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, therefore, this equation is not an exact differential equation. Let us check if it has an integrating factor depending only on x :

$$\frac{d\ln\mu}{dx} = \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \frac{2x(x - y)}{x^2(y - x)} = -\frac{2}{x}.$$

There is a function of x on the right-hand side of the equation, thus, such a factor exists and we have:

$$\frac{d\ln\mu}{dx} = -\frac{2}{x} \text{ or } d\ln\mu = -\frac{2dx}{x},$$

$$\text{there fore, } \mu = \frac{1}{x^2}.$$

By multiplying the original equation by this function, we obtain

$$\left(\frac{1}{x^2} - y\right) dx + (y - x)dy = 0, \quad \frac{dx}{x^2} + ydy - (ydx + xdy) = 0.$$

Therefore,

$$d\left(-\frac{1}{x}\right) + \frac{1}{2}d\left(y^2\right) - d(xy) = 0,$$

and we obtain the general solution of the equation:

$$-\frac{1}{x}+\frac{1}{2}y^2-xy=C.$$

We also need to check whether or not the function $\mu(x)$ is equal to zero, and if it exists for all x. The check shows that $x=0$ is also a solution of the original equation.

Answer: $-\frac{1}{x}+\frac{1}{2}y^2-xy=C,\ \ x=0.$

Tasks for independent work

Integrate the exact equations:

1. $\left(3x^2+6xy^2\right)dx+\left(6x^2y+4y^3\right)dy=0;$
2. $\left(\frac{x}{\sqrt{x^2+y^2}}+\frac{1}{x}+\frac{1}{y}\right)dx+\left(\frac{y}{\sqrt{x^2+y^2}}+\frac{1}{y}-\frac{x}{y^2}\right)dy=0;$
3. $x\left(2x^2+y^2\right)+y\left(x^2+2y^2\right)y'=0;$
4. $\left(2x+\frac{x^2+y^2}{x^2y}\right)dx=\frac{x^2+y^2}{xy^2}dy;$
5. $\left(\frac{\sin 2x}{y}+x\right)dx+\left(y-\frac{\sin^2 x}{y^2}\right)dy=0;$
6. $\frac{xy}{\sqrt{1+x^2}}+2xy-\frac{y}{x})dx+\left(\sqrt{1+x^2}+x^2-\ln x\right)dy=0;$
7. $\frac{xdx+ydy}{\sqrt{x^2+y^2}}+\frac{xdy-ydx}{x^2}=0;$
8. $\frac{2xdx}{y^3}+\frac{(y^2-3x^2)dy}{y^4}=0,\quad y(1)=1;$
9. $\left(3x^2-2x-y\right)dx+\left(2y-x+3y^2\right)dy=0;$
10. $\left(3x^2y+y^3\right)dx+\left(x^3+3xy^2\right)dy=0.$

Solve equations finding integrating factor

11. $\left(1-x^2y\right)dx+x^2(y-x)dy=0;$
12. $\left(2x^2y+2y+5\right)dx+\left(2x^3+2x\right)dy=0;$
13. $\left(x^4\ln x-2xy^3\right)dx+3x^2y^2dy=0;$
14. $(x+\sin x+\sin y)dx+\cos y\,dy=0;$
15. $\left(2xy^2-3y^3\right)dx+\left(7-3xy^2\right)dy=0.$

The Lagrange and Clairaut equations

The equation of the form

$$y = \varphi\left(y'\right)x + \psi\left(y'\right),$$

where $\varphi\left(y'\right) \neq y'$, y is a linear function of x with coefficients depending on y', is called **Lagrange equation**.

Denoting $y' = p$, differentiating with respect to x and replacing dy by pdx, we convert this equation to a linear equation with respect to x as a function p. Finding solution of last equation $x = f(p, C)$, we obtain the general solution of the original equation in parametric form:

$$\begin{cases} x = f(p, C); \\ y = \varphi(p) f(p, C) + \psi(p), \end{cases}$$

p is a parameter.

Remark. Lagrange equation may also have singular solutions of the form $y = \varphi\left(p_i\right)x + \psi\left(p_i\right)$ where p_i is a root of the equation $\varphi(p)x - p = 0$.

Example. Integrate the equation $y = 2xy' - 4\left(y'\right)^3$.

Solution. This equation is a Lagrange equation. Denote $y' = p$. Then the equation is written in the form $y = 2xp - 4p^3$. Differentiating, we obtain:

$$y' = 2p + 2x\frac{dp}{dx} - 12p^2\frac{dp}{dx},$$

$$p = 2p + \left(2x - 12p^2\right)\frac{dp}{dx},$$

$$p + \left(2x - 12p^2\right)\frac{dp}{dx} = 0,$$

$$\frac{dx}{dp} + \frac{2x - 12p^2}{p} = 0 \ (p \neq 0),$$

$$\frac{dx}{dp} + \frac{2}{p}x = 12p.$$

We obtain a linear nonhomogeneous equation with respect to x and $\frac{dx}{dp}$. We find its solution with the Bernoulli method. Let $x = uv$. Then $x' = u'v + uv'$.

Substitute x and x' into the equation and obtain the following system for the functions u and v:

$$\begin{cases} v' + \frac{2}{p} = 0; \\ u'v = 12p. \end{cases}$$

Solving this system, we obtain

$$v = \frac{1}{p^2},\ u = 3p^4 + C$$

and the general solution of the linear nonhomogeneous equation is

$$x = \frac{1}{p^2}\left(3p^4 + C\right),\ \text{or}\ x = 3p^2 + \frac{C}{p^2}.$$

Substitute expressions for x into the equation $y = 2xp - 4p^3$, we obtain the solution of original equation in parametric form

$$\begin{cases} x = 3p^2 + \frac{C}{p^2}; \\ y = 2p^3 + \frac{2C}{p}. \end{cases}$$

This solution is obtained in the assumption that $p \neq 0$. When $p = 0$ we obtain

$$y = 2x \cdot 0 - 4 \cdot 0^3,\ \text{i.e.}\ y = 0.$$

The check shows that this is also a solution of the equation which is not included in the general one.

Answer: $y = 0,\ \begin{cases} x = 3p^2 + \frac{C}{p^2}; \\ y = 2p^3 + \frac{2C}{p}. \end{cases}$

An equation of the form $y = xy' + \psi\left(y'\right)$ is called **Clairaut equation**.

The Clairaut equation is a particular case of the Lagrange equation when $\varphi\left(y'\right) = y'$. The method of solution is the same as for the Lagrange equation. The general solution of the Clairaut equation has the form

$$y = xC + \psi(C),$$

i. e. the general solution of the Clairaut equation is obtained by replacing y' by C.

The Clairaut equation may have singular solutions in parametric form:

$$\begin{cases} x = -\psi'(p) \\ y = -\psi'(p)p + \psi(p) \end{cases},\ p \text{ is a parameter.}$$

Example. Integrate the equation $y = xy' - y'^2$.

Solution. This is a Clairaut equation. By denoting $y' = p$, we obtain

$$y = xp - p^2.$$

Differentiating the last equation and replacing dy by pdx, we obtain

$$pdx = pdx + xdp - 2pdp, \text{ where } dp(x-2p) = 0.$$

Equating the first factor to zero, we obtain

$$dp = 0, \text{ where } p = C,$$

and the general solution of the original equation is

$$y = xC - C^2.$$

Equating the second factor to zero, we obtain $x = 2p$. Eliminating p from this equation and from the equation $y = xp - p^2$, we obtain that $y = \frac{x^2}{4}$ is also a solution of our equation (singular solution).

Answer: $y = \frac{x^2}{4}$, $y = xC - C^2$.

Tasks for independent work

Solve the equations:

1. $y = 2xy' + \ln y'$;
2. $y = \frac{3}{2}xy' + e^{y'}$;
3. $y = xy' + (y')^2$;
4. $y = xy' + a\sqrt{1+(y')^2}$;
5. $y = xy' + \frac{a}{(y')^2}$;
6. $y = x(1+y') + (y')^2$;
7. $y = x(y')^2 - \frac{1}{y'}$;
8. $y = 2xy' + (\sin^2 y')$;
9. $x(y')^2 - yy' - y' + 1 = 0$;
10. $x = \frac{y}{y'} + \frac{1}{(y')^2}$.

Глава 2

Higher order ordinary differential equations

Basic definitions and terms

Differential equation of n-th order can be written as

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$$

or, if it can be solved with respect to the highest derivative, as

$$y^{(n)} = f\left(x, y, y', \ldots, y^{(n-1)}\right). \tag{2.1}$$

A solution of the differential equation (2.1) is a differentiable function $y = y(x)$, which after substituting into the equation (2.1), converts it into an identity.

The general solution of the equation (2.1) depends on the variable x and n arbitrary constants, i.e. has the form

$$y = \varphi\left(x, C_1, C_2, \ldots, C_n\right).$$

A fixed set of the constants $C_1, C_2, \ldots, C_n$ gives a **particular solution** of the equation (2.1).

The Cauchy problem for the differential equation (2.1) is the problem of finding a solution of this equation that satisfies the initial conditions:

$$y\left(x_0\right) = y_0,\ y'\left(x_0\right) = y_0',\ y''\left(x_0\right) = y_0'', \ldots y^{(n-1)}\left(x_0\right) = y_0^{(n-1)}.$$

Differential equations, admitting reduction of order

1. Equations of the form $y^{(n)} = f(x)$. The general solution of differential equation of this kind is obtained by n-fold integration of the equation.

Example. Solve the Cauchy problem $y^{IV} = \cos^2 x$, $y(0) = \frac{1}{32}$, $y'(0) = 0$, $y''(0) = \frac{1}{8}$, $y'''(0) = 0$.

Solution. By consecutive integrating of this equation we find the general solution:

$$y''' = \int (\cos^2 x) dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C_1;$$

$$y'' = \frac{1}{2}\int \left(x + \frac{1}{2}\sin 2x + 2C_1\right) dx = \frac{1}{2}\left(\frac{x^2}{2} - \frac{1}{4}\cos 2x + 2C_1 x\right) + C_2;$$

$$y' = \frac{1}{2}\int \left(\frac{x^2}{2} - \frac{1}{4}\cos 2x + 2C_1 x + 2C_2\right) dx = \frac{1}{2}\left(\frac{x^3}{6} - \frac{1}{8}\sin 2x + C_1 x^2 + 2C_2 x\right) + C_3;$$

$$y = \int y' dx = \frac{1}{2}\left(\frac{x^4}{24} + \frac{1}{16}\cos 2x + \frac{c_1}{3}x^3 + C_2 x^2\right) + C_3 x + C_4.$$

To find a particular solution it is necessary to define constants C_1, C_2, C_3, C_4 . Substitute the initial conditions into the equations:

$$\begin{cases} \frac{1}{2}\cdot\frac{1}{16} + C_4 = \frac{1}{32}; \\ C_3 = 0; \\ -\frac{1}{2}\cdot\frac{1}{4} + C_2 = \frac{1}{8}; \\ C_1 = 0. \end{cases}$$

We see that $C_1 = 0$, $C_2 = \frac{1}{4}$, $C_3 = 0$, $C_4 = 0$. Therefore, the particular solution is written as

$$y = \frac{1}{8}\left(\frac{x^4}{6} + \frac{1}{4}\cos 2x + x^2\right).$$

Answer: $y = \frac{1}{8}\left(\frac{x^4}{6} + \frac{1}{4}\cos 2x + x^2\right)$.

2. The equations do not contain explicitly y and its derivatives up to order $(k-1)$:

$$F\left(x, y^{(k)}, y^{(k+1)}, \ldots, y^{(n)}\right) = 0. \tag{2.2}$$

The order of this equation can be reduced to k units by introducing $y^{(k)} = z(x)$, i. e. taking for a new unknown function the lowest of the derivatives in (2.2). Then the equation (2.2) is reduced to the form

$$F\left(x, z, z', \ldots, z^{(n-k)}\right) = 0.$$

From the last equation we can define a function

$$z = f\left(x, C_1, C_2, \ldots, C_{n-k}\right),$$

and then find y from the equation $y^{(k)} = f\left(x, C_1, C_2, \ldots, C_{n-k}\right)$ by n-fold integration.

Example. Solve the Cauchy problem $y'' - \frac{y'}{x-1} = x(x-1),\ y(2) = 1,\ y'(2) = -1.$

Solution. This equation is a second-order differential equation that does not contain the desired function.

Find the general solution of this equation. Let us $y' = z$, convert the equation to the form

$$z' - \frac{z}{x-1} = x(x-1).$$

This is a first-order nonhomogeneous linear equation. Let us solve it with the Lagrange method. First, we find the general solution of the corresponding homogeneous linear equation of the first order

$$z' - \frac{z}{x-1} = 0.$$

Separate the variables and integrate:

$$\frac{dz}{dx} = \frac{z}{x-1}; \quad \frac{dz}{z} = \frac{dx}{x-1};$$

$$\int \frac{dz}{z} = \int \frac{dx}{x-1};$$

$$\ln|z| = \ln|x-1| + \ln|C|;$$

$$z = C(x-1).$$

Then we find the general solution of the nonhomogeneous linear equation. Let $z = (x-1)\cdot C(x)$. By differentiating this equality, we obtain

$$z' = C(x) + (x-1)\cdot C'(x).$$

Substitution z and z' into the nonhomogeneous linear equation gives

$$C(x) + (x-1)\cdot C'(x) - \frac{C(x)\cdot(x-1)}{x-1} = x(x-1);$$

$$C'(x) = x;$$

$$C(x) = \frac{x^2}{2} + C_1.$$

Substituting the last equality into $z = (x-1)\cdot C(x)$, we obtain the general solution of the nonhomogeneous linear equation:

$$z = (x-1)\cdot\left(\frac{x^2}{2} + C_1\right).$$

Due to $z' = z$, we obtain

$$y' = (x-1)\cdot\left(\frac{x^2}{2}+C_1\right) = \frac{x^3}{2} - \frac{x^2}{2} + C_1 x - C_1.$$

It is a separable equation. By integrating, we obtain a general solution of the equation:

$$y = \frac{x^4}{8} - \frac{x^3}{6} + \frac{C_1 x^2}{2} - C_1 x + C_2.$$

To find a particular solution we must define the constants C_1, C_2. Substitute the initial conditions $y(2) = 1$, $y\prime(2) = -1$ into the equations, we obtain:

$$\begin{cases} 1 = \frac{2^4}{8} - \frac{2^3}{6} + \frac{C_1 2^2}{2} - 2C_1 + C_2; \\ -1 = 4 + 2C_1 - 2 - C_1. \end{cases}$$

By solving the system, we obtain $C_1 = -3$, $C_2 = \frac{1}{3}$. Therefore, the particular solution has the form

$$y = \frac{x^4}{8} - \frac{x^3}{6} - \frac{3x^2}{2} + 3x + \frac{1}{3}.$$

Answer: $y = \frac{x^4}{8} - \frac{x^3}{6} - \frac{3x^2}{2} + 3x + \frac{1}{3}$.

3. Equations do not contain the independent variable

$$F\left(y, y', y'', \ldots, y^{(n)}\right) = 0. \qquad (2.3)$$

The order of this equation can be reduced to the unit by replacing $y' = z$, where z is considered as a new unknown function of y: $z = z(y)$. All derivatives $y', y'', \ldots, y^{(n)}$ are expressed over derivatives of new unknown functions $z(y)$:

$$y' = \frac{dy}{dx} = z(y) = z;$$

$$y'' = \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = z\frac{dz}{dy};$$

$$y''' = \frac{d}{dx}\left(z\frac{dz}{dy}\right) = z^2\frac{d^2 z}{dy^2} + z\left(\frac{dz}{dy}\right)^2, \text{ etc.}$$

By substituting these expressions instead of $y', y'', \ldots, y^{(n)}$ into the equation (2.3), we obtain the differential equation of a $(n-1)$-th order.

Example. Find the general solution of the equation $y' \cdot y''' - 2\left(y''\right)^2 = 0$.

Solution. This equation is a third-order differential equation that does not contain the independent variable x. Let

$$y' = z;\ y'' = z \cdot z';\ y''' = z^2 z'' + z\left(z'\right)^2.$$

By substituting these expressions instead of y', y'', y''' into the original equation, we obtain a second-order equation that does not contain an independent variable y:

$$z \cdot z'' - \left(z'\right)^2 = 0.$$

Reduce the order of the equation to the unit. Next we introduce the replacement $z' = p$; $z'' = p \cdot p'$. We obtain the separable equation:

$$z \cdot p \cdot p' - p^2 = 0.$$

We deduce that either $p = 0$ or $z \cdot p' = p$. Integrate the second equation $z \cdot \frac{dp}{dz} = p$:

$$\int \frac{dp}{p} = \int \frac{dz}{z};$$

$$\ln|p| = \ln|z| + C;$$

$$p = zC_1.$$

Since $p = z'$, we obtain the separable equation:

$$z' = zC_1.$$

By separating the variables and integrating, we obtain

$$z = C_2 e^{C_1 y}.$$

Introduce the inverse substitution $z = y'$. We obtain the separable equation

$$y' = C_2 e^{C_1 y}.$$

By integrating it, we obtain the general solution of the original equation:

$$e^{C_1 y} = C_2 x + C_3.$$

The case $p = 0$ gives $z' = 0$, where $z = C$, $y' = C$, therefore, $y = Cx + C_4$ is a solution, that can be obtained from the general one at the corresponding values of C_1, C_2, C_3.

Answer: $e^{C_1 y} = C_2 x + C_3$.

4. The equation $F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$ is homogeneous with respect to $y, y', y'', \ldots, y^{(n)}$, i. e.

$$F\left(x, ty, ty', ty'', \ldots, ty^{(n)}\right) = t^m F\left(x, y, y', y'', \ldots, y^{(n)}\right).$$

The order of this equation can be reduced to the unit by replacing $y' = yz(x)$, where z is new unknown function.

Example. Find the solution of the equation $xyy'' - x\left(y'\right)^2 = yy'$.

Solution. Let us check that the equation is homogeneous with respect to y, y', y''. We obtain $F\left(x, y, y', y''\right) = xyy'' - x\left(y'\right)^2 - yy'$. Then

$$F\left(x, ty, ty', ty''\right) = x(ty)\left(ty''\right) - x\left(ty'\right)^2 - (ty)\left(ty'\right) =$$

$$= t^2\left(xyy'' - x\left(y'\right)^2 - yy'\right) = t^2 F\left(x, y, y', y''\right).$$

Therefore, the equation is homogeneous with respect to y, y', y''. Let $y' = yz$. Then we obtain:

$$y'' = y'z + yz' = y\left(z^2 + z'\right).$$

By substituting y' and y'' into the equation, we obtain:

$$xy^2\left(z^2 + z'\right) - xy^2z^2 = y^2z;$$

$$x\left(z^2 + z'\right) - xz^2 = z \quad (y \neq 0);$$

$$xz' = z;$$

$$\frac{dz}{z} = \frac{dx}{x} \quad (z \neq 0, x \neq 0);$$

$$\ln|z| = \ln|x| + \ln C_1, \quad C_1 > 0;$$

$$z = C_1 x, \quad C_1 \neq 0.$$

The condition $x \neq 0$ does not lead to the loss of solutions of the equation $xz' = z$. The condition $z \neq 0$ leads to the loss of the solution $z = 0$. But this solution can be included in the general solution when $C_1 = 0$. Therefore, we obtain

$$z = C_1 x, \forall C_1 \text{ or } \frac{y'}{y} = C_1 x, \forall C_1;$$

$$\frac{dy}{y} = C_1 x\,dx;$$

$$\ln|y| = C_1\frac{x^2}{2} + C_2 \text{ or } y = e^{C_1\frac{x^2}{2} + C_2};$$

$$y = C_2 \cdot e^{C_1 x^2}, \quad C_2 \neq 0.$$

The condition $y \neq 0$ leads to the loss of the solution $y = 0$. But this solution can be included in the general solution when $C_2 = 0$. Therefore, the general solution of the equation is

$$y = C_2 \cdot e^{C_1 x^2}.$$

Answer: $y = C_2 \cdot e^{C_1 x^2}$.

Tasks for independent work

Solve equations

1. $y''' = 6x + 1$;
2. $y^{IV} = x^3 - 2x$;
3. $y''' = x \cdot e^{-x}$;
4. $2x \cdot y''' \cdot y'' = \left(y''\right)^2 - 9$;
5. $x \cdot y''' - y'' = 0$;
6. $yy'' = \left(y'\right)^2$;
7. $\left(y'\right)^2 + 2yy'' = 0$;
8. $y'' = x \cdot e^x,\ y(0) = 0,\ y'(0) = 0$;
9. $2y'' = \frac{y'}{x} + \frac{x}{y'}, \quad y(1) = \frac{\sqrt{2}}{5}, \quad y'(1) = \frac{\sqrt{2}}{2}$;
10. $y' = y'' \ln y',\ y(0) = 0,\ y'(0) = 1$.

Higher order linear differential equations

A linear differential equation of n-th order is an equation of the form

$$y^{(n)} + a_1(x) y^{(n-1)} + \cdots + a_{n-1}(x) y' + a_n(x) y = f(x), \tag{2.4}$$

where the functions $a_1(x), a_2(x), \ldots, a_n(x), f(x)$ are defined and continuous in a certain interval (a, b).

If in the equation (2.4) $f(x) \neq 0$, it is called a **linear nonhomogeneous**, or equation with a right-hand side. If in the equation (2.4) $f(x) = 0$, it is converted to the form

$$y^{(n)} + a_1(x) y^{(n-1)} + \cdots + a_{n-1}(x) y' + a_n(x) y = 0 \tag{2.5}$$

and it is called a **linear homogeneous**, or equation without a right-hand side.

A set of n functions $y_1, y_2, \ldots, y_n$ is called **linearly dependent** on the interval (a, b), if there are numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, not all equal zero, such that there is an identity

$$\alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n \equiv 0. \tag{2.6}$$

The set of n functions $y_1, y_2, \ldots, \quad y_n$ is called **linearly independent** on the interval (a, b), if the identity (2.6) is true on this interval only for

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

The determinant, which is composed of functions $y_1, y_2, \ldots, \quad y_n$ and their derivatives

$$W\left(y_1, y_2, \ldots, y_n\right) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_2' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

is called the **Wronskian determinant.**

Theorem 1. If the set of functions $y_1, y_2, \ldots, \quad y_n$ is linearly dependent on the interval $[a, b]$, then its Wronskian determinant is identically zero on this interval.

Theorem 2. In order that the set of n solutions $y_1, y_2, \ldots, \quad y_n$ of a linear ordinary differential equation of n-th order would be linearly independent on the interval $[a, b]$, it is necessary and sufficient that its Wronskian determinant is not equal to zero at least at one point of the interval $[a, b]$.

Any set of n linearly independent solutions $y_1, y_2, \ldots, \quad y_n$ of a linear ordinary differential equation of n-th order is called a **set of fundamental solutions** of this equation.

Theorem 3. If $y_1, y_2, \ldots, \quad y_n$ are set of fundamental solutions of a linear ordinary differential equation of n-th order, then the general solution of this equation is written as

$$y = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n, \tag{2.7}$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants.

The general solution of the linear nonhomogeneous equation (2.4) is

$$y = y_{\text{g}} + y_{\text{p}},$$

where y_{g} is the general solution of the corresponding linear homogeneous equation (2.5); y_{p} is a particular solution of the linear nonhomogeneous equation (2.4).

If a set of fundamental solutions $y_1, y_2, \ldots, \quad y_n$ of the homogeneous equation (2.5) is known, then the general solution of the corresponding nonhomogeneous equation (2.4) can be also found by the method of variation of arbitrary constants.

Higher order linear equations with constant coefficients

A linear homogeneous equation with constant coefficients is an equation of the form

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1} y' + a_n y = 0, \tag{2.8}$$

where $a_1, a_2, \ldots, a_n$ are some real numbers. This is a particular case of the equation (2.5) when $a_i(x) = a_i = const$ $(i = 1, 2, \ldots, n)$.

To find particular solutions of the equation (2.8) it is necessary to write the **characteristic equation**:

$$k^n + a_1 k^{n-1} + a_2 k^{n-2} \ldots + a_{n-1} k + a_n = 0. \tag{2.9}$$

The general solution of the equation (2.8) has the form (2.7) and is constructed depending on the type of the roots of the equation (2.9) according to the following rule:

1) each simple real root k corresponds to a component of the form

$$C \cdot e^{kx}$$

in the general solution (2.7);

2) each real root of multiplicity k m corresponds to the sum of components of the form

$$\left(C_1 + C_2 x + C_3 x^2 + \cdots + C_m x^{m-1}\right) \cdot e^{kx}$$

in the general solution (2.7);

3) each pair of complex conjugate roots $k_{1,2} = \alpha \pm \beta i$ corresponds to a component of the form

$$e^{\alpha x} \cdot (C_1 \cos \beta x + C_2 \sin \beta x)$$

in the general solution (2.7);

4) each pair of conjugate roots $k_{1,2} = \alpha \pm \beta i$ of multiplicity m corresponds to components of the form the sum of

$$e^{\alpha x}\left(\left(C_1 + C_2 x + \cdots + C_m x^{m-1}\right) \cos \beta x + \left(\tilde{C}_1 + \tilde{C}_2 x + \cdots + \tilde{C}_m x^{m-1}\right) \sin \beta x\right)$$

in the general solution (2.7).

Example 1. Find the general solution of the equation $y''' - 3y'' + 2y' = 0$.

Solution. This equation is a linear homogeneous equation of the third order with constant coefficients. Write the characteristic equation for it:

$$k^3 - 3k^2 + 2k = 0.$$

Its roots are $k_1 = 0$, $k_2 = 1$, $k_3 = 2$. Therefore, $e^{0\cdot x} = 1$, $e^{1\cdot x} = e^x$, $e^{2\cdot x} = e^{2x}$ are a set of fundamental solutions, and the general solution of this equation is

$$y = C_1 + C_2 e^x + C_3 e^{2x}.$$

Answer: $y = C_1 + C_2 e^x + C_3 e^{2x}$.

Example 2. Find the general solution of the equation $y''' + 2y'' + y' = 0$.

Solution.. The characteristic equation is

$$k^3 + 2k^2 + k = 0,$$

where $k_1 = k_2 = -1$, $k_3 = 0$. The roots are real, and one of them $k_1 = -1$ appears 2 times, so a set of fundamental solutions is written as e^{-x}, xe^{-x}, 1, and the general solution is

$$y = C_1 e^{-x} + C_2 x e^{-x} + C_3.$$

Answer: $y = C_1 e^{-x} + C_2 x e^{-x} + C_3$.

Example 3. Find the general solution of the equation $y''' + 4y'' + 13y' = 0$.

Solution. The roots of characteristic equation

$$k^3 + 4k^2 + 13k = 0$$

are $k_1 = 0, k_{2,3} = -2 \pm 3i$. A set of fundamental solutions is: 1, $e^{-2x}\cos 3x$, $e^{-2x}\sin 3x$, and the general solution is

$$y = C_1 + C_2 e^{-2x}\cos 3x + C_3 e^{-2x}\sin 3x.$$

Answer: $y = C_1 + C_2 e^{-2x}\cos 3x + C_3 e^{-2x}\sin 3x$.

Example 4. Find the general solution of the equation

$$y^{VI} + 10y^{IV} + 32y'' + 32y = 0.$$

Solution. This equation is a linear homogeneous equation of the sixth order with constant coefficients. Write the characteristic equation for it:

$$k^6 + 10k^4 + 32k^2 + 32 = 0,$$

where $k_{1,2} = \pm 2i$ are the pair of conjugate roots of multiplicity 2, $k_{3,4} = \pm\sqrt{2}i$ is a pair of complex conjugate roots. Therefore, the general solution is

$$y = (C_1 + C_2 x)\cos 2x + (C_3 + C_4 x)\sin 2x + C_5 \cos\sqrt{2}x + C_6 \sin\sqrt{2}x.$$

Answer: $y = (C_1 + C_2 x)\cos 2x + (C_3 + C_4 x)\sin 2x + C_5 \cos\sqrt{2}x + C_6 \sin\sqrt{2}x.$

Tasks for independent work

Solve the equations:

1. $y'' - 2y' + 10y = 0;$
2. $y^{IV} - y'' = 0;$
3. $y^{IV} - 2y''' - y'' + 2y' = 0;$
4. $y^{V} + 4y^{IV} + 5y''' - 6y' - 4y = 0;$
5. $y''' + 3y'' = 0;$
6. $y^{IV} - 2y''' + 2y'' - 2y' + y = 0;$
7. $y''' - 3y'' + 3y' - y = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 4;$
8. $y''' + 3y'' + y' + 3y = 0, \quad y(0) = -1, \quad y'(0) = 0, \quad y''(0) = 1;$
9. $y''' - 2y'' + 9y' - 18y = 0, \quad y(0) = -2,5, \quad y'(0) = 0, \quad y''(0) = 0;$
10. $y''' + 2y'' + y' = 0, \quad y(0) = 0, \quad y'(0) = 2, \quad y''(0) = -3.$

Higher order linear nonhomogeneous equations with constant coefficients

A **linear nonhomogeneous equation of n-th order with constant coefficients** is an equation of the form

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1} y' + a_n y = f(x), \qquad (2.10)$$

where $a_1, a_2, \ldots, a_n$ are some real numbers. This is a particular case of the equation (2.4), when $a_i(x) = a_i = const$ $(i = 1, 2, \ldots, n)$.

The general solution of the linear nonhomogeneous equation (2.10) can be found with two methods.

1. Method of undetermined coefficients

The general solution of (2.10) is $y = y_g + y_*$, where y_g is the general solution of the corresponding linear homogeneous equation (2.8); y_* is a particular solution of the equation (2.10).

We find a particular solution y_* of this linear nonhomogeneous equation (2.10) with the method of undetermined coefficients. This method is only used if the right-hand side of the equation (2.10) has the form

$$f(x) = e^{\alpha x} \cdot \left(P_n(x)\cos\beta x + Q_m(x)\sin\beta x\right), \tag{2.11}$$

where $P_n(x), Q_m(x)$ are polynomials of x of n and m degrees; α, β are constants.

Then a particular solution y_* of the equation (2.10) is

$$y_* = x^s \cdot e^{\alpha x} \cdot \left(P_k'(x)\cos\beta x + Q_k'(x)\sin\beta x\right), \tag{2.12}$$

where s is the multiplicity of the roots $\alpha \pm \beta i$ of characteristic equation (if $\alpha \pm \beta i$ are not roots of the characteristic equation, then $s = 0$); $k = \max(n, m)$; $P_k'(x)$, $Q_k'(x)$ are polynomials of x, of n and m degrees, of the general form with undetermined coefficients.

Some simplest types of right-hand sides $f(x)$ of equation (2.10), particular cases of expression (2.11) and corresponding them particular solutions y_*, particular cases of the formula (2.12) are shown in the table.

Particular solutions of the linear nonhomogeneous equation of n-th order with constant coefficients

№	Right hand side $f(x)$ of equation (2.10)	The form of a particular solution y_* of equation (2.10)	
		λ is not root of the characteristic equation (2.9)	λ is root of the characteristic equation (2.9)
1	$f(x) = P_n(x)$ – n-th degree polynomial	$\lambda = 0$ – is not root $\Longrightarrow$ $y_* = Q_n(x)$ – n-th degree polynomial	$\lambda = 0$ – is s-multiple root$\Longrightarrow$ $y_* = x^s \cdot Q_n(x) - (n+s)$ degree polynomial
2	$f(x) = e^{\alpha x} \cdot P_n(x)$	$\lambda = \alpha$ – is not root $\Longrightarrow$ $y_* = e^{\alpha x} \cdot Q_n(x)$	$\lambda = \alpha$ – is s-multiple root$\Longrightarrow$ $y_* = e^{\alpha x} \cdot x^s \cdot Q_n(x)$
3	$f(x) = P_n(x) \cdot \cos\beta x + Q_m(x)\sin\beta x$	$\lambda = \pm\beta i$ – is not root $\Longrightarrow$ $y_* = P_k'(x)\cos\beta x + +Q_k'(x)\sin\beta x$	$\lambda = \pm\beta i$ – is s-multiple root $\Longrightarrow$ $y_* = x^s \cdot (P_k'(x) \cdot \cdot \cos\beta x + +Q_k'(x)\sin\beta x)$
4	$f(x) = e^{\alpha x} \cdot (P_n(x) \cdot \cdot \cos\beta x + Q_m(x)\sin\beta x)$	$\lambda = \alpha \pm \beta i$ – is not root $\Longrightarrow$ $y_* = e^{\alpha x} \cdot (P_k'(x) \cdot \cdot \cos\beta x + +Q_k'(x)\sin\beta x)$	$\lambda = \alpha \pm \beta i$ – is s-multiple root $\Longrightarrow$ $y_* = x^s \cdot e^{\alpha x} \cdot (P_k'(x) \cdot \cdot \cos\beta x + +Q_k'(x)\sin\beta x)$

Example 1. Find the general solution of the equation $y^{IV} - y = 10\cos x$.

Solution. This equation is a linear nonhomogeneous equation of the fourth order with constant coefficients. Find the general solution of the corresponding linear homogeneous equation

$$y^{IV} - y = 0.$$

We write the characteristic equation for it

$$k^4 - k = 0.$$

Its roots are $k_{1,2} = \pm 1$; $k_{3,4} = \pm i$. Therefore, the general solution of the homogeneous equation is

$$y_g = C_1 e^x + C_2 e^{-x} + C_3\cos x + C_4\sin x.$$

We find a particular solution of this linear nonhomogeneous equation with the method of undetermined coefficients. The right-hand side of the equation $f(x) = 10\cos x$ is a particular case of expression (2.11), where $P_n(x) = 10$, $\beta = 1$, $Q_m(x) = 0$. We obtain a particular solution of this linear nonhomogeneous equation

$$y_* = x \cdot (A_1\cos x + A_2\sin x).$$

Find the unknown coefficients A_1, A_2. Differentiate four times the equation, we obtain:

$$y'_* = A_1 \cos x + A_2 \sin x + x \cdot (-A_1 \sin x + A_2 \cos x);$$

$$y''_* = 2A_2 \cos x - 2A_1 \sin x + x \cdot (A_1 \cos x + A_2 \sin x);$$

$$y'''_* = -3A_1 \cos x - 3A_2 \sin x - x \cdot (-A_1 \sin x + A_2 \cos x);$$

$$y^{IV}_* = 4A_1 \sin x - 4A_2 \cos x + x \cdot (A_1 \cos x + A_2 \sin x).$$

Substitute the expression for y^{IV}, y into the original equation, we have

$$4A_1 \sin x - 4A_2 \cos x + x(A_1 \cos x + A_2 \sin x) - x(A_1 \cos x + A_2 \sin x) = 10 \cos x.$$

After transformations we obtain the equation

$$4A_1 \sin x - 4A_2 \cos x = 10 \cos x.$$

Equalize the coefficients in the left hand side and right hand side before $\sin x$ and $\cos x$:

$$4A_1 = 0; -4A_2 = 10;$$

$$A_1 = 0, A_2 = -\frac{5}{2}.$$

Substitute these values into the particular solution of this linear nonhomogeneous equation:

$$y_* = -\frac{5}{2} x \sin x.$$

Hence, the general solution of this linear nonhomogeneous equation with constant coefficients is

$$y = y_{\text{g}} + y_* = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x - \frac{5}{2} x \sin x.$$

Answer: $y = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x - \frac{5}{2} x \sin x.$

Example 2. Find the general solution of the equation $y'' - 3y' + 2y = x^2 + 3x$.

Solution. This equation is a linear nonhomogeneous equation of the second order with constant coefficients. Find the general solution of the corresponding linear homogeneous equation

$$y'' - 3y' + 2y = 0.$$

We write the characteristic equation for it:

$$k^2 - 3k + 2 = 0,$$

roots of which are $k_1 = 1$, $k_2 = 2$. Hence, the general solution of linear homogeneous equation is

$$y_g = C_1 e^x + C_2 e^{2x}.$$

We find a particular solution of this linear inhomogeneous equation with the method of undetermined coefficients. The right-hand side of the equation $f(x) = x^2 + 3x$ is a polynomial of 2-nd degree (a particular case of the expression (2.11)).

0 is not a root of the characteristic equation, therefore, we obtain a particular solution of this linear nonhomogeneous equation:

$$y_* = Q_2(x) = A_1 x^2 + A_2 x + A_3.$$

To find the unknown coefficients A_1, A_2, A_3 differentiate the equation two times:

$$y'_{\text{p}} = 2A_1 x + A_2;\; y''_* = 2A_1.$$

Substitute the expression for y'', y' , y into the original equation:

$$2A_1 - 6A_1 x - 3A_2 + 2A_1 x^2 + 2A_2 x + 2A_3 = x^2 + 3x.$$

After the transformations we obtain the equation:

$$2A_1 x^2 + (2A_2 - 6A_1)\, x + (2A_1 - 3A_2 + 2A_3) = x^2 + 3x.$$

Equalize the coefficients on the left-hand side and on right-hand side of equality before the same degrees of x:

$$\begin{cases} 2A_1 = 1; \\ 2A_2 - 6A_1 = 3; \\ 2A_1 - 3A_2 + 2A_3 = 0. \end{cases}$$

By solving the system, we obtain $A_1 = \frac{1}{2}$, $A_2 = 3$, $A_3 = 4$. Substitute these values into the particular solution of this linear nonhomogeneous equation:

$$y_* = \frac{1}{2} x^2 + 3x + 4.$$

Thus, the general solution of this linear nonhomogeneous equation with constant coefficients is

$$y = y_g + y_* = C_1 e^x + C_2 e^{2x} + \frac{1}{2} x^2 + 3x + 4.$$

Answer: $y = y_g + y_* = C_1 e^x + C_2 e^{2x} + \frac{1}{2} x^2 + 3x + 4.$

2. Method of variation of arbitrary constants

If a set of fundamental solutions $y_1, y_2, \dots, y_n$ of the homogeneous equation (2.5) is known, then the general solution of the corresponding nonhomogeneous equation (2.4) can be found with the **method of variation of an arbitrary constants (Lagrange method)**. This method can be used in solving the linear nonhomogeneous equation (2.4) with variables and constant coefficients. If the right hand side of the nonhomogeneous equation with constant coefficients (2.10) is not a particular case of the formula (2.11), then this method allows us to find a solution.

The idea of the method is us follows. We write the general solution of equation (2.10) in the form

$$y = C_1(x)y_1 + C_2(x)y_2 + \dots + C_n(x)y_n,$$

where $y_1, y_2, \dots, y_n$ are linearly independent particular solutions (a set of fundamental solutions) of equation (2.5), and the functions $C_1(x), C_2(x), \dots, C_n(x)$ are obtained from a system of equations:

$$\begin{cases} C_1'(x)\cdot y_1 + C_2'(x)\cdot y_2 + \dots + C_n'(x)\cdot y_n = 0; \\ C_1'(x)\cdot y_1' + C_2'(x)\cdot y_2' + \dots + C_n'(x)\cdot y_n' = 0; \\ \dots \\ C_1'(x)\cdot y_1^{(n-2)} + C_2'(x)\cdot y_2^{(n-2)} + \dots + C_n'(x)\cdot y_n^{(n-2)} = 0; \\ C_1'(x)\cdot y_1^{(n-1)} + C_2'(x)\cdot y_2^{(n-1)} + \dots + C_n'(x)\cdot y_n^{(n-1)} = f(x). \end{cases}$$

Example 1. Find the general solution of the equation $y' + x\cdot y'' = x^2$.

Solution. This equation is a linear nonhomogeneous equation of the second order. We solve it with the method of variation of arbitrary constants. Find the general solution of the corresponding linear homogeneous equation

$$y' + x\cdot y'' = 0.$$

This equation is a second-order differential equation that does not contain the desired function y. Let $y' = z$, therefore, $y'' = z'$. Convert the linear homogeneous equation to the form

$$x\cdot z' + z = 0.$$

Thus, we reduce to the unit the order of the equation and obtain a separable equation. Solve it:

$$x\cdot\frac{dz}{dx} = -z;$$

$$\frac{dz}{z} = -\frac{dx}{x};$$
$$\ln|z| = \ln\left|\frac{C_1}{x}\right|;$$
$$z = \frac{C_1}{x}.$$

Returning to the variable y, we obtain the separable equation:

$$y' = \frac{C_1}{x},$$

where $y = C_1 \ln|x| + C_2$ is the general solution of the linear homogeneous equation.

Basing on the general solution we obtained, assume that $y_1 = \ln|x|$, $y_2 = 1$ are partial solutions of the linear homogeneous equation. Find the general solution of this linear nonhomogeneous equation in the form

$$y = C_1(x)\ln|x| + C_2(x). \tag{2.13}$$

Note that if we divide the original equation by $x \neq 0$, we obtain the equation

$$y'' + y'\frac{1}{x} = x.$$

Then $f(x) = x$, $y_1 = \ln|x|$, $y_2 = 1$. Write a system of equations to find the functions $C_1(x), C_2(x)$:

$$\begin{cases} C_1'(x)\cdot\ln|x| + C_2'(x)\cdot 1 = 0; \\ C_1'(x)\cdot\frac{1}{x} + C_2'(x)\cdot 0 = x; \end{cases} \quad \begin{cases} C_1'(x)\cdot\ln|x| = -C_2'(x); \\ C_1'(x)\cdot\frac{1}{x} = x. \end{cases}$$

By double differentiating the equation (2.13) and taking into account the equality of the system, we obtain:

$$y' = C_1'(x)\cdot\ln|x| + C_1(x)\cdot\frac{1}{x} + C_2'(x) = C_1(x)\cdot\frac{1}{x}.$$

Here $C_1'(x)\cdot\frac{1}{x} = x$; $C_1'(x) = x^2$; $C_1(x) = \frac{x^3}{3} + C_1$.

The first equation of the system is $x^2\ln|x| + C_2'(x) = 0$. By integrating it, we obtain

$$C_2(x) = -\frac{x^3}{3}\ln|x| + \frac{x^3}{9} + C_2.$$

Thus, by substituting the values of the functions $C_1(x), C_2(x)$ into the equation (2.13), we obtain the general solution of this linear nonhomogeneous equation:

$$y = C_1\ln|x| + \frac{x^3}{9} + C_2.$$

Answer: $y = C_1\ln|x| + \frac{x^3}{9} + C_2$.

Example 2. Find the general solution of the equation

$$y''' - 2y'' - y' + 2y = \frac{e^{2x}}{e^x + 1}.$$

Solution. This equation is a linear nonhomogeneous equation of the third order. We solve it with the method of variation of an arbitrary constants. Find the general solution of the corresponding linear homogeneous equation

$$y''' - 2y'' - y' + 2y = 0.$$

This equation is a linear homogeneous equation of the third order with constant coefficients. Write the characteristic equation for it

$$k^3 - 2k^2 - k + 2 = 0.$$

Its roots are $k_1 = 1, \quad k_2 = -1, \quad k_3 = 2$. Therefore, the general solution of the homogeneous equation is

$$y = C_1 e^x + C_2 e^{-x} + C_3 e^{2x}.$$

Basing on the general solution we obtained, assume that $y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = e^{2x}$ are partial solutions of the linear homogeneous equation. Find the general solution of this linear nonhomogeneous equation in the form

$$y = C_1(x)e^x + C_2(x)e^{-x} + C_3(x)e^{2x}. \qquad (2.14)$$

Write a system of equations to find the functions $C_1(x), C_2(x), C_3(x)$:

$$\begin{cases} C_1'(x) \cdot e^x + C_2'(x) \cdot e^{-x} + C_3'(x) \cdot e^{2x} = 0; \\ C_1'(x) \cdot e^x - C_2'(x) \cdot e^{-x} + 2C_3'(x) \cdot e^{2x} = 0; \\ C_1'(x) \cdot e^x + C_2'(x) \cdot e^{-x} + 4C_3'(x) \cdot e^{2x} = \frac{e^{2x}}{e^x+1}. \end{cases}$$

Solving the system of three equations with three unknowns (with the Gauss method), we obtain:

$$C_1'(x) = -\frac{1}{2}\frac{e^x}{e^x + 1};$$

$$C_2'(x) = \frac{1}{6}\frac{e^{3x}}{e^x + 1};$$

$$C_3'(x) = \frac{1}{3}\frac{1}{e^x + 1}.$$

By integrating these expressions, we obtain:

$$C_1(x) = -\frac{1}{2}\ln\left(e^x+1\right)+C_1;$$

$$C_2(x) = \frac{1}{6}\left(\frac{e^{2x}}{2}-e^x+\ln\left(e^x+1\right)\right)+C_2;$$

$$C_3(x) = \frac{1}{3}\left(x-\ln\left(e^x+1\right)\right)+C_3.$$

Thus, by substituting the values of the functions $C_1(x)$, $C_2(x)$, $C_3(x)$ into the equation (**??**), we obtain the general solution of this linear nonhomogeneous equation:

$$y = C_1e^x + C_2e^{-x} + C_3e^{2x} + \frac{1}{12}\left(e^x-2+4xe^{2x}\right)+\frac{1}{6}\left(e^{-x}-3e^x-2e^{2x}\right)\ln\left(e^x+1\right).$$

Answer:

$$y = C_1e^x + C_2e^{-x} + C_3e^{2x} + \frac{1}{12}\left(e^x-2+4xe^{2x}\right)+\frac{1}{6}\left(e^{-x}-3e^x-2e^{2x}\right)\ln\left(e^x+1\right).$$

Theorem of superposition of particular solutions of linear nonhomogeneous differential equation

Let the right hand side $f(x)$ of a linear nonhomogeneous differential equation with constant coefficients (or with variables ones) be the sum of some functions

$$f(x) = f_1(x)+\ldots+f_m(x),$$

and, for each $k = 1,\ldots,m$, the function $y_k(x)$ is a particular solution of the linear nonhomogeneous differential equation. Then the function

$$y(x) = y_1(x)+\ldots+y_m(x)$$

is a particular solution of the linear nonhomogeneous differential equation with constant coefficients (or with variables ones).

Example. Solve the equation$y''-3y' = 9x^2-5-12\sin 3x$.
Solution. There is the characteristic equation

$$k^2-3k=0$$

corresponding to the homogeneous equation $y'' - 3y' = 0$.

Its roots are $k_1 = 0$, $k_2 = 3$. Then the general solution is

$$y_g = C_1 + C_2 \cdot e^{3x}.$$

The right-hand side of the equation can be represented as the sum of two functions, $f_1(x) = 9x^2 - 5$ and $f_2(x) = -12\sin 3x$. Each of them is a function of a special type.

Consider the equation with the first summand on the right-hand side

$$y'' - 3y' = 9x^2 - 5.$$

Solving it with the method of undetermined coefficients, we obtain a first particular solution:

$$y_1^* = -x^3 - x^2 + x.$$

A particular solution of the equation

$$y'' - 3y' = -12\sin 3x$$

is also found with the method of undetermined coefficients. A second particular solution is:

$$y_2^* = -\frac{2}{3}\cos 3x + \frac{2}{3}\sin 3x.$$

A particular solution of the original nonhomogeneous equation is equal to the sum of the obtained particular solutions:

$$y = C_1 + C_2 \cdot e^{3x} - x^3 - x^2 + x - \frac{2}{3}\cos 3x + \frac{2}{3}\sin 3x.$$

Answer: $y = C_1 + C_2 \cdot e^{3x} - x^3 - x^2 + x - \frac{2}{3}\cos 3x + \frac{2}{3}\sin 3x$.

Tasks for independent work

Solve the equations:

1. $y'' - 5y' + 4y = 4x^2 e^{2x}$;
2. $y'' + 3y' - 4y = \mathrm{e}^{-4x} + x\mathrm{e}^{-x}$;
3. $y'' - 4y' + 8y = e^{2x} + \sin 2x$;
4. $y'' - 9y = e^{3x}\cos x$;
5. $y'' + y = x\sin x$;
6. $y'' - 5y' = 3x^2 + \sin 5x$;

7. $y'' - 2y' + y = \frac{e^x}{x}$;

8. $y'' - 2y' + y = 0; y(2) = 1, \quad y'(2) = -2$;

9. $y'' - 2y' = 2\mathrm{e}^x; \quad y(1) = -1, \quad y'(1) = 0$;

10. $y''' - y' = 0; \quad y(0) = 3, \quad y'(0) = -1, \quad y''(0) = 1$.

Глава 3

Systems of differential equations

Basic definitions and terms

A **system of differential equations** is a set of differential equations (equations containing independent variables, unknown functions and their derivatives or differentials). A system of n differential equations of the first order, which are solved with respect to all derivatives, is called **normal**. It has the form

$$\begin{cases} \frac{dy_1}{dx} = f_1\left(x, y_1, y_2, \ldots, y_n\right); \\ \frac{dy_2}{dx} = f_2\left(x, y_1, y_2, \ldots, y_n\right); \\ \ldots \\ \frac{dy_n}{dx} = f_\pi\left(x, y_1, y_2, \ldots, y_n\right), \end{cases} \tag{3.1}$$

where $y_1, y_2, \ldots, y_n$ are unknown functions of an independent variable x; $f_1, f_2, \ldots, f_n$ are known functions dependent on x, $y_1, y_2, \ldots, y_n$, which are specified and continuous in some domain.

To solve a system or to integrate it in some interval $[a, b]$ means to find a set of n functions $y_1, y_2, \ldots, y_n$ which are defined and continuously differentiable in the specified interval, and convert each equation of the system into an identity.

The **Cauchy problem** for the system of differential equations (3.1) is the problem of finding the solution

$$y_1 = y_1(x), y_2 = y_2(x), \ldots, y_n = y_n(x)$$

satisfying the initial conditions:

$$y_1\left(x_0\right) = y_1^{(0)}, \quad y_2\left(x_0\right) = y_2^{(0)}, \ldots, y_n\left(x_0\right) = y_n^{(0)}.$$

The geometric meaning of the Cauchy problem is in finding those curves, among all integrals, that pass through the points $\left(x_0, y_1^0\right), \ldots, \left(x_0, y_n^0\right)$.

The general solution of the system (3.1) is a set of n functions:

$$\begin{cases} y_1 = y_1\left(x, C_1, C_2, \ldots, C_n\right); \\ y_2 = y_2\left(x, C_1, C_2, \ldots, C_n\right); \\ \ldots \\ y_n = y_n\left(x, C_1, C_2, \ldots, C_n\right), \end{cases}$$

which depend on the variable x and arbitrary constants C_1, C_2, …, C_n and satisfy the following conditions:

1) the functions $y_1, y_2, \ldots, y_n$ are defined in the domain of change of variables and have continuous partial derivatives with respect to the variable x;

2) the functions y_1, y_2, …, y_n are solutions of the system (3.1) for any fixed values of C_1, C_2, …, C_n.

A **particular solution** of the system (3.1) is a solution obtained from the general solution for some particular values of C_1, C_2, …, C_n.

An **integral** of a normal system (3.1) is a function

$$\psi\left(x, y_1, y_2, \ldots, y_n\right),$$

which, in the domain of changing of variable,

1) is defined and continuous with its partial derivatives

$$\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y_1}, \ldots, \frac{\partial \psi}{\partial y_n};$$

2) takes a constant value for any x after substituting an arbitrary solution of the system into it.

The function $\psi\left(x, y_1, y_2, \ldots, y_n\right)$ depends only on the choice of solution $y_1, y_2, \ldots, y_n$, and does not depend on the variable x.

A **first integral** of a normal system (3.1) is the equality

$$\psi\left(x, y_1, y_2, \ldots, y_n\right) = C,$$

where $\psi\left(x, y_1, y_2, \ldots, y_n\right)$ is an integral of a normal system (3.1); C is an arbitrary constant.

Sometimes a first integral of the system (3.1) is simply called an integral of this system.

A system of differential equations (3.1) can be converted to one n-th order differential equation:

$$y^{(n)} = f\left(x, y, y', y'', y''', \ldots, y^{(n-1)}\right),$$

and, conversely, an n-th order differential equation can be converted to a system of differential equations (3.1).

Example 1. Convert the differential equation $y'' + 2y - 3 = 0$ to the normal system of differential equations.

Solution. Let $y' = z$. Then $y'' = z'$, and the equation is converted to a normal system.

$$\begin{cases} y' = z; \\ z' = 3 - 2y. \end{cases}$$

Answer: $\begin{cases} y' = z; \\ z' = 3 - 2y. \end{cases}$

Example 2. Convert the normal system of differential equations

$$\begin{cases} y' = 3 - z; \\ z' = 1 + 6y + z \end{cases}$$

to one differential equation.

Solution. From the first equation of the system we find $z = 3y - y'$; $z' = 3y' - y''$. Substituting the values z and z' into the second equation of the system, we obtain

$$3y' - y'' = 1 + 6y + 3y - y';$$

$y'' - 4y' + 9y + 1 = 0$ – linear differential equation of the 2nd order.

Answer: $y'' - 4y' + 9y + 1 = 0$.

A system is called **linear** if unknown functions and their derivatives (or differentials) are included in each of the equations only in the first order. The normal system of linear differential equations of the first order has the form

$$\begin{cases} \frac{dy_1}{dx} = a_{11}(x)y_1 + a_{12}(x)y_2 + \ldots + a_{1n}(x)y_n + f_1(x); \\ \frac{dy_2}{dx} = a_{21}(x)y_1 + a_{22}(x)y_2 + \ldots + a_{2n}(x)y_n + f_2(x); \\ \ldots \\ \frac{dy_n}{dx} = a_{n1}(x)y_1 + a_{n2}(x)y_2 + \ldots + a_{nn}(x)y_n + f_n(x), \end{cases} \quad (3.2)$$

where the functions $a_{ij}(x)$, $f_i(x)$ $(i = 1, \ldots, n; j = 1, \ldots, n)$ are continuous in a certain interval.

If all $f_i(x) = 0$, then the system (3.2) is called **homogeneous**, otherwise it is **nonhomogeneous**.

If $a_{ij}(x) = a_{ij} = const$, then the system (3.2) is called **linear with constant coefficients**.

Elimination method

A normal system of n differential equations of the first order is equivalent to one differential equation of n-th order. Therefore, it is possible to eliminate all unknown functions in the system (3.1) but one and obtain one differential equation of n-th order.

It is possible to convert the normal system (3.1) to one equation by differentiating one of the equations of the system and the consequent elimination of all unknowns but one.

Example. Find the general solution of the system of differential equations

$$\begin{cases} y_1' = 3y_1 - y_2 + y_3 + e^x; \\ y_2' = y_1 + y_2 + y_3 + x; \\ y_3' = 4y_1 - y_2 + 4y_3. \end{cases}$$

Solution. We solve the system with the elimination method. Differentiate with respect to the variable x the first equation of the system and substitute, instead of y_1', y_2', y_3', their expressions from this system. We obtain:

$$y_1'' = 3y_1' - y_2' + y_3' + e^x =$$

$$= 3\left(3y_1 - y_2 + y_3 + e^x\right) - \left(y_1 + y_2 + y_3 - x\right) + 4y_1 - y_2 + 4y_3 + e^x =$$

$$= 12y_1 - 5y_1 + 6y_3 + 4e^x + x.$$

Differentiate y_1'' with respect to x and substitute, instead of y_1', y_2', y_3', their expressions from this system:

$$y_1''' = 12y_1' - 5y_2' + 6y_3' + 4e^x + 1 =$$

$$= 12\left(3y_1 - y_2 + y_3 + e^x\right) - 5\left(y_1 + y_2 + y_3 - x\right) + 6\left(4y_1 - y_2 + 4y_3\right) + 4e^x + x =$$

$$= 55y_1 - 23y_2 + 31y_3 + 16e^x + 6x.$$

Hence, we obtain a system of differential equations:

$$\begin{cases} y_1' = 3y_1 - y_2 + y_3 + e^x; \\ y_1'' = 12y_1 - 5y_2 + 6y_3 + 4e^x + x; \\ y_1'' = 55y_1 - 23y_2 + 31y_3 + 16e^x + 6x. \end{cases} \qquad (3.3)$$

From the first two equations we obtain y_2 and y_3:

$$y_2 = y_1'' - 6y_1' + 6y_1 + 2e^x - x;$$

$$y_3 = y_1'' - 5y_1' + 3y_1 + e^x - x. \tag{3.4}$$

We substitute the expressions for y_2 and y_3 into the third equation of the system (3.3):

$$y_1''' = 55y_1 - 23\left(y_1'' - 6y_1' + 6y_1 + 2e^x - x\right) + 31\left(y_1'' - 5y_1' + 3y_1 + e^x - x\right) + 16e^x + 6x =$$

$$= 8y_1'' - 17y_1' + 1 + 10y_1 + e^x - 2x.$$

Thus we obtain nonhomogeneous linear equation of the third order with constant coefficients:

$$y_1''' - 8y_1'' + 17y_1' - 10y_1 = e^x - 2x \tag{3.5}$$

Its general solution is obtained by the formula $y_1 = y_1^g + y_1^*$, where y_1^g is the general solution of the corresponding linear homogeneous equation; y_1^* is a particular solution of the linear nonhomogeneous equation (3.5).

Find the general solution y_1^g of the corresponding linear homogeneous equation

$$y_1''' - 8y_1'' + 17y_1' - 10y_1 = 0.$$

We write the characteristic equation for it

$$k^3 - 8k^2 + 17k - 10 = 0.$$

Its roots are: $k_1 = 1$, $k_2 = 2$, $k_3 = 5$. Therefore, the general solution of the homogeneous equation is

$$y_1^g = C_1e^x + C_2e^{2x} + C_3e^{5x}.$$

Find a particular solution y_1^* of linear nonhomogeneous equation (3.4) with the method of undetermined coefficients.

The right-hand side of the equation (3.5) $f(x) = e^x - 2x$ is the sum of two functions, $f_1(x) = e^x$ and $f_2(x) = -2x$, therefore, a particular solution of the equation (3.5) is

$$y_1^* = y_1^{*1} + y_1^{*2},$$

where y_1^{*1} and y_1^{*2} are particular solutions of nonhomogeneous linear equations

$$y_1''' - 8y_1'' + 17y_1' - 10y_1 = f_1(x) \text{ and } y_1''' - 8y_1'' + 17y_1' - 10y_1 = f_2(x).$$

For $f_1(x) = e^x$ we have $k = 1$, for $f_2(x) = -2x$ the value $k = 0$, then we obtain a particular solution y_1^* of linear nonhomogeneous equation (3.5):

$$y_1^* = A_1xe^x + A_2x + A_3.$$

Find the unknown values A_1, A_2, A_3. Differentiate three times the equation:

$$y_1^{*\prime} = A_1 x e^x + A_1 e^x + A_2;$$

$$y_1^{*\prime\prime} = A_1 x e^x + 2A_1 e^x;$$

$$y_1^{*\prime\prime\prime} = A_1 x e^x + 3A_1 e^x.$$

Substitute the expressions for $y_1^{*\prime\prime\prime}$, $y_1^{*\prime}$, $y_1^{*\prime\prime}$, y_1^* into the equation (3.5):

$$A_1 x e^x + 3A_1 e^x - 8\left(A_1 x e^x + 2A_1 e^x\right) + 17\left(A_1 x e^x + A_1 e^x + A_2\right) -$$

$$-10\left(A_1 x e^x + A_2 x + A_3\right) = e^x - 2x.$$

We equate the coefficients on the left-hand side and right-hand side before the the corresponding expressions, obtain:

$$A_1 = \frac{1}{4};\;\; A_2 = \frac{1}{5};\;\; A_3 = \frac{17}{50}.$$

Substitute these values into the particular solution y_1^* of the linear nonhomogeneous equation (3.5):

$$y_1^* = \frac{1}{4} x e^x + \frac{1}{5} x + \frac{17}{50}.$$

Hence, the general solution of a linear nonhomogeneous equation with constant coefficients (3.5) has the form

$$y_1 = y_1^g + y_1^* = C_1 e^x + C_2 e^{2x} + C_3 e^{5x} + \frac{1}{4} x e^x + \frac{1}{5} x + \frac{17}{50}.$$

Find derivatives $y_{1,}'$, $y_{1,}''$:

$$y_1' = C_1 e^x + 2C_2 e^{2x} + 5C_3 e^{5x} + \frac{1}{4} x e^x + \frac{1}{4} e^x + \frac{1}{5};$$

$$y_1'' = C_1 e^x + 4C_2 e^{2x} + 25C_3 e^{5x} + \frac{1}{4} x e^x + \frac{1}{2} e^x.$$

Substitute the found values $y_{1,}'$, $y_{1,}''$ into the equalities (3.4):

$$y_2 = y_1'' - 6y_1' + 6y_1 + 2e^x - x = C_1 e^x + 4C_2 e^{2x} + 25C_3 e^{5x} + \frac{1}{4} x e^x + \frac{1}{2} e^x -$$

$$-6\left(C_1 e^x + 2C_2 e^{2x} + 5C_3 e^{5x} + \frac{1}{4} x e^x + \frac{1}{4} e^x + \frac{1}{5}\right) +$$

$$+6\left(C_1 e^x + C_2 e^{2x} + C_3 e^{5x} + + \frac{1}{4} x e^x + \frac{1}{5} x + \frac{17}{50}\right) + 2e^x - x =$$

$$= C_1 e^x - 2C_2 e^{2x} + C_3 e^{5x} + \frac{1}{4} x e^x + \frac{6}{5} x + \frac{21}{25} - e^x;$$

$$y_3 = y_1'' - 5y_1' + 3y_1 + e^x - x = C_1e^x + 4C_2e^{2x} + 25C_3e^{5x} + \frac{1}{4}xe^x + \frac{1}{2}e^x -$$
$$-5\left(C_1e^x + 2C_2e^{2x} + 5C_3e^{5x} + \frac{1}{4}xe^x + \frac{1}{4}e^x + \frac{1}{5}\right) +$$
$$+3\left(C_1e^x + C_2e^{2x} + C_3e^{5x} + \frac{1}{4}xe^x + \frac{1}{5}x + \frac{17}{50}\right) + e^x - x =$$
$$= C_1e^x - 3C_2e^{2x} + 3C_3e^5 - \frac{1}{4}xe^x - \frac{2}{5}x + \frac{1}{4}e^x + \frac{1}{50}.$$

Thus, the general solution of the original system has the form

$$y_1 = C_1e^x + C_2e^{2x} + C_3e^{5x} + \frac{1}{4}xe^x + \frac{1}{5}x + \frac{17}{50};$$
$$y_2 = C_1e^x - 2C_2e^{2x} + C_3e^{5x} + \frac{1}{4}xe^x + \frac{6}{5}x + \frac{21}{25} - e^x;$$
$$y_3 = -C_1e^x - 3C_2e^{2x} + 3C_3e^{5x} - \frac{1}{4}xe^x - \frac{2}{5}x + \frac{1}{4}e^x + \frac{1}{50}.$$

Answer:

$$y_1 = C_1e^x + C_2e^{2x} + C_3e^{5x} + \tfrac{1}{4}xe^x + \tfrac{1}{5}x + \tfrac{17}{50};$$
$$y_2 = C_1e^x - 2C_2e^{2x} + C_3e^{5x} + \tfrac{1}{4}xe^x + \tfrac{6}{5}x + \tfrac{21}{25} - e^x;$$
$$y_3 = -C_1e^x - 3C_2e^{2x} + 3C_3e^{5x} - \tfrac{1}{4}xe^x - \tfrac{2}{5}x + \tfrac{1}{4}e^x + \tfrac{1}{50}.$$

Tasks for independent work

Find the solutions of the systems of differential equations

1. $\begin{cases} \frac{dx}{dt} = -9y; \\ \frac{dy}{dt} = x; \end{cases}$

2. $\begin{cases} \frac{dx}{dt} + 3x + 4y = 0, \quad x(0) = 1; \\ \frac{dy}{dt} + 2x + 5y = 0, \quad y(0) = 4; \end{cases}$

3. $\begin{cases} 4\frac{dx}{dt} - \frac{dy}{dt} + 3x = \sin t; \\ \frac{dx}{dt} + y = \cos t; \end{cases}$

4. $\begin{cases} \frac{d^2x}{dt^2} = y; \\ \frac{d^2y}{dt^2} = x; \end{cases}$

5. $\begin{cases} \frac{dx}{dt} = x + 5y, \quad x(0) = -2; \\ \frac{dy}{dt} = -x - 3y, \quad y(0) = 1. \end{cases}$

Method of integrable combinations

This method consists in the fact that, by various transformations, equations of the original system (3.1) are converted to a simple form. It allows to integrate them easily and obtain a solution of the system. The equations obtained with this way are called integrable combinations.

The integrable combination is obtained by adding, subtracting, multiplying or dividing the original equations of the system. Each integrable combination gives one first integral. If n independent first integrals of the system (3.1) are found, its integration is complete. If m independent first integrals are found, where $n > m$, then the system (3.1) is converted to a system with a less number of unknown functions.

Example. Find a solution of the Cauchy problem:

$$\begin{cases} \frac{dx}{dt} = \frac{x}{x+y}; \\ \frac{dy}{dt} = \frac{y}{x+y}, \end{cases} \qquad x(0) = 2, y(0) = 4.$$

Solution. This system is a normal system of two differential equations. We solve the system with the method of integrable combinations.

Find a general solution of the system. Compose a first integrable combination. Dividing the second equation by the first, we obtain a separable equation

$$\frac{dy}{dx} = \frac{y}{x}.$$

By separating the variables and integrating, we obtain

$$x = C_1 y.$$

Compose a second integrable combination. By adding both the equations of the system, we obtain

$$\frac{dx + dy}{dt} = 1,$$

$$dx + dy = dt,$$

$$x + y = t + C_2.$$

Therefore, we obtain the system: $\begin{cases} x = C_1 y; \\ x + y = t + C_2. \end{cases}$

The general solution of the system has the form:

$$\begin{cases} x = C_1 \frac{t+C_2}{1+C_1}; \\ y = \frac{t+C_2}{1+C_1}. \end{cases}$$

Find a particular solution of the system. Substitute the initial conditions into the general solution of the system:

$$\begin{cases} 2 = C_1 \frac{C_2}{1+C_1}; \\ 4 = \frac{C_2}{1+C_1} \end{cases},$$

where $C_1 = \frac{1}{2}, C_2 = 6$.

The solution of the Cauchy problem of original system is: $\begin{cases} x = \frac{t}{3} + 2; \\ y = \frac{2t}{3} + 4. \end{cases}$

Answer: $\begin{cases} x = \frac{t}{3} + 2; \\ y = \frac{2t}{3} + 4. \end{cases}$

Tasks for independent work

Find solutions of systems of differential equations

1. $\begin{cases} \frac{dx}{dt} = 2x + y, \quad x(0) = 1; \\ \frac{dy}{dt} = x + 2y, \quad y(0) = -1; \end{cases}$

2. $\begin{cases} \frac{dx}{dt} = \frac{x}{y}; \\ \frac{dy}{dt} = \frac{y}{x}; \end{cases}$

3. $\begin{cases} \frac{dx}{dt} = \frac{y}{x-y}; \\ \frac{dy}{dt} = \frac{x}{x-y}; \end{cases}$

4. $\begin{cases} \frac{dx}{dt} = 2x - y, \quad x(0) = -1; \\ \frac{dy}{dt} = -x + 2y, \quad y(0) = 3; \end{cases}$

5. $\begin{cases} \frac{dx}{dt} = x^2 + y^2; \\ \frac{dy}{dt} = 2xy. \end{cases}$

Euler method

The Euler method is applied for solving linear homogeneous systems of differential equations with constant coefficients. Consider a system of n linear homogeneous differential equations with n unknown functions whose coefficients are constant:

$$\begin{cases} \frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n; \\ \frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n; \\ \ldots \\ \frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n. \end{cases}$$

We write the system in the matrix form $\frac{dX}{dt} = A \cdot X$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}, \quad \frac{dX}{dt} = \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \dots \\ \frac{dx_n}{dt} \end{pmatrix}.$$

The solution of the system is obtained in the form:

$$\begin{cases} x_1 = p_1 e^{\lambda t}; \\ x_2 = p_2 e^{\lambda t}; \\ \quad \dots \\ x_n = p_n e^{\lambda t}, \end{cases}$$

where λ, p_i $(i = 1, 2, \dots, n)$ are constants.

By substituting the values of x_i $(i = 1, 2, \dots, n)$ into the system of differential equations, we obtain a system of linear algebraic equations with respect to p_i:

$$\begin{cases} (a_{11} - \lambda)\, p_1 + a_{12} p_2 + \dots + a_{1n} p_n = 0; \\ a_{21} p_1 + (a_{22} - \lambda)\, p_2 + \dots + a_{2n} p_n = 0; \\ \dots \\ a_{n1} p_1 + a_{n2} p_2 + \dots + (a_{nn} - \lambda)\, p_n = 0. \end{cases}$$

Since the system has a nonzero solution if and only if the determinant of the main matrix is equal to zero, we obtain the following equation of n-th degree:

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = 0.$$

This equation allows to find λ. It is the characteristic equation of the matrix A and, at the same time, the characteristic equation of the system. Let the characteristic equation have n different roots λ_i $(i = 1, 2, \dots, n)$, which are eigenvalues of the matrix A. There are corresponding eigenvectors to each eigenvalue.

Let an eigenvector $(p_{1k}; p_{2k}; \dots; p_{nk})$, where $k = 1, 2, ..., n$, correspond to an eigenvalue λ_k. Then the system of differential equations has n solutions:

1-st solution corresponding to the root $\lambda = \lambda_1$:

$$x_{11} = p_{11} e^{\lambda_1 t}, x_{21} = p_{21} e^{\lambda_1 t}, \dots, x_{n1} = p_{n1} e^{\lambda_1 t};$$

2-nd solution corresponding to the root $\lambda = \lambda_2$:

$$x_{12} = p_{12} e^{\lambda_2 t}, x_{22} = p_{22} e^{\lambda_2 t}, \dots, x_{n2} = p_{n2} e^{\lambda_2 t};$$

n-th solution corresponding to the root $\lambda = \lambda_n$:

$$x_{1n} = p_{1n}e^{\lambda_n t}, x_{2n} = p_{2n}e^{\lambda_n t}, \ldots, x_{nn} = p_{nn}e^{\lambda_n t}.$$

Thus, we obtain a fundamental set of solutions. The general solution of the system has the form:

$$\begin{cases} x_1 = C_1 x_{11} + C_2 x_{12} + \ldots + C_n x_{1n}; \\ x_2 = C_1 x_{21} + C_2 x_{22} + \ldots + C_{2n} x_{2n}; \\ \ldots \\ x_n = C_1 x_{n1} + C_2 x_{n2} + \ldots + C_n x_{nn}. \end{cases}$$

Example 1. Find the general solution of the system of differential equations

$$\begin{cases} \frac{dx_1}{dt} = -2x_2; \\ \frac{dx_2}{dt} = -2x_1. \end{cases}$$

Solution. This system is a linear homogeneous system of differential equations of second order with constant coefficients. Solve it with Euler method. We write the characteristic equation of the matrix of the system

$$\begin{vmatrix} 0-\lambda & -2 \\ -2 & 0-\lambda \end{vmatrix} = 0.$$

Its roots $\lambda_1 = -2; \lambda_2 = 2$ are eigenvalues of the matrix. There are equations for determining the eigenvector when $\lambda_1 = -2$:

$$\begin{cases} 2p_1 - 2p_2 = 0; \\ -2p_1 + 2p_2 = 0, \end{cases}$$

where $p_1 = p_2$, hence $(1; 1)$ is an eigenvector.

There are equations for determining an eigenvector when $\lambda_2 = 2$:

$$\begin{cases} -2p_1 - 2p_2 = 0; \\ -2p_1 - 2p_2 = 0, \end{cases}$$

where $p_1 = -p_2$, hence $(1; -1)$ is an eigenvector.

We obtain a set of fundamental solutions:
for $\lambda_1 = -2$: $x_{11} = e^{-2t}$, $x_{21} = e^{-2t}$;
for $\lambda_2 = 2$: $x_{12} = e^{2t}$, $x_{22} = -e^{2t}$.

The general solution of the system is: $\begin{cases} x_1 = C_1 e^{-2t} + C_2 e^{2t}; \\ x_2 = C_1 e^{-2t} - C_2 e^{2t}. \end{cases}$

Answer: $\begin{cases} x_1 = C_1 e^{-2t} + C_2 e^{2t}; \\ x_2 = C_1 e^{-2t} - C_2 e^{2t}. \end{cases}$

Example 2. Find the general solution of the system of differential equations

$$\begin{cases} \frac{dx_1}{dt} = x_1 - x_2 + x_3; \\ \frac{dx_2}{dt} = x_1 + x_2 - x_3; \\ \frac{dx_3}{dt} = 2x_1 - x_2. \end{cases}$$

Solution. This system is a linear homogeneous system of differential equations of the third order with constant coefficients. Solve it with the Euler method. We write the characteristic equation for the matrix of the system

$$\begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 2 & -1 & 0-\lambda \end{vmatrix} = 0,$$

where $\lambda_1 = -1$; $\lambda_2 = 1$; $\lambda_3 = 2$ are eigenvalues of the matrix.

There are equations for determining an eigenvector when $\lambda_1 = -1$:

$$\begin{cases} 2p_1 - p_2 + p_3 = 0; \\ p_1 + 2p_2 - p_3 = 0, \\ 2p_1 - p_2 + p_3 = 0; \end{cases} \quad \begin{cases} p_2 = -3p_1; \\ p_3 = -5p_1. \end{cases}$$

Hence, $(1; -3; -5)$ is an eigenvector.

There are equations for determining an eigenvector when $\lambda_2 = 1$:

$$\begin{cases} -p_2 + p_3 = 0; \\ p_1 - p_3 = 0; \\ 2p_1 - p_2 - p_3 = 0, \end{cases}$$

where $p_1 = p_2 = p_3$. Hence, $(1; 1; 1)$ is an eigenvector.

There are equations for determining an eigenvector when $\lambda_2 = 2$:

$$\begin{cases} -p_1 - p_2 + p_3 = 0; \\ p_1 - p_2 - p_3 = 0; \\ 2p_1 - p_2 - 2p_3 = 0; \end{cases} \quad \begin{cases} p_2 = 0 \\ p_3 = p_1 \end{cases}.$$

Hence, $(1; 0; 1)$ is an eigenvector.

We obtain a fundamental set of solutions:

for $\lambda_1 = -1$: $x_{11} = e^{-t}$,$x_{21} = -3e^{-t}$, $x_{31} = -5e^{-t}$;

for $\lambda_2 = 1$: $x_{12} = e^{t}$,$x_{22} = e^{t}$, $x_{32} = e^{t}$;

for $\lambda_3 = 2$: $x_{13} = 0$,$x_{23} = e^{t}$, $x_{33} = e^{2t}$.

The general solution of the system is:

$$\begin{cases} x_1 = C_1e^{-t} + C_2e^t + C_3e^{2t}; \\ x_2 = -3C_1e^{-t} + C_2e^t; \\ x_3 = -5C_1e^{-t} + C_2e^t + C_3e^{2t}. \end{cases}$$

Answer: $\begin{cases} x_1 = C_1e^{-t} + C_2e^t + C_3e^{2t}; \\ x_2 = -3C_1e^{-t} + C_2e^t; \\ x_3 = -5C_1e^{-t} + C_2e^t + C_3e^{2t}. \end{cases}$

Example 3. Find the general solution of the system of differential equations:

$$\begin{cases} \frac{dx_1}{dt} = 4x_1 - 3x_2; \\ \frac{dx_2}{dt} = 3x_1 + 4x_2. \end{cases}$$

Solution. This system is a linear homogeneous system of differential equations of the second order with constant coefficients. Solve it with the Euler method. We write the characteristic equation for the matrix of the system:

$$\begin{vmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = 0$$

Its roots $\lambda_{1,2} = 4 \pm 3i$ are complex eigenvalues of the matrix. Let us find eigenvectors for them.

There are equations for determining an eigenvector when $\lambda_1 = 4 + 3i$:

$$\begin{cases} 3i \cdot p_1 - 3p_2 = 0; \\ 3p_1 + 3i \cdot p_2 = 0, \end{cases}$$

where $p_2 = ip_1$. Hence, $(1; i)$ is an eigenvector.

There are equations for determining eigenvector when $\lambda_1 = 4 - 3i$:

$$\begin{cases} -3i \cdot p_1 - 3p_2 = 0; \\ 3p_1 - 3i \cdot p_2 = 0, \end{cases}$$

where $p_1 = ip_2$. Hence, $(1; -i)$ is an eigenvector.

We obtain a set of fundamental solutions:

for $\lambda_1 = 4 + 3i$:

$$x_{11} = e^{(4+3i)t} = e^{4t}(\cos 3t + i \sin 3t),$$

$$x_{21} = ie^{(4+3i)t} = e^{4t}(-\sin 3t + i \cos 3t);$$

for $\lambda_2 = 4 - 3i$:

$$x_{12} = e^{(4-3i)t} = e^{4t}(\cos 3t - i \sin 3t),$$

$$x_{22} = ie^{(4-3i)t} = e^{4t}(-\sin 3t - i\cos 3t).$$

The general solution of the system is:

$$\begin{cases} x_1 = e^{4t}\left((C_1 + C_2)\cos 3t + (C_1 - C_2)\, i \sin 3t\right) \\ x_2 = e^{4t}\left(-(C_1 + C_2)\sin 3t + (C_1 - C_2)\, i \cos 3t\right) \end{cases}.$$

By substitution $C_1 + C_2 = C_1^*$; $(C_1 - C_2)\, i = C_2^*$, we obtain

$$\begin{cases} x_1 = e^{4t}\left(C_1^* \cos 3t + C_2^* \sin 3t\right) \\ x_2 = e^{4t}\left(-C_1^* \sin 3t + C_2^* - \cos 3t\right) \end{cases}.$$

The general solution in the case of complex roots of the characteristic equation can be found by other way. We separate the real and imaginary parts in solutions corresponding to one of the complex characteristic numbers:

$$e^{(4+3i)t} = e^{4t}(\cos 3t + i\sin 3t) = e^{4t}\cos 3t + ie^{4t}\sin 3t;$$

$$ie^{(4+3i)t} = -e^{4t}\sin 3t + ie^{4t}\cos 3t.$$

We obtain linearly independent particular solutions:

$$x_{11} = e^{4t}\cos 3t;\ x_{21} = -e^{4t}\sin 3t;\ x_{12} = e^{4t}\sin 3t;\ x_{22} = e^{4t}\cos 3t.$$

The general solution is:

$$\begin{cases} x_1 = C_1 x_{11} + C_2 x_{12}; \\ x_2 = C_1 x_{21} + C_2 x_{22}; \end{cases} \quad \begin{cases} x_1 = e^{4t}(C_1 \cos 3t + C_2 \sin 3t); \\ x_2 = e^{4t}(-C_1 \sin 3t + C_2 - \cos 3t). \end{cases}$$

Answer: $\begin{cases} x_1 = e^{4t}(C_1 \cos 3t + C_2 \sin 3t); \\ x_2 = e^{4t}(-C_1 \sin 3t + C_2 - \cos 3t). \end{cases}$

Remark. Note that we do not consider the conjugate characteristic number because the solutions, corresponding to the root $a-bi$, are linearly dependent with the solutions for the root $a+bi$.

Tasks for independent work

Find solutions of the systems of differential equations:

1. $\begin{cases} \frac{dx}{dt} = 8y - x; \\ \frac{dy}{dt} = x + y; \end{cases}$

2. $\begin{cases} \frac{dx}{dt} = 2x - y + z \\ \frac{dy}{dt} = x + 2y - z; \\ \frac{dz}{dt} = x - y + 2z; \end{cases}$

3. $\begin{cases} \frac{dx}{dt} = x + y, \quad x(0) = 0; \\ \frac{dy}{dt} = 4y - 2x, \quad y(0) = -1; \end{cases}$

4. $\begin{cases} \frac{dx}{dt} = 4x - 5y, \quad x(0) = 0; \\ \frac{dy}{dt} = x, \quad y(0) = 1; \end{cases}$

5. $\begin{cases} \frac{dx}{dt} = 2x - y + z, \quad x(0) = 0; \\ \frac{dy}{dt} = x + z, \quad y(0) = 0; \\ \frac{dz}{dt} = y - 2z - 3x, \quad z(0) = 1. \end{cases}$

Lagrange method

The Lagrange method or the method of variation of arbitrary constants is used to solve linear nonhomogeneous systems of differential equations with constant coefficients.

Consider the idea of the Lagrange method on the example of a system of three nonhomogeneous differential equations with constant coefficients. There is a system of three linear nonhomogeneous differential equations with three unknown functions whose coefficients are constant:

$$\begin{cases} x' + a_1x + b_1y + e_1z = f_1(t); \\ y' + a_2x + b_2y + e_2z = f_2(t); \\ z' + a_3x + b_3y + e_3z = f_3(t). \end{cases} \tag{3.6}$$

Let the general solution of the corresponding homogeneous system be found and have the form:

$$\begin{cases} x = C_1x_1 + C_2x_2 + C_3x_3; \\ y = C_1y_1 + C_2y_2 + C_3y_3; \\ z = C_1z_1 + C_2z_2 + C_3z_3. \end{cases} \tag{3.7}$$

We find a solution of the nonhomogeneous system in the form

$$\begin{cases} x = C_1(t)x_1 + C_2(t)x_2 + C_3(t)x_3; \\ y = C_1(t)y_1 + C_2(t)y_2 + C_3(t)y_3; \\ z = C_1(t)z_1 + C_2(t)z_2 + C_3(t)z_3, \end{cases} \tag{3.8}$$

where $C_1(t)$, $C_2(t)$, $C_3(t)$ are unknown functions.

Substitute (3.8) into (3.6), then the first equation of the system (3.6) is converted to the form:

$$C_1'x_1 + C_2'x_2 + C_3'x_3 + C_1\left(x_1' + a_1x_1 + b_1y_1 + e_1z_1\right) +$$

$$+C_2\left(x_2' + a_1x_2 + b_1y_2 + e_1z_2\right) + C_3\left(x_3' + a_1x_3 + b_1y_3 + e_1z_3\right) = f_1(t)$$

All the expressions in brackets are equal to zero, because (3.7) is the solution of the corresponding homogeneous system. Then we obtain

$$C_1' x_1 + C_2' x_2 + C_3' x_3 = f_1(t).$$

Similarly, by substituting (3.8) into (3.6), the second and third equations of the system (3.6) are converted to the form:

$$C_1' y_1 + C_2' y_2 + C_3' y_3 = f_2(t);$$

$$C_1' z_1 + C_2' z_2 + C_3' z_3 = f_3(t).$$

Thus, we obtain a system of three linear equations with respect to C_1', C_2', C_3'. It has a solution because its determinant

$$\Delta = \begin{vmatrix} x_1 x_2 x_3 \\ y_1 y_2 y_3 \\ z_1 z_2 z_3 \end{vmatrix} \neq 0$$

due to the linear independence of the particular solutions of the corresponding homogeneous system.

Calculate C_1', C_2', C_3'. Then, by integrating these expressions, we obtain $C_1(t)$, $C_2(t)$, $C_3(t)$ and, consequently, the solution (3.8) of the nonhomogeneous system (3.6).

Example. Find the general solution of the system of differential equations

$$\begin{cases} \frac{dx}{dt} = y; \\ \frac{dy}{dt} = -x + \frac{1}{\cos t}. \end{cases}$$

Solution. This system is a linear nonhomogeneous system of differential equations of the second order with constant coefficients. Solve it with the Lagrange method.

First, we solve the corresponding homogeneous system: $\begin{cases} \frac{dx}{dt} - y = 0; \\ \frac{dy}{dt} + x = 0. \end{cases}$

From the second equation of the system we obtain:

$$x = -\frac{dy}{dt};$$

$$\frac{dx}{dt} = -\frac{d^2 y}{dt^2}.$$

Substitute these expressions into the first equation of the homogeneous system:

$$-\frac{d^2y}{dt^2} - y = 0;$$

$$\frac{d^2y}{dt^2} + y = 0.$$

We obtain a second-order linear homogeneous differential equation with constant coefficients. We write the characteristic equation for it

$$k^2 + 1 = 0.$$

Its roots are $k_{1,2} = \pm i$ – complex conjugate numbers. Therefore, the general solution of this homogeneous equation is

$$y = e^{0t}(C_1 \cos t + C_2 \sin t) = C_1 \cos t + C_2 \sin t.$$

Since $x = -\frac{dy}{dt}$, we obtain $x = C_1 \sin t - C_2 \cos t$. Thus, the general solution of the corresponding homogeneous system is:

$$\begin{cases} x = C_1 \sin t - C_2 \cos t; \\ y = C_1 \cos t + C_2 \sin t. \end{cases}$$

We find a general solution of the nonhomogeneous system in the form:

$$\begin{cases} x = C_1(t) \sin t - C_2(t) \cos t; \\ y = C_1(t) \cos t + C_2(t) \sin t. \end{cases}$$

Substitute these expressions and their derivatives into this nonhomogeneous system. After the transformations we obtain:

$$\begin{cases} C_1' \sin t - C_2' \cos t = 0; \\ C_1' \cos t + C_2' \sin t = \frac{1}{\cos t}; \end{cases} \qquad \begin{cases} C_2' = \operatorname{tg} t; \\ C_1' = 1. \end{cases}$$

By integrating, we obtain:

$$\begin{cases} C_2(t) = -\ln(\cos t) + C_2; \\ C_1(t) = t + C_1, \end{cases}$$

where C_1, C_2 are arbitrary constants. Substituting these values into the general solution of the nonhomogeneous system, we obtain:

$$\begin{cases} x = (t + C_1) \sin t + (\ln(\cos t) - C_2) \cos t; \\ y = (t + C_1) \cos t - (\ln(\cos t) - C_2) \sin t. \end{cases}$$

Answer: $\begin{cases} x = (t + C_1) \sin t + (\ln(\cos t) - C_2) \cos t; \\ y = (t + C_1) \cos t - (\ln(\cos t) - C_2) \sin t. \end{cases}$

Tasks for independent work

Find solutions of the systems of differential equations

1. $\begin{cases} \frac{dx}{dt} = -2x + y - e^{2t}; \\ \frac{dy}{dt} = -3x + 2y + 6e^{2t}; \end{cases}$

2. $\begin{cases} \frac{dx}{dt} = y + \operatorname{tg}^2 t - 1; \\ \frac{dy}{dt} = \operatorname{tg} t - x; \end{cases}$

3. $\begin{cases} \frac{dx}{dt} = y; \\ \frac{dy}{dt} = -x + \frac{1}{\cos t}; \end{cases}$

4. $\begin{cases} \frac{dx}{dt} = x + y - \cos t, \quad x(0) = 1; \\ \frac{dy}{dt} = -y - 2x + \cos t + \sin t, \quad y(0) = -2; \end{cases}$

5. $\begin{cases} \frac{dx}{dt} = -4x - 2y + \frac{2}{e^t - 1}; \\ \frac{dy}{dt} = 6x + 3y - \frac{3}{e^t - 1}. \end{cases}$

Method of undetermined coefficients

The method of undetermined coefficients is used to solve linear nonhomogeneous systems of differential equations with constant coefficients (3.2) when the functions $f_i\,(x)(i = 1, \ldots, n)$ on the right-hand side of the system have a special form: polynomials, exponential functions, sine, cosines, and the sum or the product of these functions. A particular solution of the nonhomogeneous system is found depending on the type of the right hand side.

The method of undetermined coefficients is based on the following theorem: the general solution of a linear nonhomogeneous system (3.2) is equal to the sum of the general solution of the corresponding homogeneous system and any particular solution of this nonhomogeneous system.

Example 1. Find the general solution of the system of differential equations

$$\begin{cases} \frac{dx}{dt} = x + 2y; \\ \frac{dx}{dt} = x - 5\sin t. \end{cases}$$

Solution. This system is a linear nonhomogeneous system of differential equations of the second order with constant coefficients. Let us solve it with the method of undetermined coefficients.

First, we find the general solution of the corresponding homogeneous system with the Euler method:

$$\begin{cases} \frac{dx}{dt} - x - 2y = 0; \\ \frac{dy}{dt} - x = 0. \end{cases}$$

We write the characteristic equation of the matrix of the system

$$\begin{vmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = 0; \quad \lambda^2 - \lambda - 2 = 0.$$

Its roots (the eigenvalues of the matrix) $\lambda_1 = -1; \quad \lambda_2 = 2$ are real numbers. The general solution of the homogeneous system is:

$$\begin{cases} x^g = C_1 e^{-t} + 2C_2 e^{2t}; \\ y^g = -C_1 e^{-t} + C_2 e^{2t}. \end{cases}$$

We find a particular solution of the nonhomogeneous system in the form (because $f_1(t) = 0$, $f_2(t) = -5\sin t$):

$$x^* = A\cos t + B\sin t;$$
$$y^* = C\cos t + D\sin t.$$

Substitute these expressions and their derivatives into this nonhomogeneous system:

$$-A\sin t + B\cos t = A\cos t + B\sin t + 2(C\cos t + D\sin t);$$
$$-C\sin t + D\cos t = A\cos t + B\sin t - 5\sin t.$$

By equating the coefficients before $\cos t$ and $\sin t$ in both the sides of the equations, we obtain:

$$\begin{cases} -A = B + 2D; \\ B = A + 2C; \\ -C = B - 5; \\ D = A, \end{cases}$$

where $A = -1$, $B = 3$, $C = 2$, $D = -1$.

Then the particular solution of the nonhomogeneous system is:

$$x^* = -\cos t + 3\sin t;$$
$$y^* = 2\cos t - \sin t.$$

We obtain the general solution of the original nonhomogeneous system as the sum of the general solution of the corresponding homogeneous system and the particular solution of this nonhomogeneous system:

$$\begin{cases} x = x^g + x^* = C_1 e^{-t} + 2C_2 e^{2t} - \cos t + 3\sin t; \\ y = y^g + y^* = -C_1 e^{-t} + C_2 e^{2t} + 2\cos t - \sin t. \end{cases}$$

$$Answer: \begin{cases} x = x^g + x^* = C_1 e^{-t} + 2C_2 e^{2t} - \cos t + 3\sin t; \\ y = y^g + y^* = -C_1 e^{-t} + C_2 e^{2t} + 2\cos t - \sin t. \end{cases}$$

Example 2. Find the general solution of the system of differential equations

$$\begin{cases} \frac{dx}{dt} = y + t; \\ \frac{dy}{dt} = -2x + 3y - t^2. \end{cases}$$

Solution. This system is a linear nonhomogeneous system of differential equations of the second order with constant coefficients. Let us solve it with the method of undetermined coefficients.

First, we find the general solution of the corresponding homogeneous system with the Euler method:

$$\begin{cases} \frac{dx}{dt} = y; \\ \frac{dy}{dt} = -2x + 3y. \end{cases}$$

We write the characteristic equation of the matrix of the system:

$$\begin{vmatrix} 0-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = 0; \quad \lambda^2 - 3\lambda + 2 = 0.$$

Its roots (the eigenvalues of the matrix): $\lambda_1 = 1; \lambda_2 = 2$.

There are equations for determining an eigenvector when $\lambda_1 = 1$:

$$\begin{cases} -p_1 + p_2 = 0; \\ -2p_1 + 2p_2 = 0, \end{cases}$$

where $p_2 = 2p_1$. Therefore, we obtain an eigenvector $(1;2)$.

We obtain the set of fundamental solutions:
for $\lambda_1 = 1$: $x_{11} = e^t$, $x_{21} = e^t$;
for $\lambda_2 = 2$: $x_{12} = e^{2t}$, $x_{22} = 2e^{2t}$.

The general solution of the homogeneous system is:

$$\begin{cases} x^g = C_1 e^t + C_2 e^{2t}; \\ y^g = C_1 e^t + 2C_2 e^{2t}. \end{cases}$$

We find a particular solution of the nonhomogeneous system in the form (because $f_1(t) = t$, $f_2(t) = -t^2$):

$$x^* = At^2 + Bt + C;$$
$$y^* = Dt^2 + Et + F.$$

Substitute these expressions and their derivatives into this nonhomogeneous system:

$$2At + B = Dt^2 + Et + F + t;$$

$$2Dt+E=-2\left(At^2+Bt+C\right)+3\left(Dt^2+Et+F\right)-t^2.$$

$$2At+B=Dt^2+(E+1)t+F;$$

$$2Dt+E=(-2A+3D-1)t^2+(-2B+3E)t-2C+3F.$$

By equating the coefficients at the same degrees of the unknown t in both the sides of the equations, we obtain:

$$\begin{cases} 0=D; \\ 2A=E+1; \\ B=F; \\ 0=-2A+3D-1; \\ 2D=-2B+3E; \\ E=-2C+3F, \end{cases}$$

where $A=-\frac{1}{2}$, $B=F=-3$, $C=-\frac{7}{2}$, $D=0$, $E=-2$.

Then the particular solution of the nonhomogeneous system is:

$$\begin{cases} x^* = -\frac{1}{2}t^2-3t-\frac{7}{2}; \\ y^* = -2t-3. \end{cases}$$

We obtain the general solution of the original nonhomogeneous system as the sum of the general solution of the corresponding homogeneous system and a particular solution of this nonhomogeneous system:

$$\begin{cases} x=x^g+x^*=C_1e^t+C_2e^{2t}-\frac{1}{2}t^2-3t-\frac{7}{2}; \\ y=y^g+y^*=C_1e^t+2C_2e^{2t}-2t-3. \end{cases}$$

Answer: $\begin{cases} x=C_1e^t+C_2e^{2t}-\frac{1}{2}t^2-3t-\frac{7}{2}; \\ y=C_1e^t+2C_2e^{2t}-2t-3. \end{cases}$

Tasks for independent work

Find solutions of systems of differential equations

1. $\begin{cases} \frac{dx}{dt}=3-2y; \\ \frac{dy}{dt}=2x-2t; \end{cases}$

2. $\begin{cases} \frac{dx}{dt}=4x-5y+4t-1, \quad x(0)=0; \\ \frac{dy}{dt}=x-2y+t, y(0)=0; \end{cases}$

3. $\begin{cases} \frac{dx}{dt}+y=t^2; \\ \frac{dy}{dt}-x=t; \end{cases}$

4. $\begin{cases} \frac{dx}{dt}=-y+\sin t; \\ \frac{dy}{dt}=x+\cos t; \end{cases}$

5. $$\begin{cases} \dfrac{dx}{dt} + x + 2y = 2e^{-t}, x(0) = 1; \\ \dfrac{dy}{dt} + y + z = 1, \quad y(0) = 1; \\ \dfrac{dz}{dt} + z = 1, \quad z(0) = 1. \end{cases}$$

Глава 4

Tasks leadind to differential equations

Task 1. A boat moved on a lake at a speed of 32 km/h and 1 minute, after the engine was turned off, its speed became equal to 8 km/h. What speed of the boat will be in 2 minutes after stopping the engine, if the water resistance is proportional to the speed of the boat? What the distance will it pass in 1 minute after turning off the motor? What is the distance will it pass in 2 minutes after turning off the engine?

Solution. Let v be the speed of the boat and k is the proportionality factor. According to the condition of the problem, the force acting on the moving boat is equal $F = -k \cdot v$. On the other hand, according to Newton's second law, this force is $F = m \cdot \frac{dv}{dt}$, where m is the mass and $\frac{dv}{dt}$ is the acceleration. Therefore,

$$m \cdot \frac{dv}{dt} = -k \cdot v \tag{4.1}$$

is a differential equation (mathematical model) describing the motion of boat. By separating the variables and integrating, we obtain

$$\frac{dv}{v} = -\frac{k}{m} \cdot dt,$$

$$\ln|v| = \ln e^{-\frac{k}{m}t} + \ln C.$$

The general solution of the differential equation (4.1) is:

$$v = C \cdot e^{-\frac{k}{m}t}. \tag{4.2}$$

Since at time $t = 0$ sec. the speed of the boat was $v = 32$ km/h, and after a minute, i.e. at $t = 1$ min$= \frac{1}{60}$h it was $v = 8$ km/h, from the general solution (4.2), we obtain: 32 = C and $8 = C \cdot e^{-\frac{k}{m} \cdot \frac{1}{60}}$.

Therefore, $8 = 32 \cdot e^{-\frac{k}{m} \cdot \frac{1}{60}}$, i.e. $4^{-1} = e^{-\frac{k}{m} \cdot \frac{1}{60}}$ or $e^{-\frac{k}{m}} = 4^{-60}$. By substituting it into the (4.2), we obtain:

$$v = 32 \cdot 4^{-60t}. \tag{4.3}$$

At $t = 2\text{min} = \frac{1}{30}\text{h}$ we obtain from (4.3)

$$v = 32 \cdot 4^{-60 \cdot \frac{1}{30}} = 32 \cdot 4^{-2} = 2.$$

Thus, the speed of the boat in 2 minutes after turning off the engine is equal to 2 km/h.

Denote by S the distance, which the boat passed after turning off the engine. Obviously, it depends on time, i.e. $S = S(t)$, and at the moment of turning off the engine $S(0) = 0$. Since the speed is the derivative of the distance with respect to the time, using the formula (4.3), we obtain

$$S' = 32 \cdot 4^{-60t}.$$

By integrating, and taking into account that $S(0) = 0$, we obtain

$$S = \int_0^t 32 \cdot 4^{-60x} dx = -\frac{32}{60} \int_0^t 4^{-60x} d(-60x) = -\frac{8}{15} \frac{4^{-60x}}{\ln 60} \Big|_0^t = \frac{8}{15 \cdot \ln 60} \left[1 - 4^{-60t}\right].$$

For simplicity of calculation we pat $\ln 60 = 4$. Therefore, from the preceding equality we obtain:

$$S = \frac{2}{15} \left[1 - 4^{-60t}\right]. \tag{4.4}$$

At $t = 1$ min $= \frac{1}{60}$ h, from (4.4) obtain

$$S = \tfrac{2}{15} \left[1 - 4^{-1}\right] = \tfrac{2}{15} \cdot \tfrac{3}{4} = \tfrac{1}{10} \text{ km},$$

i.e., in a minute after turning off the engine, the boat passes 100 meters.

At $t =$min$= \frac{1}{30}$ h from (4.4) we obtain

$$S = \tfrac{2}{15} \left[1 - 4^{-2}\right] = \tfrac{2}{15} \cdot \tfrac{15}{16} = \tfrac{1}{8}\text{km},$$

i.e., in 2 minutes after turning off the engine, the motorboat passes 125 meters.

Answer: In 2 minutes after turning off the engine the speed of the boat is 2 km/h and it passes the distance of 125 meters, and in 1 minute after turning off the engine it will move the distance of 100 meters.

Task 2. Within 20 minutes, the temperature of a bread taken out of an oven and placed in a storehouse falls from 100° to 60°. The air temperature in the storehouse is equal to 20°. How long from the moment of cooling the temperature of the bread will drop to 1) 40°? 2) 30°?

Solution. Denote by S the temperature of the bread. By the condition of the problem it depends on time t, i.e. $S = S(t)$. Since, by Newton's law, the rate of heat loss of a body is directly proportional to the difference in the temperatures between the body and its surroundings, we obtain the differential equation:

$$\frac{dS}{dt} = k \cdot (S-20), \tag{4.5}$$

where k is a proportionality factor. By separating the variables, from (4.5) we have

$$\frac{dS}{S-20} = k \cdot dt,$$

and, by integrating, we obtain

$$\ln|S-20| = k \cdot t + \ln 20; S-20 = C \cdot e^{k \cdot t}.$$

According to the condition of the problem, at $t = 0$ we obtain $S = 100$. Therefore, from the preceding equality we find C: $100-20 = C \cdot e^0$, i.e. $C = 80$. Therefore,

$$S-20 = 80 \cdot e^{k \cdot t}. \tag{4.6}$$

According to the condition of the problem, at $t = 20$ we have $S = 60$. Therefore, from (4.6) we obtain

$$60-20 = 80 \cdot e^{k \cdot 20}, \text{ i.e. } e^k = \left(\frac{1}{2}\right)^{\frac{1}{20}}.$$

Thus, from (4.6) we deduce that

$$S-20 = 80 \cdot \left(\frac{1}{2}\right)^{\frac{t}{20}}. \tag{4.7}$$

From the formula (4.7) at $S = 40$ we obtain $t = 40$, and at $S = 30$ we obtain $t = 60$.

Answer: The temperature of the bread will drop to 40° in 40 minutes and to 30° in 60 minutes.

Task 3. A motorboat moves in a calm water at a speed of $v_0 = 20 = 20$km/h. At full speed the engine is turned off and after 40 seconds the boat speed is reduced to $v_1 = 8$km/h. The water resistance is proportional to the speed of the boat. Determine the speed of the boat in 2 minutes after stopping the engine.

Solution. There is the water resistance force $F = -kv$ acting on the moving motorboat, where $k > 0$ is a proportionality factor. On the other hand, according to Newton's second law we have $F = ma$ and, hence, $ma = -kv$ or $v' = -\frac{k}{m}v$.

We solve this differential equation by separating the variables. By integrating, we obtain:

$$\frac{dv}{dt} = -\frac{k}{m}v;$$

$$\frac{dv}{v} = -\frac{k}{m}dt;$$

$$\int \frac{dv}{v} = -\frac{k}{m}\int dt;$$

$$\ln v = -\frac{k}{m}\cdot t + \ln C;$$

$$\ln v - \ln C = -\frac{k}{m}\cdot t;$$

$$\ln\frac{v}{C} = -\frac{kt}{m}.$$

Thus,we have $\frac{v}{C} = e^{-\frac{kt}{m}}$; $v = C\cdot e^{-\frac{kt}{m}}$.

Find C, using the initial condition $v_0 = 20$km/h at $t = 0$: $20 = C\cdot e^0$; $C = 20$. Therefore,

$$v = 20\cdot e^{-\frac{kt}{m}}.$$

Now, using an additional condition $t = 40$c= $\frac{1}{90}$h $v = 8$km/h, we obtain $8 = 20e^{-\frac{k}{m}\cdot\frac{1}{90}}$ or $e^{-\frac{k}{m}} = \left(\frac{5}{2}\right)^{90}$.

Therefore, $v = 20\cdot\left(\frac{5}{2}\right)^{-90t}$. Hence, the desired speed is:

$$v = 20\cdot\left(\frac{5}{2}\right)^{-90\cdot\frac{1}{30}} = 20\cdot\left(\frac{5}{2}\right)^{-3} = 20\cdot\frac{8}{125} = \frac{32}{25} \approx 1,28(\text{km/h}).$$

Answer: the speed of the motorboat in 2 minutes after the stopping will be equal to $1,28$ km/h.

Task 4. What amount will be in a bank account in five hundred years, if today you open a savings deposit of one ruble at 55% per annum with continuous accrual of interest?

Solution. Let $a(t)$ be the amount of the deposit at the time t. Then, by solving the Cauchy problem

$$a' = 0,05a,\ a(0) = t,$$

we obtain

$$a = e^{0,05t}.$$

By substituting $t = 500$, we obtain the answer

$$a(500) = e^{25} \approx 72004899337 \text{ rubles.}$$

Answer: 72004899337 rubles will be on the account in five hundred years.

Task 5. A load of 320 g is suspended on a light spring and removed from the resting state by stretching the spring by 8 cm, while experience the power elasticity is 1 H. Then the load was thrown vertically upwards, giving it the initial velocity of 0.5 m/s. Find the period and the amplitude of the free fluctuation of the load if the movement is without resistance.

Solution. Denote by $s(t)$ the deviation of the load from the equilibrium position. Vibrations are caused by the elastic force $F_e = ks$. Since when the lengthening of the spring by 8 cm the elastic force is $1H$, the stiffness factor $k = \frac{1}{0{,}08} = 12{,}5$ kg/s^2. Write down Newton's second law $ma = -ks$. Then we obtain a linear homogeneous equation of the second order

$$0{,}32s'' + 12{,}5s = 0.$$

The characteristic equation

$$0{,}32\lambda^2 + 12{,}5 = 0$$

has roots $\lambda_{1,2} = \pm 6{,}25i$, therefore, the general solution is written in the form

$$s = c_1 \cos 6{,}25t + c_2 \sin 6{,}25t.$$

For finding constants, note that

$$s(0) = c_1 = 0{,}08 \text{ and } v(0) = s'(0) = 6{,}25c_2 = 0{,}5.$$

Therefore, $c_1 = 0{,}08$, $c_2 = 0{,}08$ and the equation of the oscillatory motion has the form $s = 0{,}08\cos 6{,}25t + 0{,}08\sin 6{,}25t$, or, if we enter an additional angle,

$$s = 0{,}08\sqrt{2}\sin\left(6{,}25t + \frac{\pi}{4}\right).$$

Thus, the amplitude of the free fluctuation equals $0{,}08\sqrt{2} \approx 0{,}113$m, and the period is $\frac{2\pi}{6{,}25} \approx 1$s.

Answer: the period is 1 s; the amplitude is 11 sm 3 mm.

Глава 5

Solving ordinary differential equations and systems in SCM Maple

Maple allows to solve differential equations of different types. You can use the following commands in Maple to do it: **dsolve**, **pdsolve**; packages: **DEtools** (solution of ordinary differential equations) and **PDEtools** (solution of partial differential equations). In addition, there are many functions that allow to classify differential equations, to make substitutions in differential equations, to transform solutions, to visualize the solutions (to plot graphs of different types).

The command **diff** is used in writing of differential equations to denote the derivative

diff(y(x),x) – the first derivative over x from the function $y(x)$;

diff(y(x),x$n) – n-th derivative over x from the function $y(x)$.

For example, the differential equation of the form $y'' + 2y' + 5y = 0$ is written as

> diff(y(x),x$2)+2*diff(y(x),x)+5*y(x)=0.

The command **dsolve** is used to solve ordinary differential equations and systems of differential equations in different forms of writing:

dsolve(ODE,y(x));

dsolve(ODE, y(x), options);

dsolve({ODE, ICs}, y(x), options);

dsolve({sys},{x(t), y(t), ... });

dsolve({sys,ICs},{x(t), y(t), ... }, options);

where **ODE** – ordinary differential equation, **y(x)** – any indeterminate function of one variable, representing the unknown of the ODE problem, **ICs** – initial conditions, **x(t), y(t)** – a set or list of functions of one variable, representing the unknowns of the ODE problem, **options** – option specifying the method of solution, for example:

type=exact (analytical solution);

type=series (approximate solution in the form of a power series);

type=numeric (numerical solution);

output=basis (set of fundamental solutions);

method=laplace (solution using integral transforms).

The general solution of the differential equation depends on arbitrary constants, the number of which is equal to the order of the differential equation. In Maple, such constants are usually denoted as **_C1**, **_C2**, and so on.

For solving the Cauchy problem it is necessary to include the initial conditions in the parameters **dsolve** , and the differential equation and the expression that specifies the initial conditions are specified in round brackets:
dsolve({ODE, ICs}, y(x)).

Derivatives in the initial conditions are written using the differential operator **D**: **D(y)(x_0), (D@@n)(y)(x_0)**.

In the output line, the solution of an nonhomogeneous linear ODE always consists of summands that contain arbitrary constants (this is the general solution of the corresponding homogeneous ODE), and summands without arbitrary constant (this is a particular solution of the same nonhomogeneous ODE).

To check the obtained solution, use the command **odetest(sol,ODE)**, where **sol** is the solution of the equation.

The command **dsolve** gives the solution of the differential equation in the non-computable form. If you want to work with solution further (for example, plot graph of solution) It is necessary to separate the right-hand side of the solution by command **rhs(%)**.

Example 1. Find the general solution of the differential equation

$$y' + y\cos x = \sin x \cos x.$$

```
de1:=diff(y(x),x)+y(x)*cos(x)=sin(x)*cos(x);
```

$$de1 := \frac{d}{dx} y(x) + y(x)\cos(x) = \sin(x)\cos(x)$$

```
dsolve(de1,y(x));
```

$$y(x) = \sin(x) - 1 + e^{-\sin(x)}\_C1$$

Example 2. Find the solution of the Cauchy problem

$$y^{(4)} + y'' = 2\cos x, y(0) = -2, y'(0) = -2, y''(0) = 0, y'''(0) = 0.$$

```
de2:=diff(y(x),x$4)+diff(y(x),x$2)=2*cos(x);
```

$$de2 := \frac{d^4}{dx^4}y(x) + \frac{d^2}{dx^2}y(x) = 2\cos(x)$$

```
init2:=y(0)=-2,D(y)(0)=1, (D@@2)(y)(0)=0,(D@@3)(y)(0)=0;
```

$$init2 := y(0) = -2, D(y)(0) = 1, D^{(2)}(y)(0) = 0, D^{(3)}(y)(0) = 0$$

```
sol2:=dsolve({de2,init2},y(x));
```

$$sol2 := y(x) = -2\cos(x) - \sin(x)\,x + x$$

```
odetest(sol2,de2);
```

$$0$$

Example 3. Find the general solution of the differential equation of the 2-nd order $y'' - 2y' + y = \sin x + e^{-x}$.

```
de3:=diff(y(x),x$2)-2*diff(y(x),x)+y(x)=sin(x)+exp
(-x);
```

$$de3 := \frac{d^2}{dx^2}y(x) - 2\left(\frac{d}{dx}y(x)\right) + y(x) = \sin(x) + e^{-x}$$

```
dsolve(de3,y(x));
```

$$y(x) = e^x\_C2 + e^x x\_C1 + \frac{1}{4}e^{-x}\left(2\cos(x)\,e^x + 1\right)$$

Example 4. Find the set of fundamental solutions of the differential equation

$$y^{(4)} + 2y'' + y = 0.$$

```
de4:=diff(y(x),x$4)+2*diff(y(x),x$2)+y(x)=0;
```

$$de4 := \frac{d^4}{dx^4}y(x) + 2\left(\frac{d^2}{dx^2}y(x)\right) + y(x) = 0$$

```
dsolve(de4,y(x),output=basis);
```

$$[\cos(x), \sin(x), x\cos(x), x\sin(x)]$$

Example 5. Find the solution of the Cauchy problem $y^{(4)} + y'' = 2x\sin x$, $y(0) = -2$, $y'(0) = -2$, $y''(0) = 0$, $y'''(0) = 0$ and plot its graph.

```
de5:=diff(y(x),x$4)+diff(y(x),x$2)=2*x*sin(x);
```

$$de5 := \frac{d^4}{dx^4}\,y(x) + \frac{d^2}{dx^2}\,y(x) = 2\sin(x)\,x$$

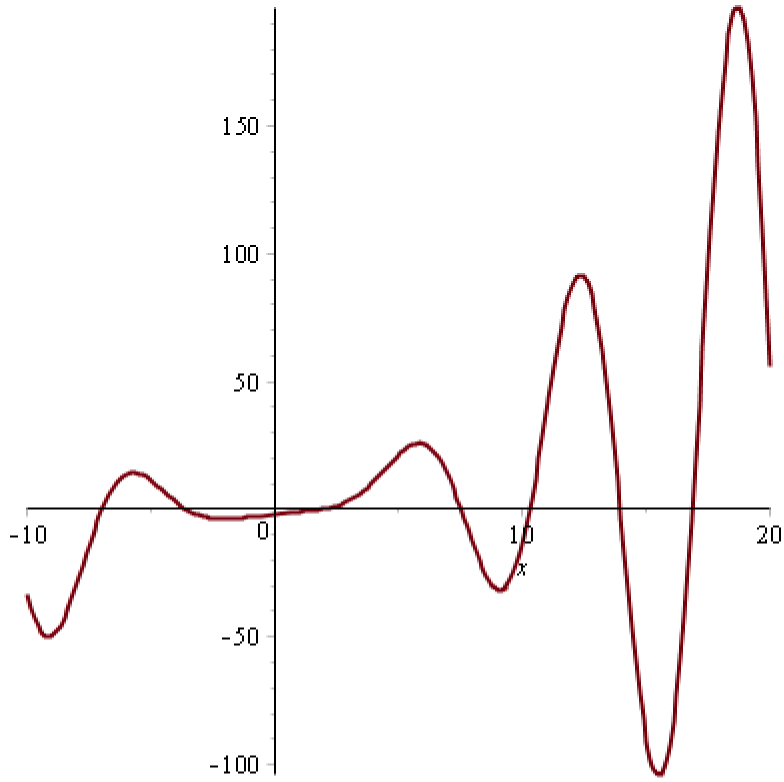
```
init5:=y(0)=-2,D(y)(0)=1, (D@@2)(y)(0)=0,(D@@3)(y)(0)=0;
```

$$init5 := y(0) = -2, D(y)(0) = 1, D^{(2)}(y)(0) = 0, D^{(3)}(y)(0) = 0$$

```
dsolve({de5,init5},y(x));
```

$$y(x) = 2 - 4\cos(x) - \frac{5}{2}\sin(x)\,x + \frac{1}{2}\cos(x)\,x^2 + x$$

```
y5:=rhs(%):
plot(y5,x=-10..20,thickness=2);
```



Example 6. Find the solution of the system of differential equations

$$\begin{cases} x' = -4x - 2y + \frac{2}{e^t - 1}; \\ y' = 6x + 3y - \frac{3}{e^t - 1}. \end{cases}$$

```
> sys:=diff(x(t),t)=-4*x(t)-2*y(t)+2/(exp(t)-1),diff(y(t),t)
  =6*x(t)+3*y(t)-3/(exp(t)-1):
> solsys:=dsolve({sys},{x(t),y(t)});
```

$$solsys := \left\{x(t) = \frac{2 + 2\ln(e^t - 1) - \_C1}{e^t} + \_C2, y(t) = -2\_C2 - 3e^{-t}\ln(e^t - 1) + \frac{3}{2}e^{-t}\_C1 - 3e^{-t}\right\}$$

```
> odetest(solsys,{sys});
```

$$\{0\}$$

```
>
```

Consider the package **DEtools** which contains commands for research of ODE, visualization of solutions of ODE, transformations, etc.

```
> with(DEtools);
```

[*AreSimilar, Closure, DEnormal, DEplot, DEplot3d, DEplot_polygon, DFactor, DFactorLCLM, DFactorsols, Dchangevar, Desingularize, FunctionDecomposition, GCRD, Gosper, Heunsols, Homomorphisms, IVPsol, IsHyperexponential, LCLM, MeijerGsols, MultiplicativeDecomposition, ODEInvariants, PDEchangecoords, PolynomialNormalForm, RationalCanonicalForm, ReduceHyperexp, RiemannPsols, Xchange, Xcommutator, Xgauge, Zeilberger, abelsol, adjoint, autonomous, bernoullisol, buildsol, buildsym, canoni, caseplot, casesplit, checkrank, chinisol, clairautsol, constcoeffsols, convertAlg, convertsys, dalembertsol, dcoeffs, de2diffop, dfieldplot, diff_table, diffop2de, dperiodic_sols, dpolyform, dsubs, eigenring, endomorphism_charpoly, equinv, eta_k, eulersols, exactsol, expsols, exterior_power, firint, firtest, formal_sol, gen_exp, generate_ic, genhomosol, gensys, hamilton_eqs, hypergeomsols, hyperode, indicialeq, infgen, initialdata, integrate_sols, intfactor, invariants, kovacicsols, leftdivision, liesol, line_int, linearsol, matrixDE, matrix_riccati, maxdimsystems, moser_reduce, muchange, mult, mutest, newton_polygon, normalG2, ode_int_y, ode_y1, odeadvisor, odepde, parametricsol, particularsol, phaseportrait, poincare, polysols, power_equivalent, rational_equivalent, ratsols, redode, reduceOrder, reduce_order, regular_parts, regularsp, remove_RootOf, riccati_system, riccatisol, rifread, rifsimp, rightdivision, rtaylor, separablesol, singularities, solve_group, super_reduce, symgen, symmetric_power, symmetric_product, symtest, transinv, translate, untranslate, varparam, zoom*]

odeadvisor – classify ODE and suggest solution methods;

```
> with(DEtools):
> ode1:=x*diff(y(x),x)+3*(y(x))^2-y(x)+7*x^2;
```

$$ode1 := x\left(\frac{d}{dx}y(x)\right) + 3\,y(x)^2 - y(x) + 7\,x^2$$

```
> odeadvisor(ode1);
```

[[*_homogeneous, class D*], *_rational, _Riccati*]

```
> ode2:=diff(y(x),x)=(1-y(x))/x;
```

$$ode2 := \frac{d}{dx}y(x) = \frac{1-y(x)}{x}$$

```
> odeadvisor(ode2);
```

[*_separable*]

```
> ode3:=diff(y(x),x)+2*x*y(x)=2*x*exp(-x^2);
```

$$ode3 := \frac{d}{dx} y(x) + 2 x y(x) = 2 x e^{-x^2}$$

```
> odeadvisor(ode3);
```

[_linear]

```
> ode4:=diff(y(x),x$4)-y(x)=10*cos(x);
```

$$ode4 := \frac{d^4}{dx^4} y(x) - y(x) = 10 \cos(x)$$

```
> odeadvisor(ode4);
```

[[_high_order, _linear, _nonhomogeneous]]

DEplot – plots solutions of differential equations or systems of differential equations, as well as phase portraits and field directions. This command is similar to the command **odeplot** , but is more functional.

The most commonly used graphical parameters of command **DEplot** are:

linecolor – color of the line ;

color – color of the arrows;

arrows=SMALL, MEDIUM, LARGE, LINE or **NONE** Џ type of arrows;

stepsize – number equal to the distance between points on the graph;

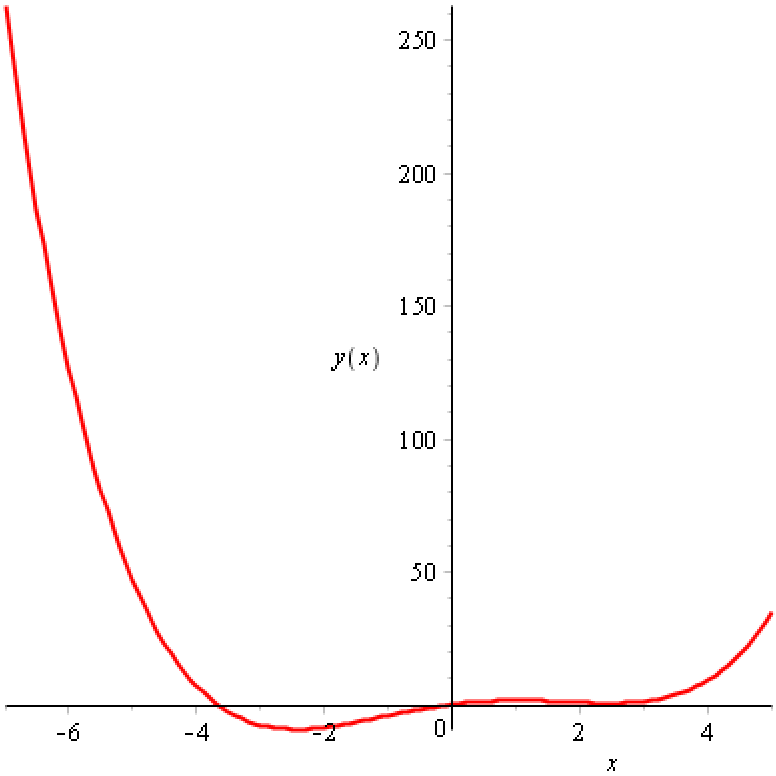
obsrange = true/false – indicates whether the integrator should stop once the solution curve has passed outside the specified range.

The graph of solution of the Cauchy problem (integral curve) with the command **DEplot**

```
> restart:
> with(DEtools):
> ode:=diff(y(x),x$2)-(diff(y(x),x))/(x-1)=x*(x-1);
```

$$ode := \frac{d^2}{dx^2} y(x) - \frac{\frac{d}{dx} y(x)}{x-1} = x (x-1)$$

```
> DEplot(ode,{y(x)},x=-7..5,[[y(2)=1,D(y)(2)=-1]],stepsize=.1,
  linecolor=red,thickness=2);
```

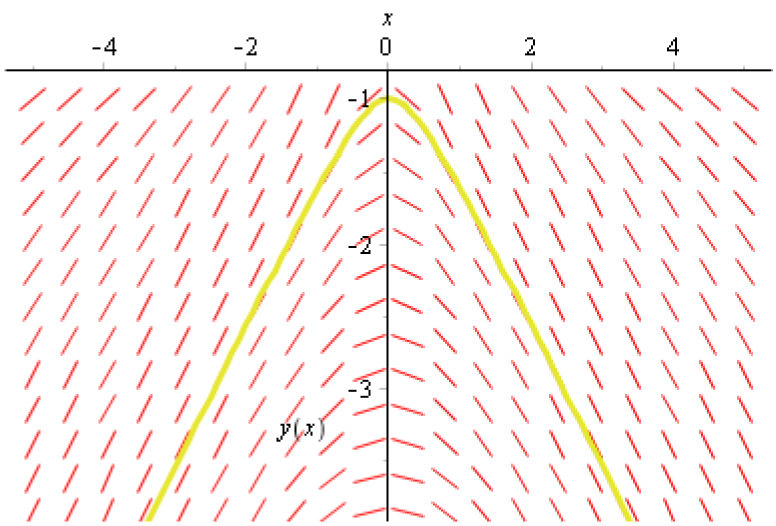


The graph of solution of the Cauchy problem (integral curve and directions field) with the command **DEplot**

```
> with(DEtools):
> ode:=diff(y(x),x)=2*x*y(x)/(x^2+(y(x))^2);
```

$$ode := \frac{d}{dx}\, y(x) = \frac{2\, x\, y(x)}{x^2 + y(x)^2}$$ (

```
> DEplot(ode,{y(x)},x=-5..5,[[y(0)=-1]],stepsize=.1,arrows=LINE);
```

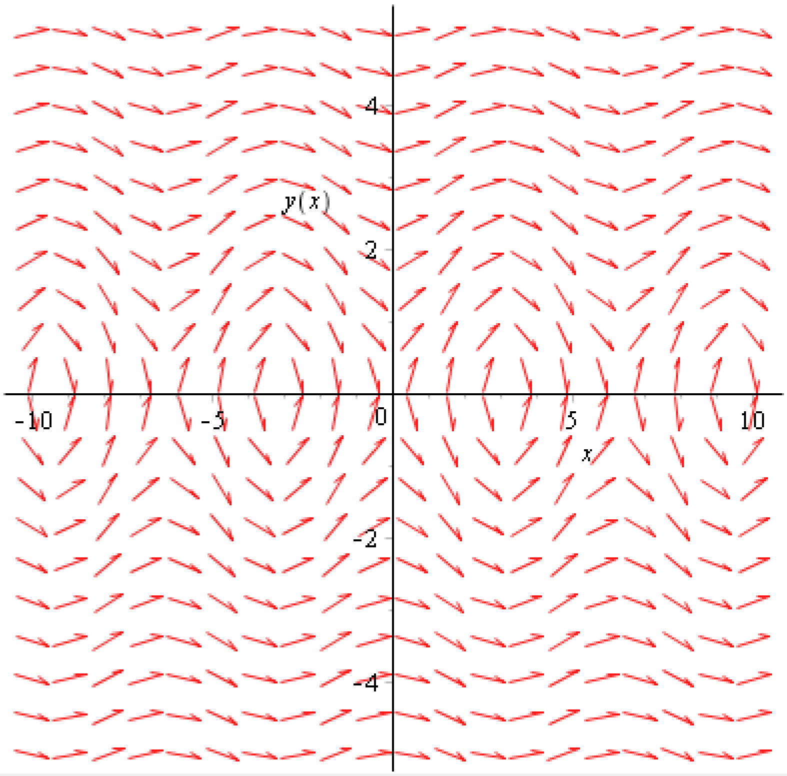


Direction field of nonlinear ODE of the first order with the command **DEplot**

```
> restart:
> with(DEtools):
> nonlin_ode:=diff(y(x),x)=sin(x)/y(x);
```

$$nonlin\_ode := \frac{d}{dx} y(x) = \frac{\sin(x)}{y(x)}$$

```
> DEplot(nonlin_ode,y(x),x=-10..10,y=-5..5,stepsize=0.05);
```

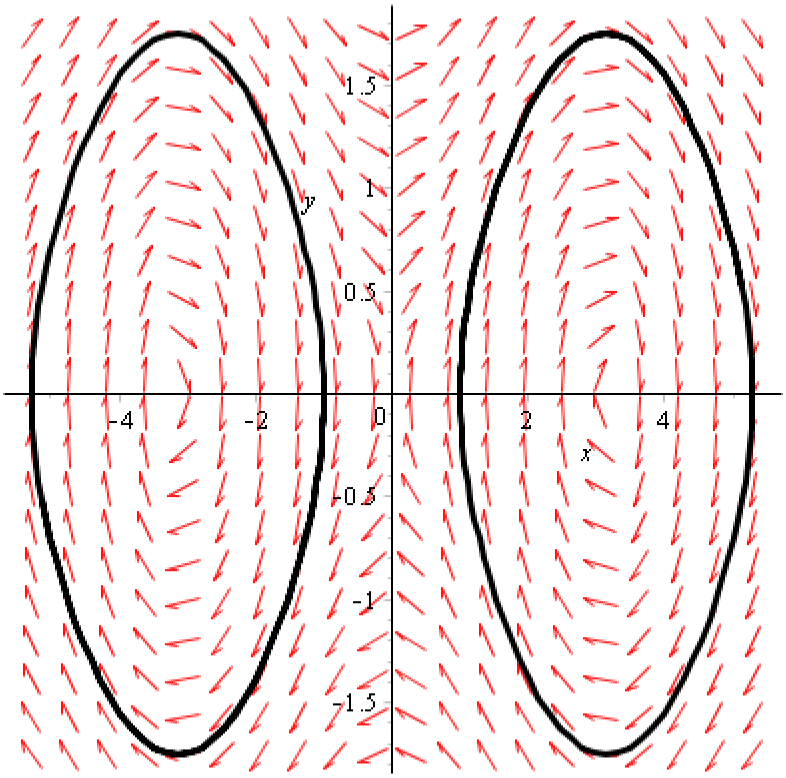


Phase trajectories with direction field for nonlinear ODE system with initial conditions with the command **DEplot**

```
> restart:
> with(DEtools):
> sys:=diff(x(t),t)=y(t), diff(y(t),t)=sin(x(t));
```

$$sys := \frac{d}{dt} x(t) = y(t), \frac{d}{dt} y(t) = \sin(x(t))$$

```
> DEplot({sys},[x(t),y(t)],t=0..4*Pi,[[0,1,0],[0,-1,0]],stepsize=
  0.1,linecolor=black);
```



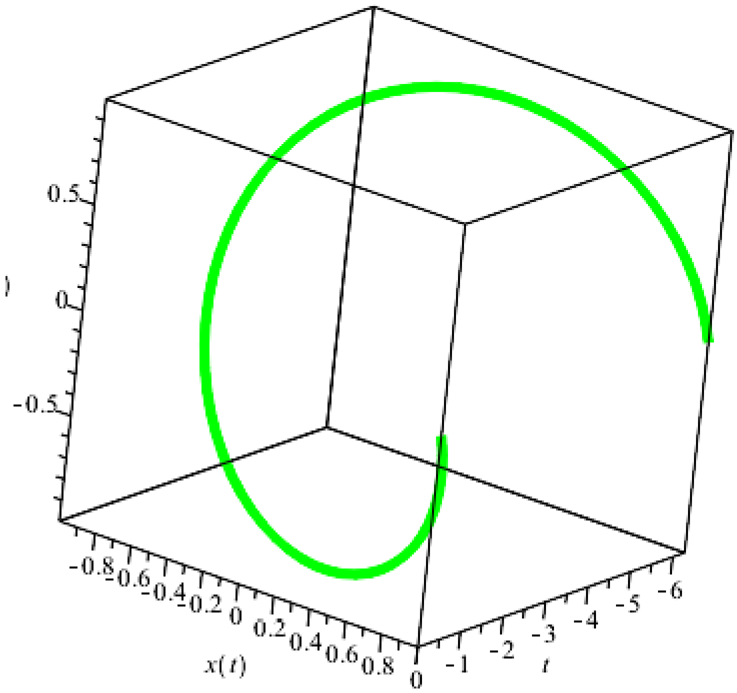
DEplot3d – plot the 3-D solutions to a system of DEs or ODE;

The graph of the solution of the Cauchy problem for systems of 2 differential equations with the command **DEplot3d**

```
> restart:
> with(DEtools):
> sys:=diff(x(t),t)=-sin(t),diff(y(t),t)=cos(t);
```

$$sys := \frac{d}{dt}x(t) = -\sin(t),\ \frac{d}{dt}y(t) = \cos(t)$$

```
> DEplot3d({sys},{x(t),y(t)},t=-2*Pi..0,[[y(0)=0,x(0)=1]],stepsize=
  .1,thickness=5,linecolor=green);
```

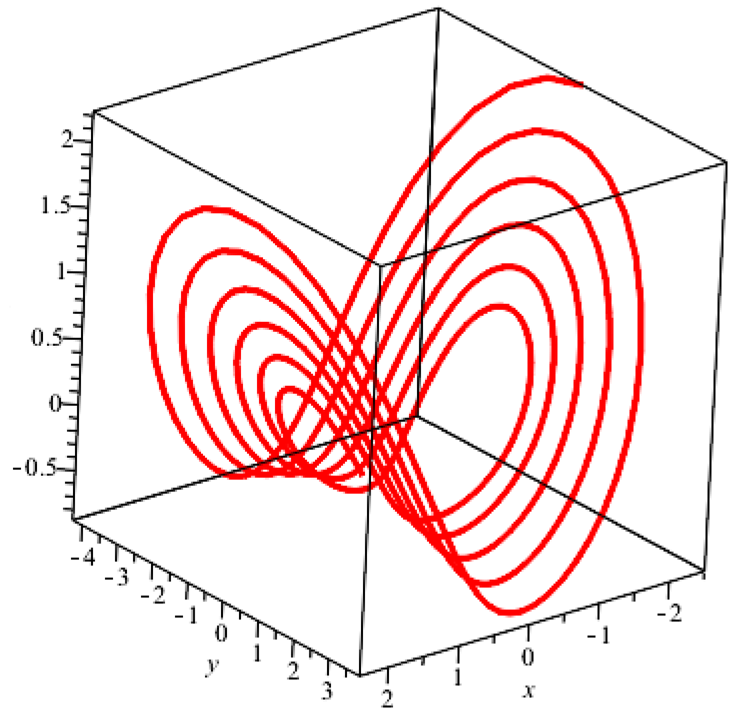


The graph of the solution of the Cauchy problem for systems of 3 differential equations with the command **DEplot3d**

```
> restart:
> with(DEtools):
> sys:=diff(x(t),t)=y(t),diff(z(t),t)=-0.1*z(t)+x(t)*y(t),diff(y(t),
  t)=0.1*y(t)-(z(t)-1)*x(t)-x(t)^3;
```

$$sys := \frac{d}{dt}x(t) = y(t),\ \frac{d}{dt}z(t) = -0.1\,z(t) + x(t)\,y(t),\ \frac{d}{dt}y(t) = 0.1\,y(t) - (z(t) - 1)\,x(t) - x(t)^3$$

```
> DEplot3d([sys],[x,y,z],t=0..25,[[y(0)=1,x(0)=1,z(0)=0]],stepsize=
0.05,thickness=3,linecolor=red);
```



phaseportrait – phase portrait for a system of first order ordinary differential equations or single first order ordinary differential equation; a system of two first order differential equations also produces a direction field plot, provided the system is determined to be autonomous ;

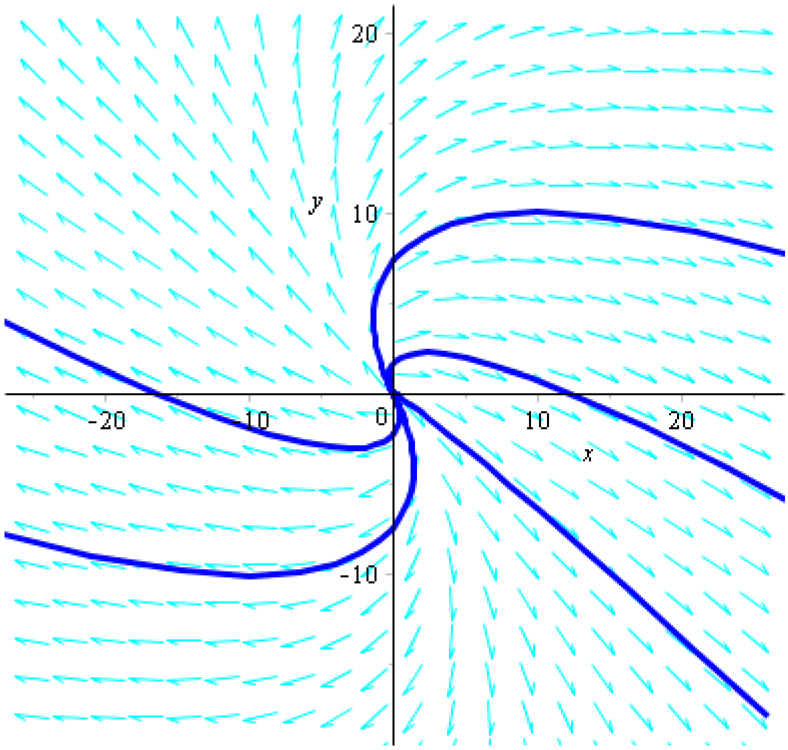
Phase trajectories and the direction field for linear ODE system with initial conditions with the command **phaseportrait**

```
restart:
with(DEtools):
sys:=diff(x(t),t)=3*x(t)+y(t), diff(y(t),t)=-x(t)+y(t);
inits:=x(0)=1,y(0)-2;
```

$$sys := \frac{d}{dt}x(t) = 3x(t) + y(t), \frac{d}{dt}y(t) = -x(t) + y(t)$$

$$inits := x(0) = 1, y(0) - 2$$

```
phaseportrait({sys},{x(t),y(t)},t=-10..10,[[0,1,-2],[0,-3,-3],[0,
5,-3],[0,5,2],[0,-1,2]],x=-25..25,y=-20..20,stepsize=0.1,color=
cyan,linecolor=blue);
```



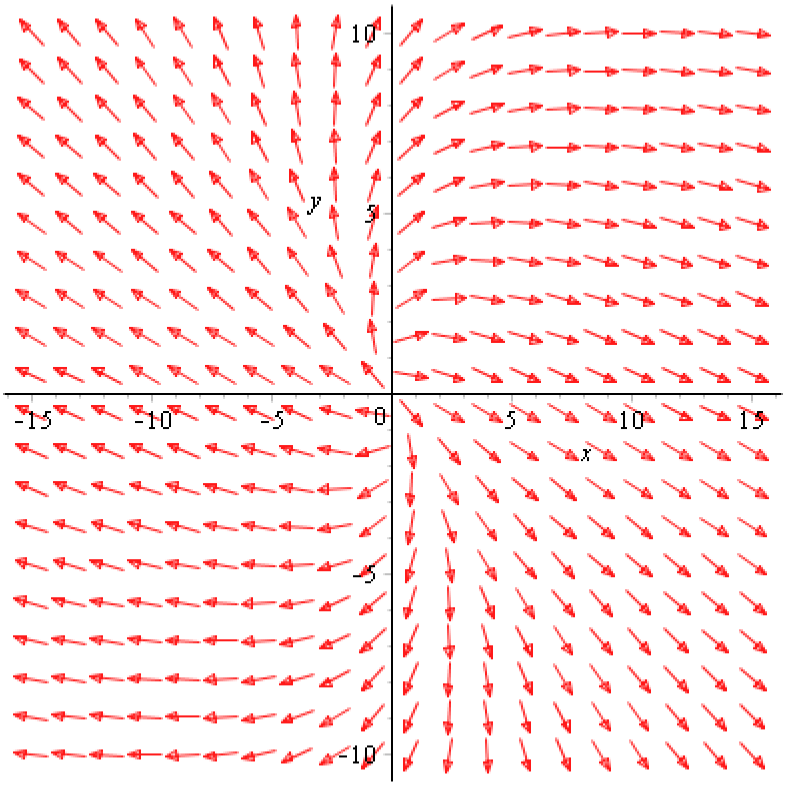
dfieldplot – plot direction field to a system of DEs;

The direction field for linear system of ODE with the command **dfieldplot**

```
> restart:
> with(DEtools):
> sys:=diff(x(t),t)=3*x(t)+y(t), diff(y(t),t)=-x(t)+y(t);
```

$$sys := \frac{d}{dt}x(t) = 3\,x(t) + y(t),\ \frac{d}{dt}y(t) = -x(t) + y(t)$$

```
> dfieldplot([sys],[x(t),y(t)],t=-10..10,x=-15..15,y=-10..10,
  stepsize=0.1,arrows=SLIM);
```



DEnormal – return to the normalized form of a DE;

```
> with(DEtools):
> de:=2*(-2*x^3)/(x-5)*y(x)+4*(x^2)/(x+1)*x*D(y)(x)+5*x^3/(x-1)^2*
(D@@2)(y)(x)=x;
```

$$de := -\frac{4\,x^3\,y(x)}{x-5}+\frac{4\,x^3\,\mathrm{D}(y)(x)}{x+1}+\frac{5\,x^3\,\mathrm{D}^{(2)}(y)(x)}{(x-1)^2}=x$$

```
> DEnormal(de,x,y(x));
```

$$\left(5\,x^2-20\,x-25\right)\left(\frac{d^2}{dx^2}\,y(x)\right)+\left(4\,x^3-28\,x^2+44\,x-20\right)\left(\frac{d}{dx}\,y(x)\right)-4\,y(x)\,x^3$$
$$+4\,y(x)\,x^2+4\,y(x)\,x-4\,y(x)=x^2-6\,x+4+\frac{6}{x}-\frac{5}{x^2}$$

autonomous – determine if a set of DEs is strictly autonomous;

```
> restart:
> with(DEtools):
> de:=(sin(z(t)-z(t)^2)*(D@@4)(z)(t)-cos(z(t))-5);
```

$$de := -\sin\left(-z(t)+z(t)^2\right)\,\mathrm{D}^{(4)}(z)(t)-\cos(z(t))-5$$

```
> autonomous(de,z,t);
```

$$true$$

convertAlg – return to the coefficient list form for a linear ODE;

```
> with(DEtools):
> de:=diff(x(t),t)*sin(t)+5*x(t)=cos(t);
```

$$de := \left(\frac{d}{dt} x(t)\right) \sin(t) + 5\, x(t) = \cos(t)$$

```
> convertAlg(de,x(t));
```

$$[[5, \sin(t)], \cos(t)]$$

convertsys – convert a system of differential equations to a first-order system;

```
> with(DEtools):
> de:=diff(y(t),t$2)=y(t)-x(t),diff(x(t),t)=x(t);
```

$$de := \frac{d^2}{dt^2} y(t) = y(t) - x(t),\ \frac{d}{dt} x(t) = x(t)$$

```
> inits:=y(0)=1,D(y)(0)=2,x(0)=3;
```

$$inits := y(0) = 1, D(y)(0) = 2, x(0) = 3$$

```
> convertsys({de},{inits},{x(t),y(t)},t,y,y_p);
```

$$\left[[y\_p_1 = y_1, y\_p_2 = y_3, y\_p_3 = y_2 - y_1], \left[y_1 = x(t), y_2 = y(t), y_3 = \frac{d}{dt} y(t)\right], 0, [3, 1, 2]\right]$$

reduceOrder – apply the method of reduction of order to an ODE;

Calling sequence: **reduceOrder(des,dvar,partsol,solutionForm)**, where **des** – ordinary differential equation, or its list form, **dvar** – the dependent variable for an equation, **partsol** – particular solution, or list of particular solutions, **solutionForm** – flag to indicate the DE should be solved explicitly.

```
> with(DEtools):
> de:=diff(y(x),x$3)-6*diff(y(x),x$2)+11*diff(y(x),x)-6*y(x);
```

$$de := \frac{d^3}{dx^3} y(x) - 6\left(\frac{d^2}{dx^2} y(x)\right) + 11\left(\frac{d}{dx} y(x)\right) - 6\, y(x)$$

```
> reduceOrder(de,y(x),exp(x));
```

$$\frac{d^2}{dx^2} y(x) - 3\left(\frac{d}{dx} y(x)\right) + 2\, y(x)$$

```
> reduceOrder(de,y(x),exp(x),basis);
```

$$\left[e^x, e^{2x}, \frac{1}{2} e^{3x}\right]$$

intfactor – look for integrating factors for a given ODE;

```
> with(DEtools):
> de:=sin(x)*diff(y(x),x)-cos(x)*y(x)=0;
```

$$de := \sin(x)\left(\frac{d}{dx}y(x)\right) - \cos(x)\,y(x) = 0$$

```
> m:=intfactor(de);
```

$$m := \frac{1}{\sin(x)^2}$$

```
> dsolve(m*de,y(x));
```

$$y(x) = \_C1\sin(x)$$

mutest – test a given integrating factor;

```
> with(DEtools):
> de:=diff(y(x),x$2)=-((diff(y(x),x))^2+2*x*(diff(y(x),x))
  +2*y(x))/y(x);
```

$$de := \frac{d^2}{dx^2}y(x) = -\frac{\left(\frac{d}{dx}y(x)\right)^2 + 2x\left(\frac{d}{dx}y(x)\right) + 2y(x)}{y(x)}$$

```
> M:=1/(diff(y(x),x)+2*x);
```

$$M := \frac{1}{\frac{d}{dx}y(x) + 2x}$$

```
> mutest(M,de);
```

$$0$$

particularsol – find a particular solution to a nonlinear ODE, or a linear non-homogeneous ODE, without computing its general solution;

```
> with(DEtools):
> de:=diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=x^2+3*x;
```

$$de := \frac{d^2}{dx^2}y(x) - 3\left(\frac{d}{dx}y(x)\right) + 2y(x) = x^2 + 3x$$

```
> particularsol(de,y(x));
```

$$y(x) = \frac{1}{2}x^2 + 3x + 4$$

varparam – find the general solution of an ODE by the method of variation of parameters;

Calling sequence: **varparam(sols, v, ivar)**, where **sols** – list of solutions to the corresponding homogeneous equation; **v** – right hand side of the original ODE; **ivar** – independent variable;

```
> with(DEtools):
> de:=diff(y(x),x$3)+2*diff(y(x),x$2)-diff(y(x),x)+2*y(x)=exp
  (2*x)/(exp(x)+1);
```

$$de := \frac{d^3}{dx^3}y(x) + 2\left(\frac{d^2}{dx^2}y(x)\right) - \left(\frac{d}{dx}y(x)\right) + 2y(x) = \frac{e^{2x}}{e^x+1}$$

```
> sols:=[exp(x),exp(-x),exp(-2*x)];
```

$$sols := [e^x, e^{-x}, e^{-2x}]$$

```
> Rhs:=exp(2*x)/(exp(x)+1);
```

$$Rhs := \frac{e^{2x}}{e^x+1}$$

```
> varparam(sols,Rhs,x);
```

$$\_C_1 e^x + \_C_2 e^{-x} + \_C_3 e^{-2x} + \frac{1}{6}\ln(e^x+1)\,e^x - \frac{5}{36}e^x + \frac{1}{3} - \frac{1}{2}e^{-x}\ln(e^x+1) + \frac{1}{3}e^{-x} - \frac{1}{3}e^{-2x}\ln(e^x+1)$$

Keys

1. Ordinary differential equations of the first order

Basic definitions and terms

1. no; 2. no; 3. yes; 4. yes; 5. yes; 6. yes; 7. yes; 8. yes; 9. yes; 10. yes.

Separated variables equations and separable equations

1. $x+y=C(1-xy)$; 2. $\sqrt{1+x^2}+\sqrt{1+y^2}=c$;
3. $e^x=C\left(1-e^{-y}\right)$; 4. $y=\sin\left[C+\ln\left(1+x^2\right)\right]$;
5. $x^2\left(1+y^2\right)=C$; 6. $y=\operatorname{tg}\ln Cx$;
7. $1+e^y=C\left(1+x^2\right)$; 8. $\operatorname{arctg} e^x=\frac{1}{2\sin^2 y}+C$;
9. $x+C=\operatorname{ctg}\left(\frac{y-x}{2}+\frac{\pi}{4}\right)$; 10. $y=(1+Cy+\ln y)\cos x$.

Homogeneous differential equations

1. $y=xe^{1+Cx}$; 2. $y^2-3xy+2x^2=C$; 3.$(x-y)\ln Cx=x$;
4. $2x=(x-y)\ln Cx$; 5. $(x+y-1)^5(x-y-1)^2=C$; 6. $y^2+3xy+x^2-5x-5y=C$;
7. $y^2-2xy-x^2+4y=c$; 8. $y^2=x\ln Cy^2$; 9. $\sqrt{x^2y^4+1}=Cx^2y^2-1$;
10. $Cx^4=y^6+x^3$.

Linear differential equation of first order. The Bernoulli equations

1. $x=Cy-\frac{y^2}{2}$; 2. $y=\left(C+x^2\right)e^{e^x}$; 3. $y=(C+x)e^{-x^2}$; 4. $y=Cx^2+x^2\sin x$;
5. $y=\left(C+x^3\right)\ln x$; 6. $y=1$; 7. $x=(C+y)e^{-y^2/2}$; 8. $y=\frac{\sin x}{\cos^2 x}$;
9. $y=\frac{x^2}{\cos x}$; 10. $y=(C+x)e^{(1-x)e^x}$; 11. $x^3e^{-y}=C+y$; 12. $y^2(C-x)\sin x=1$;
13. $y=\frac{e^{-x^2}}{C-x}$; 14. $y^2\ln x=C+\sin x$; 15. $y^4+2x^2y^2+2y^2=C$.

Exact Differential Equations. Integrating factor

1. $x^3+3x^2y^2+y^4=C$; 2. $\sqrt{x^2+y^2}+\ln|xy|+\frac{x}{y}=C$;
3. $x^4+x^2y^2+y^4=C$; 4. $x^3y+x^2-y^2=Cxy$;
5. $\frac{\sin^2 x}{y}+\frac{x^2+y^2}{2}=C$; 6. $y\sqrt{1+x^2}+x^2y-y\ln x=C$;
7. $\sqrt{x^2+y^2}\frac{y}{x}=C$; 8. $y=x$;
9. $x^3+y^3-x^2-xy+y^2=C$; 10. $xy\left(x^2+y^2\right)=C$;
11. $xy^2-2x^2y-2=Cx$; $\quad \mu=\frac{1}{x^2}$;
12. $5\operatorname{arctg}x+2xy=C$, $\quad x=0$; $\quad \mu=\frac{1}{1+x^2}$;
13. $y^3+x^3(\ln x-1)=Cx^2$; $\quad \mu=\frac{1}{x^4}$;
14. $2e^x\sin y+2e^x(x-1)+e^x(\sin x-\cos x)=C$; $\quad \mu=e^x$;
15. $x^2-\frac{7}{y}-3xy=C$; $\quad \mu=\frac{1}{y^2}$.

The Lagrange and Clairaut equations

1. $\begin{cases} x=\frac{C}{p^2}-\frac{1}{p}; \\ y=\frac{2C}{p}+\ln p-2; \end{cases}$ 2. $\begin{cases} x=\frac{C}{p^3}-2e^p\left(\frac{1}{p}-\frac{2}{p^2}+\frac{2}{p^3}\right); \\ y=\frac{3C}{2p^2}-2e^p\left(1-\frac{3}{p}+\frac{3}{p^2}\right); \end{cases}$

3.$y=Cx+C^2$, $\quad y=-\frac{x^2}{4}$; 4. $y=Cx+a\sqrt{1+C^2}, x^2+y^2=a^2$;
5. $y=Cx+\frac{a}{C^2}$, $\quad 4y^3=27ax^2$;

6. $\begin{cases} x=2(1-p)+Ce^{-p}; \\ y=\left[2(1-p)+Ce^{-p}\right](1+p)+p^2; \end{cases}$ 7. $\begin{cases} x=\frac{Cp^2+2p-1}{2p^2(p-1)^2}; \\ y=\frac{Cp^2+2p-1}{2(p-1)^2}-\frac{1}{p}; \end{cases}$

8. $\begin{cases} x=\frac{C}{p^2}-\frac{\cos p}{p^2}-\frac{\sin p}{p}; \\ y=\frac{2C}{p}-\frac{2\cos p}{p}-\sin p; \quad y=0; \end{cases}$

9. $y=Cx-\frac{C-1}{C}$, $\quad (y+1)^2=4x$; 10. $x=Cy+C^2$, $\quad 4x=-y^2$.

2. Higher order ordinary differential equations

Differential equations, allowing reduction of order

1. $y=\frac{1}{4}x^4+\frac{1}{6}x^3+\frac{1}{2}x^2C_1+C_2x+C_3$;
2. $y=\frac{1}{840}x^7-\frac{1}{60}x^5+\frac{1}{6}x^3C_1+\frac{1}{2}C_2x^2+C_3x+C_4$;
3. $y=-\frac{1}{3e^x}x-\frac{3}{4e^x}+\frac{1}{2}x^2C_1+C_2x+C_3$;
4. $y=\pm\frac{4}{15}\frac{(9+C_1x)^{\frac{5}{2}}}{C_1^2}+C_2x+C_3$;
5. $y=C_1+C_2x+C_3x^3$;
6. $x=y-y\ln y-C_1y-C_2$;
7. $y=0; \frac{2}{3}y^{\frac{3}{2}}-C_1x-C_2=0$;
8. $y=xe^x-2e^x+x+2$;

9. $y=\frac{1}{4}\left(2x-\frac{1}{2}\right)\cdot\sqrt{x^2-\frac{1}{2}x}-\frac{1}{32}\ln\left|x-\frac{1}{4}+\sqrt{x^2-\frac{1}{2}x}\right|+\frac{1}{80}\sqrt{2}-$
$-\frac{1}{32}\ln 2+\frac{1}{32}\ln\left(\frac{3}{2}+\sqrt{2}\right);$
10. $y=\left(\pm e^{\sqrt{2x}}\sqrt{2x}\right)-e^{\sqrt{2x}}+1.$

Higher order linear equations with constant coefficients

1. $y=e^xC_1\sin 3x+C_2e^x\cos 3x;$
2. $y=C_1+C_2x+C_3e^{-x}+C_4e^x;$
3. $y=C_1+C_2e^{-x}+C_3e^x+C_4e^{2x};$
4. $y=C_1+C_2e^x+C_3e^{-0,52x}-C_4e^{-2,24x}\sin(1,63)x+C_5e^{-2,24x}\cos(1,63)x;$
5. $y=C_1+C_2x+C_3e^{-3x};$
6. $y=C_1e^x+C_2e^xx+C_3\sin x+C_4\cos x;$
7. $y=2e^xx;$
8. $y=-\cos x;$
9. $y=\frac{-45}{26}e^{2x}+\frac{15}{13}\sin 3x-\frac{10}{13}\cos 3x;$
10. $y=1-e^{-x}+e^xx.$

Higher order linear nonhomogeneous equations with constant coefficients

1. $y=C_1\mathrm{e}^x+C_2\mathrm{e}^{4x}-\left(2x^2-2x+3\right)\mathrm{e}^{2x};$
2. $y=C_1\mathrm{e}^x+C_2\mathrm{e}^{-4x}-\frac{x}{5}\mathrm{e}^{-4x}-\left(\frac{x}{6}+\frac{1}{36}\right)\mathrm{e}^{-x}\ ;$
3. $y=e^{2x}\left(C_1\cos 2x+C_2\sin 2x\right)+0,25\mathrm{e}^{2x}+0,1\cos 2x+0,05\sin 2x;$
4. $y=C_1\mathrm{e}^{3x}+C_2\mathrm{e}^{-3x}+\mathrm{e}^{3x}\left(\frac{6}{37}\sin x-\frac{1}{37}\cos x\right);$
5. $y=\left(C_1-\frac{x^2}{4}\right)\cos x+\left(C_2+\frac{x}{4}\right)\sin x;$
6. $y=C_1+C_2\mathrm{e}^{5x}-0,2x^3-0,12x^2-0,048x+0,02(\cos 5x-\sin 5x);$
7. $y=\mathrm{e}^x\left(x\ln|x|+C_1x+C_2\right);$
8. $y=(7-3x)e^{x-2};$
9. $y=\mathrm{e}^{2x-1}-2\mathrm{e}^x+\mathrm{e}-1;$
10. $y=2+\mathrm{e}^{-x}.$

3. Systems of differential equations

Elimination Method

1. $\begin{cases} x=3C_1\cos 3t-3C_2\sin 3t; \\ y=C_2\cos 3t+C_1\sin 3t; \end{cases}$ 2. $\begin{cases} x=-2e^{-t}+3e^{-7t}; \\ y=e^{-t}+3e^{-7t}; \end{cases}$

3. $\begin{cases} x=C_1e^{-t}+C_2e^{-3t}; \\ y=C_1e^{-t}+3C_2e^{-3t}+\cos t; \end{cases}$

4. $\begin{cases} x=C_1e^t+C_2e^{-t}+C_3\sin t+C_4\cos t; \\ y=C_1e^t+C_2e^{-t}-C_3\sin t-C_4\cos t; \end{cases}$ 5. $\begin{cases} x=(\sin t-2\cos t)e^{-t}; \\ y=e^{-t}\cos t; \end{cases}$

Method of integrable combinations

1. $\begin{cases} x = e^t; \\ y = -e^{-t}; \end{cases}$ 2. $\begin{cases} \frac{1}{x} - \frac{1}{y} = C_1; \\ 1 + C_1 x = C_2 e^{C_1 t}; \end{cases}$ 3. $\begin{cases} x^2 - y^2 = C_1; \\ x - y + t = C_2; \end{cases}$

4. $\begin{cases} x = e^t - 2e^{3t}; \\ y = e^t + 2e^{3t}; \end{cases}$ 5. $\begin{cases} \frac{1}{x+y} + t = C_1; \\ \frac{1}{x-y} + t = C_2. \end{cases}$

Euler method

1. $\begin{cases} x = 2C_1 e^{3t} - 4C_2 e^{-3t}; \\ y = C_1 e^{3t} + C_2 e^{-3t}; \end{cases}$ 2. $\begin{cases} x = C_1 e^{2t} - C_2 e^{3t}; \\ y = C_1 e^{2t} - C_3 e^{t}; \\ z = C_1 e^{2t} - C_2 e^{3t} - C_3 e^{t}; \end{cases}$

3. $\begin{cases} x = e^{2t} - e^{3t}; \\ y = e^{2t} - 2e^{3t}; \end{cases}$ 4. $\begin{cases} x = -5e^{2t} \sin t; \\ y = e^{2t}(\cos t - 2\sin t); \end{cases}$ 5. $\begin{cases} x = 1 - e^{-t}; \\ y = 1 - e^{-t}; \\ z = 2e^{-t} - 1. \end{cases}$

Lagrange method

1. $\begin{cases} x = 2e^{2t} + C_1 e^t + C_2 e^{-t}; \\ y = 9e^{2t} + 3C_1 e^t + C_2 e^{-t}; \end{cases}$ 2. $\begin{cases} x = C_1 \cos t + C_2 \sin t + \operatorname{tg} t; \\ y = -C_1 \sin t + C_2 \cos t + 2; \end{cases}$

3. $\begin{cases} x = C_1 \cos t + C_2 \sin t + \cos t \ln|\cos t| + t \sin t; \\ y = -C_1 \sin t + C_2 \cos t - \sin t \ln|\cos t| + t \cos t; \end{cases}$

4. $\begin{cases} x = (1-t)\cos t - \sin t; \\ y = (t-2)\cos t + t \sin t; \end{cases}$ 5. $\begin{cases} x = C_1 + 2C_2 e^{-t} + 2e^{-t} \ln\left|e^t - 1\right|; \\ y = -2C_1 - 3C_2 e^{-t} - 3e^{-t} \ln\left|e^t - 1\right|. \end{cases}$

Method of undetermined coefficients

1. $\begin{cases} x = C_1 \cos 2t + C_2 \sin 2t + t; \\ y = C_1 \sin 2t - C_2 \cos 2t + 1; \end{cases}$ 2. $\begin{cases} x = -t; \\ y = 0; \end{cases}$

3. $\begin{cases} x = -C_1 \sin t + C_2 \cos t + t; \\ y = C_1 \cos t + C_2 \sin t + t^2 - 2; \end{cases}$ 4. $\begin{cases} x = -C_1 \sin t + (C_2 - 1)\cos t; \\ y = C_1 \cos t + C_2 \sin t; \end{cases}$

5. $\begin{cases} x = e^{-t}; \\ y = e^{-t}; \\ z = 1. \end{cases}$

Литература

[1] Вельмисов П. А. Дифференциальные уравнения: учебное пособие / П. А. Вельмисов, Ю. В. Покладова, Т. Б. Распутько. – Ульяновск: УлГТУ, 2013. – 90 с.

[2] Гарипов И.Б. Обыкновенные дифференциальные уравнения: учеб.-методич. пособие / И.Б. Гарипов, Н.В. Зайцева, Р.М. Мавлявиев. – Казань: Казан. ун-т, 2018. – 49 с.

[3] Гриншпон Я.С. Геометрические, физические и экономические задачи, сводящиеся к дифференциальным уравнениям: учеб. пособие / Я.С. Гриншпон. – Томск: Изд-во Томск. гос. ун-та систем упр. и радиоэлектроники, 2011. – 74 с.

[4] Гунько В.Д. Дифференциальные уравнения. Примеры и типовые задания: Учебное пособие / В.Д. Гунько, Л.Ю. Суховеева, В.М. Смоленцев. – Краснодар: КубГАУ, 2005. – 105 с.

[5] Демидович Б.П. Дифференциальные уравнения / Б.П. Демидович, В.П. Моденов. – СПб.: Издательство Лань, 2008. – 288 с.

[6] Жарова Н.Р. Дифференциальные уравнения: Учебное пособие / Н.Р. Жарова, Л.Г. Кузнецова. – Изд. 3-е, испр. и доп. – Нижневартовск: Изд-во Нижневарт. гос. ун-та, 2014. – 147 с.

[7] Зголич М.В. Дифференциальные уравнения / М.В. Зголич, Т.А. Шалыгина. – Томск: Изд-во Том. гос. архит.-строит. ун-та, 2014. – 32 с.

[8] Имас О.Н. Лекции по дифференциальным уравнениям: учебное пособие / О.Н. Имас, Е.Г. Пахомова, С.В. Рожкова, И.Г. Устинова. – Томск: Изд-во Томского политехнического университета, 2012. – 193 с.

[9] Киясов С. Н. Дифференциальные уравнения. Основы теории, методы решения задач: Учебное пособие / С.Н. Киясов, В.В. Шурыгин. – Казань: Казанский федеральный университет, 2011. – 112 с.

[10] Краснов М.Л. Обыкновенные дифференциальные уравнения: задачи и примеры с подробными решениями / М.Л. Краснов, А.И. Киселев, И.Г. Макаренко. – Изд. 4-е, испр. – М.: Едиториал УРСС, 2002. – 256 с.

[11] Куликов Г.М. Дифференциальные уравнения. Тестовые задания: учебное пособие / Г.М. Куликов, И.В. Жигулина, А.Д. Нахман. – Тамбов: Изд-во ФГБОУ ВПО ТГТУ, 2011. – 80 с.

[12] Михеев А. В. Сборник задач по дифференциальным уравнениям: учеб. пособие [Текст] / А. В. Михеев. – СПб.: Санкт-Петербургский филиал Нац. исслед. ун-та кВысшая школа экономикињ. – Отдел оперативной полиграфии НИУ ВШЭ, 2012. – 68 с.

[13] Соболев С.К. Дифференциальные уравнения. Методические указания к решению задач / С.К. Соболев. – Москва: МГТУ им. Баумана, 2008. – 25 с.

[14] Треногин В.А. Обыкновенные дифференциальные уравнения / В.А. Треногин. – М.: ФИЗМАТЛИТ, 2009. – 312 с.

[15] Филиппов А.Ф. Сборник задач по дифференциальным уравнениям / А.Ф. Филиппов. – Ижевск: НИЦ Регулярная и хаотическая динамика, 2000. – 176 с.

[16] Лобанова Н.И. Элементы теории дифференциальных уравнений в системе дополнительного образования/ Н.И. Лобанова // Интернет-журнал "Мир науки". – 2016. – Том 4. – №6. [Электронный ресурс] – Режим доступа: http://mir-nauki.com/PDF/32PDMN616.pdf (Дата обращения 16.02.2019)

[17] Интернет-ресурс http://old.exponenta.ru/educat/systemat/SAVOTCHENKO/6 1.asp (Дата обращения 11.02.2019)

Russian-English dictionary

А

алгоритм – agorithm;

В

выносить за скобки – factor out;

Г

график решения – graph of a solution;

Д

действительный – real;
действительная часть – real part;
действительные числа – real numbers;
дифференциал – differential;
-полный дифференциал - total differential;
дифференцировать – differentiate;
дифференцируемый – differentiable;
дробь – fraction;

Е

единственность – uniqueness;

З

задача Коши – Cauchy problem;
замена – replacement;
знаменатель – denominator;

И

интеграл – integral;
-общий интеграл – general integral;
-частный интеграл – particular integral;
-первый интеграл – first integral;
интегральная кривая – integral curve;
интегрирование – integration;
-n-кратное интегрирование – n-fold integration;
интегрировать – integrate;
интегрирующий множитель – integrating factor;
интегрируемые комбинации – integrable combinations;
интервал – interval;
исключить – eliminate;

К

корень – root;
-комплексные корни – complex roots;
-сопряженные корни – conjugate roots;
кратность корня – multiplicity of root;
коэффициент – coefficient;
коэффициент пропорциональности – proportionality factor;

Л

левая часть – left hand side;
линейно зависимый – linearly dependent;
линейно независимый – linearly independent;

М

масса – mass;
математическая модель – mathematical model;
матрица – matrix;
матричное дифференциальное уравнение – matrix differential equation;
мнимый – imaginary;
-мнимая часть – imaginary part;
многочлен – polynomial;
множитель – factor;
метод – method;

-метод вариации произвольной постоянной – method of variation of an arbitrary constant;

-метод Лагранжа – Lagrange method;

-метод подстановки – method of substitution;

-метод Бернулли – Bernoulli method;

-метод неопределенных коэффициентов – method of undetermined coefficients;

-метод исключения – elimination method;

-метод вариации произвольных постоянных – method of variation of an arbitrary constants;

-метод интегрируемых комбинаций – method of integrable combinations;

-метод Эйлера – Euler method;

Н

неопределенное – undefined;

непрерывный – continuous;

-непрерывная функция – continuous function;

неравенство – inequality;

неявный вид – implicit form;

О

область – domain;

определитель – determinant;

определитель Вронского – Wronskian determinant;

П

параметр – parameter;

параметрический вид – parametric form;

переменная – variable;

-зависимая переменная – dependent variable;

-независимая переменная – independent variable;

подстановка – substitution;

подставлять – substitute;

понижение порядка - reduction of order;

порядок уравнения – order of equation;

постоянная – constant;

-произвольная постоянная – arbitrary constant;

правая часть – right hand side;
приводить подобные слагаемые – combine like terms;
приравнивать – equate;
проверить – check;
произведение – product;
производная – derivative;

Р

равен – is equal to, equals;
равенство – equality;
разделить – divide (by);
разделить переменные – to separate the variables;
разлагать на множители – factor;
раскрывать скобки – to expand brackets;
решение – solution;
-единственное решение – unique solution;
-независимые решения – independent solutions;
-общее решение – general solution;
-особое решение – singular solution;
-частное решение – particular solution;
-численное решение – numerical solution;
-аналитическое решение – analytical solution;
-приближенное решение – approximate solution;
решить – solve;

С

сила – force;
система – system;
-системы дифференциальных уравнений – system of differential equations;
-нормальная система дифференциальных уравнений- normal system of differential equations;
-линейная система дифференциальных уравнений – linear system of differential equations;
-линейная неоднородная система дифференциальных уравнений с постоянными коэффициентами – linear nonhomogeneous systems of differential equations with constant coefficients;

скорость – speed;
сумма – sum;
слагаемое – summand;
собственный вектор – eigenvector;
соответствующий – corresponding;
соотношение – relation;
существование – existence;

Т

теорема – theorem;
-теорема о существовании и единственности решения задачи Коши – Theorem of existence and uniqueness of the solution of the Cauchy problem;
-теорема о наложении частных решений НЛДУ – theorem of superposition of particular solutions of linear nonhomogeneous differential equation;
тождество – identity;

У

удовлетворять – satisfy;
уравнение – equation;
-линейное алгебраическое уравнение – linear algebraic equation
-дифференциальное уравнение – differential equation;
-обыкновенное дифференциальное уравнение первого порядка – ordinary differential equation of the first order;
-обыкновенное дифференциальное уравнение высшего порядка – higher order ordinary differential equation;
-дифференциальные уравнения с разделенными переменными – separated variables equations;
-дифференциальное уравнение разделяющимися переменными - separable equation;
-однородное дифференциальное уравнение – homogeneous differential equation;
-линейное дифференциальное уравнение – linear differential equation;
-линейное однородное уравнение – homogeneous linear differential equation;
-линейное неоднородное уравнение – nonhomogeneous linear differential equation;
-уравнение Бернулли – Bernoulli equation;

-уравнение в полных дифференциалах – exact differential equation;

-уравнение Лагранжа – Lagrange equation;

-уравнение Клеро – Clairaut equation;

-дифференциальное уравнение, допускающее понижение порядка – differential equation allowing reduction of order;

-линейное дифференциальное уравнение высшего порядка – higher order linear differential equation;

-линейное однородное уравнение высшего порядка с постоянными коэффициентами – higher order linear equation with constant coefficients;

умножить – multiply (by);

ускорение – acceleratio;

условие – condition;

-начальное условие – initial condition;

-условие Коши-Римана – Cauchy-Riemann condition;

Ф

формула – formula;

фундаментальная система решений – set of fundamental solutions

Функция – function;

-неизвестная функция – unknown function;

-непрерывная функция – continuous function;

Х

характеристическое уравнение – characteristic equation;

характеристическое число матрицы – eigenvalue of the matrix;

Ч

числитель – numerator;

Э

эквивалентный – equivalent.

English-Russian dictionary

A

agorithm (*n*) – алгоритм;
acceleratio (*n*) – ускорение;

C

Cauchy problem – задача Коши;
characteristic equation – характеристическое уравнение;
check (*v*) – проверить;
coefficient (*n*) – коэффициент;
combine like terms – приводить подобные слагаемые;
condition (*n*) – условие;
-initial condition – начальное условие;
-Cauchy-Riemann condition – условие Коши-Римана;
constant (*n*) – постоянная;
-arbitrary constant – произвольная постоянная;
continuous (*a*) – непрерывный;
-continuous function – непрерывная функция;
corresponding (*a*) – соответствующий;

D

denominator (*n*) – знаменатель;
derivative (*n*) – производная;
determinant (*n*) – определитель;
-Wronskian determinant – определитель Вронского;
differential (*n*) – дифференциал;
-total differential – полный дифференциал;
-differentiate (*v*) – дифференцировать;
-differentiable (*a*) – дифференцируемый;
divide by (*v*) – разделить;
domain (*n*) – область;

E

equals (is equal to) (*v*) – равен;

equality (*n*) – равенство;

equate (*v*) – приравнивать;

equation (*n*) – уравнение;

-linear algebraic equation – линейное алгебраическое уравнение;

-differential equation – дифференциальное уравнение;

-ordinary differential equation of the first order – обыкновенное дифференциальное уравнение первого порядка;

-higher order ordinary differential equation – обыкновенное дифференциальное уравнение высшего порядка;

-separated variables equations – дифференциальные уравнения с разделенными переменными;

-separable equation – дифференциальное уравнение разделяющимися переменными;

-homogeneous differential equation – однородное дифференциальное уравнение;

-linear differential equation – линейное дифференциальное уравнение;

-homogeneous linear differential equation – линейное однородное уравнение;

-nonhomogeneous linear differential equation – линейное неоднородное уравнение;

-Bernoulli equation – уравнение Бернулли;

-exact differential equation – уравнение в полных дифференциалах;

-Lagrange equation – уравнение Лагранжа;

-Clairaut equation – уравнение Клеро;

-differential equation allowing reduction of order –дифференциальное уравнение, допускающее понижение порядка;

-higher order linear differential equation – линейное дифференциальное уравнение высшего порядка;

-higher order linear equation with constant coefficients – линейное однородное уравнение высшего порядка с постоянными коэффициентами;

equivalent (*a*) – эквивалентный;

eigenvalue of the matrix – характеристическое число матрицы;

eigenvector (*n*) – собственный вектор;

eliminate (*v*) – исключить;

existence (*n*) – существование;
expand brackets – раскрывать скобки;

F

factor (*n*) – множитель;
factor (*v*) – разлагать на множители;
factor out (*v*) – выносить за скобки;
force (*n*) – сила;
formula (*n*) – формула;
fraction (*n*) – дробь;
function (*n*) – функция;
-unknown function – неизвестная функция;
-continuous function – непрерывная функция;

G

graph of a solution – график решения;

I

identity (*n*) – тождество;
imaginary (*a*) – мнимый;
-imaginary part – мнимая часть;
implicit form – неявный вид;
inequality (*n*) – неравенство;
integrable combinations – интегрируемые комбинации;
integral (*n*) – интеграл;
-general integral – общий интеграл;
-particular integral – частный интеграл;
-first integral – первый интеграл;
integral curve – интегральная кривая;
integration (*n*) – интегрирование;
-n-fold integration – n-кратное интегрирование;
integrate (*v*) – интегрировать;
integrating factor – интегрирующий множитель;
interval (*n*) – интервал;

L

left hand side – левая часть;

linearly dependent – линейно зависимый;

linearly independent – линейно независимый;

M

mass (*n*) – масса;

mathematical model – математическая модель;

matrix (*n*) – матрица;

matrix differential equation – матричное дифференциальное уравнение;

method (*n*) – метод;

-method of variation of an arbitrary constant – метод вариации произвольной постоянной;

-Lagrange method – метод Лагранжа;

-method of substitution – метод подстановки;

-Bernoulli method – метод Бернулли;

-method of undetermined coefficients – метод неопределенных коэффициентов;

-elimination method – метод исключения;

-method of variation of an arbitrary constants – метод вариации произвольных постоянных;

-method of integrable combinations – метод интегрируемых комбинаций;

-Euler method – метод Эйлера;

multiplicity of root – кратность корня;

multiply by (*v*) – умножить;

N

numerator (*n*) – числитель;

O

order (*n*) – порядок;

P

parameter (*n*) – параметр;

parametric form – параметрический вид;

polynomial (*n*) – многочлен;

product (*n*) – произведение;

proportionality factor – коэффициент пропорциональности;

R

real (*a*) – действительный;
real part – действительная часть;
real numbers – действительные числа;
replacement (*n*) – замена;
reduction of order – понижение порядка;
relation (*n*) – соотношение;
right hand side – правая часть;
root (*n*) — корень;
-complex roots – комплексные корни;
-conjugate roots – сопряженные корни;

S

satisfy (*v*) – удовлетворять;
separate the variables – разделить переменные;
set of fundamental solutions – фундаментальная система решений;
solve (*v*) – решить;
solution (*n*) – решение;
-unique solution – единственное решение;
-independent solutions – независимые решения;
-general solution – общее решение;
-singular solution – особое решение;
-particular solution – частное решение;
-numerical solution – численное решение;
-analytical solution – аналитическое решение;
-approximate solution – приближенное решение;
speed (*n*) – скорость;
substitute (*v*) – подставлять;
substitution (*n*) – подстановка;
sum (*n*) – сумма;
summand (*n*) – слагаемое;
system (*n*) – система;
-system of differential equations – системы дифференциальных уравнений;
-normal system of differential equations – нормальная система дифференциальных уравнений;

-linear system of differential equations – линейная система дифференциальных уравнений;

-linear nonhomogeneous systems of differential equations with constant coefficients – линейная неоднородная система дифференциальных уравнений с постоянными коэффициентами;

T

theorem (*n*) – теорема;

-Theorem of existence and uniqueness of the solution of the Cauchy problem – теорема о существовании и единственности решения задачи Коши;

-theorem of superposition of particular solutions of linear nonhomogeneous differential equation – теорема о наложении частных решений НЛДУ;

U

undefined (*a*) – неопределенный;

uniqueness (*n*) – единственность;

V

variable (*n*) – переменная;

-dependent variable – зависимая переменная;

-independent variable – независимая переменная.

Mathematical symbols and expressions

Common fractions

$\frac{1}{2}$ – one half;
$\frac{1}{3}$ – one third;
$\frac{2}{3}$ – two thirds;
$\frac{26}{38}$ – twenty six thirty eights;
$2\frac{1}{2}$ – two and a half;
$\frac{m}{n}$ – m over n;

Decimal fractions

0.2 – 1. o ([ou]) point two; 2. zero point two; 3. point two;

0.06 – 1. o ([ou] point o ([ou] six; 2. zero point zero six; 3. point o ([ou] six; 4. point zero six;

1.25 – 1. one point twenty five; 2. one point two five;

Indexes, powers and roots

a_m – a sub m (a m);
b^n – b super n (b n);
C_{mn}^{jk} – c sub m n super j k;
x^2 – 1. x squared; 2. x raised to the second power; 3. x to the second power; 4. x to the second; 5. the square of x; 6. the second power of x;
y^3 – y cubed;
z^{-10} – 1. z to the negative tenth; 2. z to the minus tenth;
$\sqrt{a}$ – the square root of a;
$\sqrt[3]{7}$ – the cube root of seven;
$\sqrt[5]{b^4}$ – the fifth root of b to the fourth;

Certain mathematical signs and expressions

$a+b$ – 1. a plus b; 2. sum of a and b;
$a-b$ – 1. a minus b; 2. difference of a and b;
$a \cdot b$ – 1. a times b; 2. a b; 3. product of a and b;
$a:b$ ($\frac{a}{b}$) – 1. a divided by b; 2. a by b; 3. ratio of a and b; 4. a over b;
$a:b=\frac{a}{b}=c$ – 1. a divided by b equals c; 2. a over b is equal to c;
$a=b$ – 1. a equals b; 2. a is equal to b;
$a \neq b$ – 1. a does not equal b; 2. a is not equal to b; 3. a is not b;
$a \equiv b$ – a is identically equal to b;
$|x|$ – absolute value of x;
$a>b$ – a is greater than b;
$a<b$ – a is less than b;
$a \leq b$ – a is less than or equal to b;
$a \geq b$ – a is greater than or equal to b;
$(a+b)c=ac+bc$ – a plus b quantity times c equals ac plus bc;
$(x+a)^2$ – x plus a quantity squared;
$\frac{a+b}{c}$ – a plus b quantity over c;
$1,2,3\ldots 25$ – one, two, three and so on to twenty five;
$1,2,3,\ldots$ – one, two, three and so on to infinity;
$a \in M$ – 1. a is an element of M; 2. a belongs to M;
$a \notin M$ – 1. a is not element of M; 2. a does not belong to M;

Calculus

$y=f(x)$ – 1. y equals f of x; 2. y is a function of x;
$y=\frac{7+x}{2+x^2}$ – y equal the fraction with the numerator 7 plus x and the denominator 2 minus x squared;
$y=sinx$ – y equals the sine of x;
$y=cosx$ – y equals the cosine of x;
f' – 1. f prime; 2. derivative of f;
f'' – 1. f double prime; 2. second derivative of f;
f''' – 1. f triple prime; 2. third derivative of f;
$f^{(n)}$ – 1. n-th derivative of f; 2. derivative of f of the order n;
f'_x – 1. f prime x; 2. derivative of f with respect to x;
dx – 1. d x; differential of x;
$\frac{dy}{dx}$ – 1. d y by d x; 2. derivative of y with respect to x;

$\frac{d^2 f}{dx^2}$ – 1. d two f by d x squared; 2. second derivative of f with respect to x;

$\frac{d^n z}{dx^n}$ – the n-th derivative of y with respect to x;

$\int f(x)dx$ – indefinite integral of f of x d x;

$\int_a^b f(x)dx$ – 1. integral of f of x d x from a to b; 2. integral of f of x d x between limits a and b;

$\log_b c = n$ – the logarithm of c to the base b is equal to n;

$\ln c = n$ – natural logarithm of c is equal to n.