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THE TOPOLOGIES OF LOCAL CONVERGENCE IN MEASURE ON THE ALGEBRAS OF MEASURABLE OPERATORS

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Abstract: Given a von Neumann algebra $\mathcal{M}$ of operators on a Hilbert space $\mathcal{H}$ and a faithful normal semifinite trace τ on $\mathcal{M}$, denote by $S(\mathcal{M},\tau)$ the $*$-algebra of τ-measurable operators. We obtain a sufficient condition for the positivity of an hermitian operator in $S(\mathcal{M},\tau)$ in terms of the topology $t_{\tau l}$ of τ-local convergence in measure. We prove that the $*$-ideal $\mathcal{F}(\mathcal{M},\tau)$ of elementary operators is $t_{\tau l}$-dense in $S(\mathcal{M},\tau)$. If t_τ is locally convex then so is $t_{\tau l}$; if $t_{\tau l}$ is locally convex then so is the topology $t_{w\tau l}$ of weakly τ-local convergence in measure. We propose some method for constructing F-normed ideal spaces, henceforth F-NIPs, on $(\mathcal{M},\tau)$ starting from a prescribed F-NIP and preserving completeness, local convexity, local boundedness, or normability whenever present in the original. Given two F-NIPs $\mathcal{X}$ and $\mathcal{Y}$ on $(\mathcal{M},\tau)$, suppose that $A\mathcal{X}\subseteq\mathcal{Y}$ for some operator $A\in S(\mathcal{M},\tau)$. Then the multiplier $\mathbf{M}_A X = AX$ acting as $\mathbf{M}_A:\mathcal{X}\to\mathcal{Y}$ is continuous. In particular, for $\mathcal{X}\subseteq\mathcal{Y}$ the natural embedding of $\mathcal{X}$ into $\mathcal{Y}$ is continuous. We inspect the properties of decreasing sequences of F-NIPs on $(\mathcal{M},\tau)$.

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Introduction

Suppose that a von Neumann operator algebra $\mathcal{M}$ acts on a Hilbert space $\mathcal{H}$ and τ is a faithful normal semifinite trace on $\mathcal{M}$ with $\tau(I)=+\infty$. This article continues the study of [1–10], of the properties of the topologies $t_{\tau l}$ of τ-local convergence in measure and $t_{w\tau l}$ of weakly τ-local convergence in measure on the $*$-algebra $S(\mathcal{M},\tau)$ of τ-measurable operators. The topologies $t_{\tau l}$ and $t_{w\tau l}$ match the order structure of the hermitian part of $S(\mathcal{M},\tau)$ better than the classical topology t_τ of convergence in measure [2, 6, 7, 9].

In this article, Theorem 1 establishes a sufficient condition for the positivity of a hermitian operator in $S(\mathcal{M},\tau)$ in terms of $t_{\tau l}$. Finding such conditions is an important problem attracting many researchers; see, for instance, [11–13] and the bibliographies therein. It is well known that each τ-measurable operator is a linear combination of four positive τ-measurable operators. In view of the operator inequality

$$|X+Y|\le U|X|U^*+V|Y|V^*,\quad X,Y\in S(\mathcal{M},\tau),\quad |X|=\sqrt{X^*X},$$

where $U,V\in\mathcal{M}$ are suitable partial isometries (see [14]), to prove the triangle inequality for an F-normed ideal space, henceforth spelled F-NIP, on $(\mathcal{M},\tau)$, it suffices to verify the triangle inequality for the F-norm just for pairs of positive τ-measurable operators.

In Theorem 2 we show that the $*$-ideal $\mathcal{F}(\mathcal{M},\tau)$ of elementary operators is $t_{\tau l}$-dense in $S(\mathcal{M},\tau)$. Therefore, the space $L_p(\mathcal{M},\tau)$ with $0<p<\infty$ is $t_{\tau l}$-dense in $S(\mathcal{M},\tau)$; see Corollary 1. Recall that the t_τ-closures of the $*$-ideal $\mathcal{F}(\mathcal{M},\tau)$ and the spaces $L_p(\mathcal{M},\tau)$ with $0<p<\infty$ coincide with the $*$-ideal $S_0(\mathcal{M},\tau)$ of all τ-compact operators.

Theorem 3 shows that if t_τ is locally convex then so is $t_{\tau l}$; if $t_{\tau l}$ is locally convex then so is $t_{w\tau l}$. We propose some method for constructing "weighted" F-NIPs on $(\mathcal{M},\tau)$ starting from a prescribed

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F-NIP and preserving completeness, local convexity, local boundedness, or normability whenever present in the original; see Theorem 4 and Corollaries 4, 5. Weighted F-NIPs on an abelian von Neumann algebra arise naturally in studying integral operators. In this case $\mathcal{M} \simeq L^\infty(\Omega, \Sigma, \mu)$ and $\tau(f) = \int_\Omega f \,\mathrm{d}\,\mu$, where (Ω, Σ, μ) is a localizable measure space.

Given two F-NIPs $\mathcal{X}$ and $\mathcal{Y}$ on $(\mathcal{M}, \tau)$, suppose that $A\mathcal{X} \subseteq \mathcal{Y}$ for some operator $A \in S(\mathcal{M}, \tau)$. Then the multiplier $\mathbf{M}_A X = AX$ acting as $\mathbf{M}_A : \mathcal{X} \to \mathcal{Y}$ is continuous. In particular, for $\mathcal{X} \subseteq \mathcal{Y}$ the natural embedding of $\mathcal{X}$ into $\mathcal{Y}$ is continuous (see Theorem 5) which yields a new proof of the key Lemma 4.3 of [15]. Theorem 6 studies the properties of decreasing sequences of F-NIPs on $(\mathcal{M}, \tau)$.

Most of the results are new even for the $*$-algebra $\mathcal{M} = \mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$ which is equipped with the canonical trace $\tau = \mathrm{tr}$.

1. Notation and Definitions

Consider a von Neumann algebra $\mathcal{M}$ of operators on a Hilbert space $\mathcal{H}$. Denote by $\mathcal{M}^{\mathrm{pr}}$ the projection lattice ($P = P^2 = P^*$) in $\mathcal{M}$; and by I, the identity of $\mathcal{M}$. So $P^\perp = I - P$ for $P \in \mathcal{M}^{\mathrm{pr}}$. Denote by $\mathcal{M}^+$ the cone of positive elements of $\mathcal{M}$. A projection $P \in \mathcal{M}$ is *minimal* or an *atom*, if $Q \in \mathcal{M}^{\mathrm{pr}}$ and $Q \le P$ implies that either $Q = 0$ or $Q = P$. A von Neumann algebra $\mathcal{M}$ is *atomic* whenever each nonzero projection in $\mathcal{M}$ majorizes a nonzero minimal projection.

A mapping $\varphi : \mathcal{M}^+ \to [0, +\infty]$ is a *trace* whenever $\varphi(X+Y) = \varphi(X)+\varphi(Y)$ and $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{M}^+$ and $\lambda \ge 0$, with $0 \cdot (+\infty) \equiv 0$, and $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{M}$. A trace φ is *faithful* whenever $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$ with $X \ne 0$, *semifinite* whenever $\varphi(X) = \sup\{\varphi(Y) : Y \in \mathcal{M}^+, Y \le X,\ \varphi(Y) < +\infty\}$ for each $X \in \mathcal{M}^+$, and *normal* whenever $X_i \nearrow X$ with $X_i, X \in \mathcal{M}^+$ implies that $\varphi(X) = \sup \varphi(X_i)$; see [16, Chapter V, § 2].

An operator in $\mathcal{H}$, not necessarily bounded or densely defined, is *affiliated to the von Neumann algebra* $\mathcal{M}$ whenever it commutes with every unitary operator in the commutator $\mathcal{M}'$ of $\mathcal{M}$. Henceforth, τ stands for a faithful normal semifinite trace on $\mathcal{M}$ and $\mathcal{M}^{\mathrm{pr}}_\tau = \{P \in \mathcal{M}^{\mathrm{pr}} :\ \tau(P) < \infty\}$.

A closed operator X affiliated to $\mathcal{M}$ whose domain $\mathcal{D}(X)$ is dense in $\mathcal{H}$ is τ*-measurable* whenever, given $\varepsilon > 0$, there exists $P \in \mathcal{M}^{\mathrm{pr}}$ such that $P\mathcal{H} \subset \mathcal{D}(X)$ and $\tau(P^\perp) < \varepsilon$. The set $S(\mathcal{M}, \tau)$ of all τ-measurable operators forms a $*$-algebra with the operations of conjugate transpose, multiplication by scalars, and strong addition and multiplication obtained by closing the ordinary operations [17, Chapter IX]. Given a family $\mathcal{L} \subset S(\mathcal{M}, \tau)$, denote by $\mathcal{L}^+$ and $\mathcal{L}^{\mathrm{h}}$ the positive and hermitian parts of $\mathcal{L}$. Denote by $\le$ the partial order on $S(\mathcal{M}, \tau)^{\mathrm{h}}$ generated by the proper cone $S(\mathcal{M}, \tau)^+$. If $X \in S(\mathcal{M}, \tau)$ and $X = U|X|$ is the polar decomposition of X then $U \in \mathcal{M}$ and $|X| = \sqrt{X^*X} \in S(\mathcal{M}, \tau)^+$.

Endow the $*$-algebra $S(\mathcal{M}, \tau)$ with the topology t_τ of convergence in measure [17, Chapter IX, § 2] whose base of neighborhoods of zero consists of the sets

$$\mathcal{U}(\varepsilon, \delta) = \{X \in S(\mathcal{M}, \tau) : \exists\, Q \in \mathcal{M}^{\mathrm{pr}}\ (\|XQ\| \le \varepsilon \ \text{ and } \ \tau(Q^\perp) \le \delta)\},\ \varepsilon > 0,\ \delta > 0.$$

It is known that $\langle S(\mathcal{M}, \tau), t_\tau\rangle$ is a complete metrizable topological $*$-algebra; furthermore, $\mathcal{M}$ is dense in $\langle S(\mathcal{M}, \tau),\ t_\tau\rangle$. To indicate the convergence of a net $\{X_j\}_{j\in J} \subset S(\mathcal{M}, \tau)$ to $X \in S(\mathcal{M}, \tau)$ in the topology t_τ, we write $X_j \xrightarrow{\tau} X$ and say that $\{X_j\}_{j\in J}$ *converges to* X *in measure* τ.

Denote by $\mu(X, t)$ a *rearrangement* of $X \in S(\mathcal{M}, \tau)$, meaning a nonincreasing right-continuous function $\mu(X, \cdot)\colon (0, \infty) \to [0, \infty)$ defined as

$$\mu(X, t) = \inf\{\|XP\| : P \in \mathcal{M}^{\mathrm{pr}},\ \tau(P^\perp) \le t\}, \quad t > 0.$$

The set of τ-compact operators $S_0(\mathcal{M}, \tau) = \{X \in S(\mathcal{M}, \tau) : \lim_{t\to\infty} \mu(X, t) = 0\}$ is an ideal of $S(\mathcal{M}, \tau)$ and the set of elementary operators $\mathcal{F}(\mathcal{M}, \tau) = \{X \in \mathcal{M} : (\exists s > 0)\ \mu(X, t) = 0\ \forall t > s\}$ is an ideal of $\mathcal{M}$. The topology t_τ is also determined by the F-norm $\rho_\tau(X) = \inf_{t>0} \max\{t, \mu(X, t)\},\ X \in S(\mathcal{M}, \tau)$.

Lemma 1 [18]. *Take $X, Y, X_j \in S(\mathcal{M}, \tau)$ for $j \in J$. Then*
(i) $\mu(X,t) = \mu(|X|,t) = \mu(X^*,t)$ *for all* $t > 0$;
(ii) $\mu(X^*X,t) = \mu(XX^*,t)$ *for all* $t > 0$;
(iii) *if* $|X| \le |Y|$ *then* $\mu(X,t) \le \mu(Y,t)$ *for all* $t > 0$;
(iv) *if* $X \in \mathcal{M}$ *then* $\lim_{t\to+0} \mu(X,t) = \sup_{t>0} \mu(X,t) = \|X\|$;
(v) $\mu(XY, t+s) \le \mu(X,t)\mu(Y,s)$ *for all* $t, s > 0$;
(vi) $\mu(|X|^\alpha, t) = \mu(X,t)^\alpha$ *for all* $\alpha > 0$ *and* $t > 0$;
(vii) $X_j \xrightarrow{\tau} X \iff \mu(X_j - X, t) \to 0$ *for each* $t > 0$.

Denote the linear Lebesgue measure on $\mathbb{R}$ by m. Define the noncommutative Lebesgue space L_p, with $0 < p < \infty$, associated with $(\mathcal{M}, \tau)$ as

$$L_p(\mathcal{M}, \tau) = \{X \in S(\mathcal{M}, \tau) : \mu(X, \cdot) \in L_p(\mathbb{R}^+, m)\}$$

with the F-norm (a norm for $1 \le p < \infty$) $\|X\|_p = \|\mu(X, \cdot)\|_p$, $X \in L_p(\mathcal{M}, \tau)$.

2. The Topologies of Local Convergence in Measure on $S(\mathcal{M}, \tau)$

The topology t_τ of convergence in measure can be localized as follows. Given $\varepsilon, \delta > 0$ and $P \in \mathcal{M}_\tau^{\mathrm{pr}}$, define the sets

$$\mathcal{V}(\varepsilon, \delta, P) = \{X \in S(\mathcal{M}, \tau) : \exists Q \in \mathcal{M}^{\mathrm{pr}}\ (Q \le P,\ \|XQ\| \le \varepsilon \text{ and } \tau(P - Q) \le \delta)\},$$

$$\mathcal{W}(\varepsilon, \delta, P) = \{X \in S(\mathcal{M}, \tau) : (\exists Q \in \mathcal{M}^{\mathrm{pr}})\ (Q \le P,\ \|QXQ\| \le \varepsilon \text{ and } \tau(P - Q) \le \delta)\}.$$

The space $S(\mathcal{M}, \tau)$ becomes a topological vector space if endowed with the topology $t_{\tau l}$ of *τ-local*, respectively $t_{w\tau l}$ of *weakly τ-local*, *convergence in measure* whose base of neighborhoods of zero consists of the family $\Theta = \{\mathcal{V}(\varepsilon, \delta, P)\}_{\varepsilon,\delta>0; P\in\mathcal{M}_\tau^{\mathrm{pr}}}$, respectively $\Theta = \{\mathcal{W}(\varepsilon, \delta, P)\}_{\varepsilon,\delta>0; P\in\mathcal{M}_\tau^{\mathrm{pr}}}$. We write $X_i \xrightarrow{\tau l} X$ and $X_i \xrightarrow{w\tau l} X$ to indicate $t_{\tau l}$-convergence and $t_{w\tau l}$-convergence. Using the standard techniques for von Neumann algebra reduction, we can show (see also [3, 6]) that $X_i \xrightarrow{\tau l} X$, respectively $X_i \xrightarrow{w\tau l} X$, if and only if $X_i P \xrightarrow{\tau} XP$, cf. [1, p. 114], respectively $PX_iP \xrightarrow{\tau} PXP$, cf. [1, p. 114; 2, p. 746], for all $P \in \mathcal{M}_\tau^{\mathrm{pr}}$. It is clear that $t_{w\tau l} \le t_{\tau l} \le t_\tau$ and $t_{w\tau l}$-convergence coincides with convergence in measure in $\langle S(P\mathcal{M}P) = PS(\mathcal{M}, \tau)P,\ t_{\tau(P\cdot P)}\rangle$ for all $P \in \mathcal{M}_\tau^{\mathrm{pr}}$. We can also define the topologies $t_{\tau l}$ and $t_{w\tau l}$ in terms of nonincreasing rearrangements. The family $\widetilde{\Theta} = \{\widetilde{\mathcal{V}}(\varepsilon, \delta, P)\}_{\varepsilon,\delta>0;\, P\in\mathcal{M}_\tau^{\mathrm{pr}}}$, where $\widetilde{\mathcal{V}}(\varepsilon, \delta, P) = \{X \in S(\mathcal{M}, \tau) : \mu(XP, \delta) < \varepsilon\}$, also determines a base of neighborhoods of zero for $t_{\tau l}$. If $\tau(I) < \infty$ then $t_\tau = t_{\tau l} = t_{w\tau l}$.

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ is the $*$-algebra of all bounded linear operators on $\mathcal{H}$ and $\tau = \mathrm{tr}$ is the canonical trace then $S(\mathcal{M}, \tau)$, $S_0(\mathcal{M}, \tau)$, $L_p(\mathcal{M}, \tau)$, and $\mathcal{F}(\mathcal{M}, \tau)$ coincide with $\mathcal{B}(\mathcal{H})$, the ideal $\mathfrak{S}_\infty(\mathcal{H})$ of compact operators, the Schatten–von Neumann ideal $\mathfrak{S}_p(\mathcal{H})$, and the ideal $\mathcal{F}(\mathcal{H})$ of finite-rank operators on $\mathcal{H}$ respectively. The topology t_τ coincides with the norm topology $\|\cdot\|$, while $t_{\tau l}$ and $t_{w\tau l}$, coincide with the topologies of strong and weak operator convergence, respectively. We have

$$\mu(X, t) = \sum_{n=1}^{\infty} s_n(X) \chi_{[n-1,n)}(t)$$

for $t > 0$, where $\{s_n(X)\}_{n=1}^\infty$ is the sequence of s-numbers of the compact operator X and χ_A is the indicator of some set $A \subset \mathbb{R}$.

If $\mathcal{M}$ is abelian, i.e., commutative; then $\mathcal{M} \simeq L^\infty(\Omega, \Sigma, \mu)$ and $\tau(f) = \int_\Omega f \,\mathrm{d}\,\mu$, where (Ω, Σ, μ) is a localizable measure space, $S(\mathcal{M}, \tau)$ coincides with the algebra of all measurable complex functions f on (Ω, Σ, μ) bounded outside a set of finite measure. The topology t_τ is the usual topology of convergence in measure and $t_{\tau l}$ coincides with $t_{w\tau l}$ and available topology of convergence in measure on the sets of finite measure.

Theorem 1. *Suppose that $X, Y \in S(\mathcal{M}, \tau)$ with $Y = Y^*$ and $X^n \xrightarrow{\tau l} 0$ as $n \to \infty$. If $X^* Y X \le Y$ then $Y \ge 0$.*

Proof. The assumption $X^*YX \le Y$ implies the chain of inequalities

$$Y \ge X^*YX \ge X^{*2}YX^2 \ge \cdots \ge X^{*n}YX^n \ge \cdots .$$

Take the Jordan decomposition $Y = Y_+ - Y_-$ into the positive and negative parts with $Y_+Y_- = 0$ and put $|Y| = Y_+ + Y_-$. Given an arbitrary projection $P \in \mathcal{M}_\tau^{\mathrm{pr}}$, write down the equalities

$$PX^{*n}YX^nP = PX^{*n}Y_+X^nP - PX^{*n}Y_-X^nP, \quad n \in \mathbb{N}.$$

For each $t > 0$ claims (ii), (vi), and (v) of Lemma 1 yield

$$\mu(PX^{*n}Y_+X^nP, t) = \mu(\sqrt{Y_+}X^nP, t)^2 \le \mu(\sqrt{Y_+}, t/2)^2 \mu(X^nP, t/2)^2 \to 0 \quad \text{as } n \to \infty.$$

Hence, $X^{*n}Y_+X^n \xrightarrow{w\tau l} 0$ as $n \to \infty$, see claim (vii) of Lemma 1. Similarly we obtain $X^{*n}Y_-X^n \xrightarrow{w\tau l} 0$ as $n \to \infty$. By the $t_{w\tau l}$-continuity of the addition $(A, B) \mapsto A + B$ from $S(\mathcal{M}, \tau) \times S(\mathcal{M}, \tau)$ to $S(\mathcal{M}, \tau)$, we have $X^{*n}YX^n \xrightarrow{w\tau l} 0$ as $n \to \infty$. Thus, $Y \ge 0$. $\square$

Theorem 2. *The $*$-ideal $\mathcal{F}(\mathcal{M}, \tau)$ is $t_{\tau l}$-dense in $S(\mathcal{M}, \tau)$.*

Proof. Step 1. Take $X \in S(\mathcal{M}, \tau)$ and its polar decomposition $X = U|X|$. If $(X_j)_{j\in J} \subset \mathcal{F}(\mathcal{M}, \tau)$ and $X_j \xrightarrow{\tau l} |X|$ then by the separate $t_{\tau l}$-continuity of multiplication [5, Theorem 1] we have $UX_j \xrightarrow{\tau l} U|X| = X$. Therefore, it suffices to show that

$$\forall X \in S(\mathcal{M}, \tau)^+ \, \exists \, (Y_j)_{j\in J} \subset \mathcal{F}(\mathcal{M}, \tau) \quad (Y_j \xrightarrow{\tau l} X).$$

Step 2. Take $X \in S(\mathcal{M}, \tau)^+$ such that $X = Y + Z$ with $Y \in \mathcal{M}^+$ and $Z \in S_0(\mathcal{M}, \tau)^+$ [19]. If $Z = \int_0^\infty \lambda \, dE_\lambda$ is the spectral representation then the operator $Z_n = \int_{1/n}^n \lambda \, dE_\lambda$ lies in the cone $\mathcal{F}(\mathcal{M}, \tau)^+$ for each $n \in \mathbb{N}$. We have

$$Z - Z_n = \int\limits_{[0,1/n)} \lambda \, dE_\lambda + \int\limits_{(n,\infty)} \lambda \, dE_\lambda \equiv Z_{n,1} + Z_{n,2}, \quad n \in \mathbb{N}.$$

It is obvious that $\|Z_{n,1}\| \le \frac{1}{n}$; hence, $Z_{n,1} \xrightarrow{\tau} 0$ as $n \to \infty$. Take the projection $s_r(A)$ onto the support of $A \in S(\mathcal{M}, \tau)^{\mathrm{h}}$. We have $s_r(Z_{n,2}) = E_n^\perp$; since by assumption Z is τ-measurable and the trace τ is normal, we infer that $\tau(E_n^\perp) \to 0$ as $n \to \infty$. Consequently, $Z_{n,2} \xrightarrow{\tau} 0$ as $n \to \infty$. Thus, $Z_n \xrightarrow{\tau} 0$ as $n \to \infty$.

Suppose that $\mathcal{M}_\tau^{\mathrm{pr}} = (P_j)_{j\in J}$. Then $I = \bigvee_{j\in J} P_j$ because the trace τ is semifinite. We have $Y_j = YP_j \in \mathcal{F}(\mathcal{M}, \tau)$ for all $j \in J$. Since $P_j \xrightarrow{\tau l} I$, for every projection $P \in \mathcal{M}_\tau^{\mathrm{pr}}$ we obtain $Y_jP \xrightarrow{\tau} YP$. Consequently, $Y_j \xrightarrow{\tau l} Y$ and the assertion follows from the $t_{\tau l}$-continuity of addition in $S(\mathcal{M}, \tau)$. $\square$

Since $\mathcal{F}(\mathcal{M}, \tau) \subset L_p(\mathcal{M}, \tau)$ for $0 < p < \infty$, we have

Corollary 1. *Every space $L_p(\mathcal{M}, \tau)$ for $0 < p < \infty$ is $t_{\tau l}$-dense in $S(\mathcal{M}, \tau)$.*

Theorem 3. (i) *If t_τ is locally convex then so is $t_{\tau l}$.*
(ii) *If $t_{\tau l}$ is locally convex then so is $t_{w\tau l}$.*

Proof. (i): If t_τ is locally convex then t_τ is determined by a family $(p_j)_{j\in J}$ of seminorms on $S(\mathcal{M}, \tau)$ [20, 1.10.1]. Then the family $(p_{j,P})_{j\in J; P\in\mathcal{M}_\tau^{\mathrm{pr}}}$ of seminorms on $S(\mathcal{M}, \tau)$, where $p_{j,P}(X) = p_j(XP)$ for all $j \in J$, with $P \in \mathcal{M}_\tau^{\mathrm{pr}}$ and $X \in S(\mathcal{M}, \tau)$, is a defining family for $t_{\tau l}$.

(ii): If $t_{\tau l}$ is locally convex then $t_{\tau l}$ is determined by a family $(q_j)_{j\in J}$ of seminorms on $S(\mathcal{M},\tau)$ [20, 1.10.1]. Then the family $(q_{j,P})_{j\in J;P\in\mathcal{M}_\tau^{\rm pr}}$ of seminorms on $S(\mathcal{M},\tau)$, where $q_{j,P}(X)=q_j(PXP)$ for all $j\in J$, with $P\in\mathcal{M}_\tau^{\rm pr}$ and $X\in S(\mathcal{M},\tau)$, is a defining family for $t_{w\tau l}$. $\square$

Recall [19] that the following are equivalent:
(i) $\mathcal{M}=S(\mathcal{M},\tau)$;
(ii) $\inf\{\tau(P):P\in\mathcal{M}^{\rm pr},\ \tau(P)\neq 0\}>0$;
(iii) t_τ coincides with the $\|\cdot\|$-topology on $\mathcal{M}$.
It is easy to verify that if these conditions hold then $\mathcal{M}$ is atomic.

Corollary 2. *For an atomic von Neumann algebra $\mathcal{M}$ with $\inf\{\tau(P):P\in\mathcal{M}^{\rm pr},\ \tau(P)\neq 0\}=0$, if*

$$\exists K>0 \qquad \sum_{\tau(P)<K,\ P \text{ is an atom}} \tau(P)<\infty \tag{1}$$

then $t_{\tau l}$ and $t_{w\tau l}$ are locally convex.

Proof. The claim follows from [21, Theorem 3.2] and Theorem 3. $\square$

Corollary 3. *Suppose that $\mathcal{M}$ is an abelian atomic von Neumann algebra and one of the following holds:* (a) $\inf\{\tau(P):P\in\mathcal{M}^{\rm pr},\ \tau(P)\neq 0\}>0$ *or* (b) $\inf\{\tau(P):P\in\mathcal{M}^{\rm pr},\ \tau(P)\neq 0\}=0$ *and condition* (1). *Then $t_{\tau l}=t_{w\tau l}$ is locally convex.*

Proof. The claim follows from [21, Corollary 3.5] and Theorem 3. $\square$

3. Sequences of F-NIPs on $(\mathcal{M},\tau)$

Consider a faithful normal semifinite trace τ on a von Neumann algebra $\mathcal{M}$. A $*$-linear space $\mathcal{X}\subset S(\mathcal{M},\tau)$ endowed with the F-norm $\|\cdot\|_{\mathcal{X}}$ is an *F-normed ideal space* (F-NIP) on $(\mathcal{M},\tau)$ whenever
(i) $\|A\|_{\mathcal{X}}=\|A^*\|_{\mathcal{X}}$ for all $A\in\mathcal{X}$;
(ii) $A\in\mathcal{X}$ and $B\in S(\mathcal{M},\tau)$ with $|B|\le|A|$ imply that $B\in\mathcal{X}$ and $\|B\|_{\mathcal{X}}\le\|A\|_{\mathcal{X}}$.
The natural embedding $\langle\mathcal{X},\|\cdot\|_{\mathcal{X}}\rangle\subset S(\mathcal{M},\tau)$ is $t_{w\tau l}$-continuous (see [22, Theorem 1] as well as [15]). Regarding the ideal spaces of τ-measurable operators, see [23, 24] and the bibliographies therein.

Example (a "weighted" F-NIP). Consider an F-NIP $\langle\mathcal{X},\|\cdot\|_{\mathcal{X}}\rangle$ on $(\mathcal{M},\tau)$. If $X\in\mathcal{X}$ and $T\in\mathcal{M}$ with $\|T\|\le 1$ then

$$\|TX\|_{\mathcal{X}}\le\|X\|_{\mathcal{X}},\quad \|XT\|_{\mathcal{X}}\le\|X\|_{\mathcal{X}}; \tag{2}$$

for $Y\notin\mathcal{X}$ we write $\|Y\|_{\mathcal{X}}=+\infty$. If $A\in S(\mathcal{M},\tau)^{\rm h}$ then for

$$\|X\|_{\mathcal{X}(A)}\equiv\left\|\frac{AX+XA}{2}\right\|_{\mathcal{X}},\qquad X\in\mathcal{X}(A)=\{Y\in S(\mathcal{M},\tau):AY+YA\in\mathcal{X}\},$$

we have

$$\|X^*\|_{\mathcal{X}(A)}=\left\|\frac{AX^*+X^*A}{2}\right\|_{\mathcal{X}}=\left\|\frac{(AX^*+X^*A)^*}{2}\right\|_{\mathcal{X}}=\|X\|_{\mathcal{X}(A)}.$$

If also $AZ=ZA$ for all $Z\in S(\mathcal{M},\tau)$ and $X,Y\in S(\mathcal{M},\tau)$ with $|X|\le|Y|$ then $|X|=T|Y|T^*$ for some $T\in\mathcal{M}$ with $\|T\|\le 1$ [25], and (2) yields

$$\begin{aligned}\||X|\|_{\mathcal{X}(A)}&=\left\|\frac{A|X|+|X|A}{2}\right\|_{\mathcal{X}}=\|A|X|\|_{\mathcal{X}}=\|AT|Y|T^*\|_{\mathcal{X}}\\&\le\|AT|Y|\|_{\mathcal{X}}=\|T|Y|A\|_{\mathcal{X}}\le\||Y|A\|_{\mathcal{X}}=\||Y|\|_{\mathcal{X}(A)}.\end{aligned}$$

If, moreover, A is invertible in $S(\mathcal{M},\tau)$ (regarding invertibility in $S(\mathcal{M},\tau)$, see [26, 27]); then $\langle\mathcal{X}(A),\|\cdot\|_{\mathcal{X}(A)}\rangle$ is an F-NIP on $(\mathcal{M},\tau)$; if $\|\cdot\|_{\mathcal{X}}$ is a norm on $\mathcal{X}$ then $\|\cdot\|_{\mathcal{X}(A)}$ is a norm on $\mathcal{X}(A)$. Given $\mathcal{U}\subset\mathcal{X}$, put $A^{-1}\cdot\mathcal{U}\equiv\{A^{-1}U:\ U\in\mathcal{U}\}$. It is obvious that $\mathcal{U}$ is convex if and only if so is $A^{-1}\cdot\mathcal{U}$.

Theorem 4. *Consider an invertible operator $A \in S(\mathcal{M},\tau)^{\mathrm{h}}$ in $S(\mathcal{M},\tau)$ with $AZ = ZA$ for all $Z \in S(\mathcal{M},\tau)$.*

(i) *An F-NIP $\langle \mathcal{X}, \|\cdot\|_{\mathcal{X}}\rangle$ is complete if and only if so is $\langle \mathcal{X}(A), \|\cdot\|_{\mathcal{X}(A)}\rangle$.*

(ii) *$(X_n)_{n=1}^{\infty}$ is a Schauder basis in an F-NIP $\langle \mathcal{X}, \|\cdot\|_{\mathcal{X}}\rangle$ if and only if $(A^{-1}X_n)_{n=1}^{\infty}$ is a Schauder basis in $\langle \mathcal{X}(A), \|\cdot\|_{\mathcal{X}(A)}\rangle$.*

(iii) *$\mathcal{U} \subset \mathcal{X}$ is a neighborhood of zero in an F-NIP $\langle \mathcal{X}, \|\cdot\|_{\mathcal{X}}\rangle$ if and only if $A^{-1}\cdot\mathcal{U}$ is a neighborhood of zero in $\langle \mathcal{X}(A), \|\cdot\|_{\mathcal{X}(A)}\rangle$.*

(iv) *$\mathcal{U} \subset \mathcal{X}$ is bounded in an F-NIP $\langle \mathcal{X}, \|\cdot\|_{\mathcal{X}}\rangle$ if and only if $A^{-1}\cdot\mathcal{U}$ is bounded in $\langle \mathcal{X}(A), \|\cdot\|_{\mathcal{X}(A)}\rangle$.*

Proof. Recall that $\mathcal{X}(A) = \{Y \in S(\mathcal{M},\tau) : \mathbf{M}_A Y = AY \in \mathcal{X}\}$, while $A, A^{-1} \in S(\mathcal{M},\tau)^{\mathrm{h}}$ with $AZ = ZA$ and $A^{-1}Z = ZA^{-1}$ for all $Z \in S(\mathcal{M},\tau)$. Note that $(\mathcal{X}(A))(A^{-1}) = \mathcal{X}$ and $\|\cdot\|_{(\mathcal{X}(A))(A^{-1})} = \|\cdot\|_{\mathcal{X}}$. Therefore, we just need to establish the sufficiency of each of conditions (i)–(iv).

(i): If $(X_n)_{n=1}^{\infty} \subset \mathcal{X}(A)$ is $\|\cdot\|_{\mathcal{X}(A)}$-fundamental then

$$\|X_n - X_m\|_{\mathcal{X}(A)} = \|A(X_n - X_m)\|_{\mathcal{X}} \to 0 \quad \text{as } n, m \to \infty,$$

i.e., $(AX_n)_{n=1}^{\infty}$ is $\|\cdot\|_{\mathcal{X}}$-fundamental. Since the F-NIP $\langle \mathcal{X}, \|\cdot\|_{\mathcal{X}}\rangle$ is complete, there is $X \in \mathcal{X}$ such that $\|AX_n - X\|_{\mathcal{X}} \to 0$ as $n \to \infty$. Consequently, $\|AX_n - A\cdot A^{-1}X\|_{\mathcal{X}} = \|X_n - A^{-1}X\|_{\mathcal{X}(A)} \to 0$ as $n \to \infty$.

(ii): Suppose that $(X_n)_{n=1}^{\infty} \subset \mathcal{X}$ is a Schauder basis for $\mathcal{X}$ [28, Chapter II, § 5], i.e., for each $X \in \mathcal{X}$ there exists a unique expansion $X = \sum_{n=1}^{\infty} \lambda_n X_n$ into a $\|\cdot\|_{\mathcal{X}}$-converging series with $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{C}$. If $Y \in \mathcal{X}(A)$ then $AY \in \mathcal{X}$, and $AY = \sum_{n=1}^{\infty} \lambda_n X_n$ by assumption. Since this series $\|\cdot\|_{\mathcal{X}}$-converges, [22, Theorem 1] shows that

$$\sum_{n=1}^{k} \lambda_n X_n \xrightarrow{w\tau l} AY \quad \text{as } k \to \infty.$$

Hence, the separate $t_{w\tau l}$-continuity of multiplication [5, Theorem 1] implies that

$$A^{-1}\sum_{n=1}^{k} \lambda_n X_n = \sum_{n=1}^{k} \lambda_n A^{-1} X_n \xrightarrow{w\tau l} Y \quad \text{as } k \to \infty; \tag{3}$$

i.e., $Y = \sum_{n=1}^{\infty} \lambda_n A^{-1} X_n$.

Uniqueness. Suppose that $Y \in \mathcal{X}(A)$ admits two distinct representations

$$Y = \sum_{n=1}^{\infty} \lambda_n A^{-1} X_n = \sum_{n=1}^{\infty} \alpha_n A^{-1} X_n.$$

Then by Theorem 1 of [22] we have

$$\sum_{n=1}^{k} \lambda_n A^{-1} X_n \xrightarrow{w\tau l} Y, \quad \sum_{n=1}^{k} \alpha_n A^{-1} X_n \xrightarrow{w\tau l} Y \quad \text{as } k \to \infty.$$

Hence, the separate $t_{w\tau l}$-continuity of multiplication [5, Theorem 1] implies that, see (3),

$$AY = \sum_{n=1}^{\infty} \lambda_n X_n = \sum_{n=1}^{\infty} \alpha_n X_n;$$

this is a contradiction.

(iii): Take an open set $\mathcal{U}$ in $\mathcal{X}$; i.e.,

$$\forall X \in \mathcal{U}\, \exists \varepsilon = \varepsilon(X) > 0\, \forall Y \in \mathcal{X} \quad (\|X - Y\|_{\mathcal{X}} < \varepsilon \Rightarrow Y \in \mathcal{U}). \tag{4}$$

We have to show that

$$\forall \widetilde{X} \in A^{-1} \cdot \mathcal{U}\, \exists \widetilde{\varepsilon} = \widetilde{\varepsilon}(\widetilde{X}) > 0\, \forall \widetilde{Y} \in \mathcal{X}(A) \quad (\|\widetilde{X} - \widetilde{Y}\|_{\mathcal{X}(A)} < \widetilde{\varepsilon} \Rightarrow \widetilde{Y} \in A^{-1} \cdot \mathcal{U}).$$

For $\widetilde{X}, \widetilde{Y} \in A^{-1} \cdot \mathcal{U}$ there are $X, Y \in \mathcal{U}$ with $\widetilde{X} = A^{-1}X$ and $\widetilde{Y} = A^{-1}U$; appreciating (4), we have

$$\|\widetilde{X} - \widetilde{Y}\|_{\mathcal{X}(A)} = \|A(A^{-1}X - A^{-1}Y)\|_{\mathcal{X}} = \|X - Y\|_{\mathcal{X}} < \varepsilon.$$

We can take $\widetilde{\varepsilon} = \varepsilon$.

(iv) A set $\mathcal{U} \subset \mathcal{X}$ is bounded in $\langle \mathcal{X}, \|\cdot\|_{\mathcal{X}}\rangle$ if and only if for each neighborhood $\mathcal{V}$ of zero there is a real $\lambda > 0$ such that $\mathcal{U} \subset \lambda\mathcal{V}$. In other words, $\|\alpha_n X_n\|_{\mathcal{X}} \to 0$ as $n \to \infty$ for all sequences $(X_n)_{n=1}^{\infty} \subset \mathcal{U}$ and $(\alpha_n)_{n=1}^{\infty} \subset \mathbb{C}$ with $\alpha_n \to 0$ as $n \to \infty$ [29, Chapter 1, Theorem 1.30]. Verify that $A^{-1} \cdot \mathcal{U}$ is bounded in $\langle \mathcal{X}(A), \|\cdot\|_{\mathcal{X}(A)}\rangle$. Take an arbitrary sequence $(Y_n)_{n=1}^{\infty} \subset A^{-1} \cdot \mathcal{U}$ and $(\alpha_n)_{n=1}^{\infty} \subset \mathbb{C}$ with $\alpha_n \to 0$ as $n \to \infty$. Then there is $(X_n)_{n=1}^{\infty} \subset \mathcal{U}$ such that $Y_n = A^{-1}X_n$ for all $n \in \mathbb{N}$ and

$$\|\alpha_n Y_n\|_{\mathcal{X}(A)} = \|\alpha_n A^{-1} X_n\|_{\mathcal{X}(A)} = \|\alpha_n A A^{-1} X_n\|_{\mathcal{X}} = \|\alpha_n X_n\|_{\mathcal{X}} \to 0$$

as $n \to \infty$. □

Obviously, if an F-normed space $\langle \mathcal{E}, \|\cdot\|_{\mathcal{E}}\rangle$ has a Schauder basis then the space is separable. Every topological vector space with a bounded neighborhood of zero is *locally bounded.*

Corollary 4. *An F-NIP $\langle \mathcal{X}, \|\cdot\|_{\mathcal{X}}\rangle$ is locally bounded, respectively locally convex, if and only if so is $\langle \mathcal{X}(A), \|\cdot\|_{\mathcal{X}(A)}\rangle$.*

Corollary 5. *An F-NIP $\langle \mathcal{X}, \|\cdot\|_{\mathcal{X}}\rangle$ is normable if and only if so is $\langle \mathcal{X}(A), \|\cdot\|_{\mathcal{X}(A)}\rangle$.*

Proof. A topological vector space is normable if and only if it includes a bounded convex neighborhood of zero [29, Chapter 1, Theorem 1.39]. □

Lemma 2. *Consider two F-NIPs $\mathcal{X}$ and $\mathcal{Y}$ on $(\mathcal{M}, \tau)$ such that $\langle \mathcal{Y}, \|\cdot\|_{\mathcal{Y}}\rangle$ is complete, and a dense linear subspace $\mathcal{Z}$ of $\mathcal{X}$. Take $A \in S(\mathcal{M}, \tau)$ with $A\mathcal{Z} \subseteq \mathcal{Y}$ and continuous multiplier $\mathbf{M}_A : \mathcal{Z} \to \mathcal{Y}$. Then the continuous extension $\overline{\mathbf{M}}_A : \mathcal{X} \to \mathcal{Y}$ of $\mathbf{M}_A$ to the whole space $\mathcal{X}$ is also a multiplier by A.*

Proof. Take $X \in \mathcal{X}$ and $X_i \in \mathcal{Z}$ such that $X_i \to X$ in $\langle \mathcal{X}, \|\cdot\|_{\mathcal{X}}\rangle$. Then $\mathbf{M}_A X_i = \overline{\mathbf{M}}_A X_i \to \overline{\mathbf{M}}_A X$ in $\langle \mathcal{Y}, \|\cdot\|_{\mathcal{Y}}\rangle$. By [5, Theorem 1] we see that $X_i \xrightarrow{w\tau l} X$, as well as $\mathbf{M}_A X_i \xrightarrow{w\tau l} \overline{\mathbf{M}}_A X$. In view of [22, Theorem 1] the multiplier $\mathbf{M}_A : S(\mathcal{M}, \tau) \to S(\mathcal{M}, \tau)$ is $t_{w\tau l}$-continuous. Therefore, $\mathbf{M}_A X_i \xrightarrow{w\tau l} \mathbf{M}_A X$, and so $\overline{\mathbf{M}}_A X = \mathbf{M}_A X = AX$. □

The analogous result holds for $\mathbf{L}_A X = XA$.

Theorem 5. *Consider two F-NIPs $\mathcal{X}$ and $\mathcal{Y}$ on $(\mathcal{M}, \tau)$ and suppose that $A\mathcal{X} \subseteq \mathcal{Y}$ for some operator $A \in S(\mathcal{M}, \tau)$. Then the multiplier $\mathbf{M}_A X = AX$, acting as $\mathbf{M}_A : \mathcal{X} \to \mathcal{Y}$, is continuous. In particular, for $\mathcal{X} \subseteq \mathcal{Y}$ the natural embedding of $\mathcal{X}$ into $\mathcal{Y}$ is continuous.*

Proof. Verify that the graph of $\mathbf{M}_A$ is closed. Indeed, if $X_n \to X$ in $\mathcal{X}$ and $AX_n \to Y$ in $\mathcal{Y}$ then $X_n \xrightarrow{w\tau l} X$ and $AX_n \xrightarrow{w\tau l} Y$ by [22, Theorem 1]. Then $Y = AX$ because of the separate $t_{w\tau l}$-continuity of multiplication [5, Theorem 1]. It remains to apply the Closed Graph Theorem [29, Chapter 2, Theorem 2.15]. □

Remark 1. The topological intersection of an at most countable family of F-NIPs $\langle \mathcal{X}_n, \|\cdot\|_{\mathcal{X}_n}\rangle$ on $(\mathcal{M}, \tau)$ is also an F-NIP on $(\mathcal{M}, \tau)$ with the F-norm

$$\|X\|_{\mathcal{X}} = \sum_{n \geq 1} 2^{-n} \frac{\|X\|_{\mathcal{X}_n}}{1 + \|X\|_{\mathcal{X}_n}}.$$

Recall that $\mathcal{X} = \bigcap_{n \geq 1} \mathcal{X}_n$; the projective, or initial, topology (see [30, p. 35]) induced by the embeddings $\mathcal{X} \subset \mathcal{X}_n$ for $n \in \mathbb{N}$ is the linear topology whose base of neighborhoods of zero consists of all sets of the form $\mathcal{U}_{n_1} \cap \cdots \cap \mathcal{U}_{n_m} \cap \mathcal{X}$, where $\mathcal{U}_n$ is a neighborhood of zero in $\mathcal{X}_n$. The space $\mathcal{X}$ endowed with this topology is the *topological intersection* of $\mathcal{X}_n$.

Theorem 6. *Consider a decreasing sequence* $(\mathcal{X}_n)_{n\geq 1}$ *of* F-NIPs *on* $(\mathcal{M},\tau)$ *such that the space* $\mathcal{X}=\bigcap_{n\geq 1}\mathcal{X}_n$ *is dense in each* $\mathcal{X}_n$*, some locally bounded* F-NIP $\mathcal{Y}$ *on* $(\mathcal{M},\tau)$*, and* $A\in S(\mathcal{M},\tau)$*. Then* $A\mathcal{X}\subseteq\mathcal{Y}$ *if and only if* $A\mathcal{X}_n\subseteq\mathcal{Y}$ *for some* $n\in\mathbb{N}$.

Proof. Assume that $A\mathcal{X}\subseteq\mathcal{Y}$. Theorem 5 and Remark 1 imply that the multiplier $\mathbf{M}_A:\mathcal{X}\to\mathcal{Y}$ is continuous. Suppose that $\mathcal{V}$ is a bounded neighborhood of zero in $\mathcal{Y}$. There exists a tuple of neighborhoods of zero $(\mathcal{U}_k(\subset\mathcal{X}_k))_{k=1}^n$ such that $\mathbf{M}_A(\mathcal{X}\cap\mathcal{U}_1\cap\dots\cap\mathcal{U}_n)\subseteq\mathcal{V}$. Since the embeddings $\mathcal{X}_n\subset\mathcal{X}_k$ for $k=1,\dots,n$ are continuous, there exist neighborhoods $\mathcal{W}_k$ of zero in $\mathcal{X}_k$ such that $\mathcal{W}_k\subset\mathcal{U}_k$ for $k=1,\dots,n$. Putting $\mathcal{U}=\bigcap_{k=1}^n\mathcal{W}_k$, we obtain a neighborhood of zero in $\mathcal{X}_n$ satisfying $\mathbf{M}_A(\mathcal{U}\cap\mathcal{X})\subseteq\mathcal{V}$. The boundedness of $\mathcal{V}$ is equivalent to the property that the system of its dilations $\varepsilon\mathcal{V}$ with $\varepsilon>0$ constitutes a base of neighborhoods of zero in $\mathcal{Y}$. Since for every $\varepsilon>0$ we have $\mathbf{M}_A(\varepsilon\mathcal{U}\cap\mathcal{X})\subseteq\varepsilon\mathcal{V}$, the mapping $\mathbf{M}_A:\mathcal{X}\to\mathcal{Y}$ is continuous in the topology on $\mathcal{X}$ induced from $\mathcal{X}_n$. Extend this mapping by continuity to some mapping from $\mathcal{X}_n$ to $\mathcal{Y}$. By Lemma 2 the extended mapping is also a multiplier by A; consequently, $A\mathcal{X}_n\subseteq\mathcal{Y}$. □

Corollary 6. *Consider the same* $(\mathcal{X}_n)_{n\geq 1}$, $\mathcal{X}$, *and* $\mathcal{Y}$ *as in Theorem* 6*. Then* $\mathcal{X}\subseteq\mathcal{Y}\iff\mathcal{X}_n\subseteq\mathcal{Y}$ *for some* $n\in\mathbb{N}$.

Proof. Apply Theorem 6 to the operator $A=I$. □

Theorem 7 (cf. [20, Chapter 6, § 6.5]). *If* $\mathcal{X}$, $\mathcal{Y}_1,\dots,\mathcal{Y}_n,\dots$ *is a sequence of* F-NIPs *on* $(\mathcal{M},\tau)$ *with* $\langle\mathcal{X},\|\cdot\|_{\mathcal{X}}\rangle$ *complete then* $\mathcal{X}\subseteq\bigcup_{n=1}^\infty\mathcal{Y}_n$ *if and only if* $\mathcal{X}\subseteq\mathcal{Y}_n$ *for some* $n\in\mathbb{N}$.

Proof. For $\mathcal{X}_n=\mathcal{X}\cap\mathcal{Y}_n$ we have $\mathcal{X}=\bigcup_{n=1}^\infty\mathcal{X}_n$. The Baire Category Theorem implies that for some $n\in\mathbb{N}$ the space $\mathcal{X}_n$ is a nonmeager set in $\mathcal{X}$. The space $\mathcal{X}_n$ regarded as the topological intersection of $\mathcal{X}$ and $\mathcal{Y}_n$ is an F-NIP on $(\mathcal{M},\tau)$ and the embedding $\mathcal{X}_n\subset\mathcal{X}$ is continuous. The Banach Theorem [31, Chapter III, Theorem 3] yields $\mathcal{X}_n=\mathcal{X}$; i.e., $\mathcal{X}\subseteq\mathcal{Y}_n$. □

References

1. Ciach L.J., "Some remarks on the convergence in measure and on a dominated sequence of operators measurable with respect to a semifinite von Neumann algebra," Colloq. Math., vol. 55, no. 1, 109–121 (1988).
2. Dodds P.G., Dodds T.K.-Y., and Pagter de B., "Noncommutative Köthe duality," Trans. Amer. Math. Soc., vol. 339, no. 2, 717–750 (1993).
3. Skvortsova G.Sh. and Tikhonov O.E., "Convex sets in noncommutative L^1-spaces that are closed in the topology of local convergence in measure," Russian Math. (Iz. VUZ), vol. 42, no. 8, 46–52 (1998).
4. Skvortsova G.Sh., "On the weak sequential completeness of quotient spaces of the space of integrable operators," Russian Math. (Iz. VUZ), vol. 46, no. 9, 68–71 (2002).
5. Bikchentaev A.M., "The continuity of multiplication for two topologies associated with a semifinite trace on von Neumann algebra," Lobachevskii J. Math., vol. 14, 17–24 (2004).
6. Dodds P.G., Dodds T.K.-Y., Sukochev F.A., and Tikhonov O.Ye., "A non-commutative Yosida–Hewitt theorem and convex sets of measurable operators closed locally in measure," Positivity, vol. 9, no. 3, 457–484 (2005).
7. Bikchentaev A.M., "Local convergence in measure on semifinite von Neumann algebras," Proc. Steklov Inst. Math., vol. 255, 35–48 (2006).
8. Bikchentaev A.M., "Local convergence in measure on semifinite von Neumann algebras. II," Math. Notes, vol. 82, no. 5, 703–707 (2007).
9. Bikchentaev A. and Sukochev F., "When weak and local measure convergence implies norm convergence," J. Math. Anal. Appl., vol. 473, no. 2, 1414–1431 (2019).
10. Bikchentaev A.M., "Convergence in measure and τ-compactness of τ-measurable operators, affiliated with a semifinite von Neumann algebra," Russian Math. (Iz. VUZ), vol. 64, no. 5, 79–82 (2020).
11. Cordero E., de Gosson M., and Nicola F., "On the positivity of trace class operators," Adv. Theor. Math. Phys., vol. 23, no. 8, 2061–2091 (2020).
12. Moslehian M.S., Kian M., and Xu Q., "Positivity of 2×2 block matrices of operators," Banach J. Math. Anal., vol. 13, no. 3, 726–743 (2019).
13. Bikchentaev A.M., "Invertibility of the operators on Hilbert spaces and ideals in C^*-algebras," Math. Notes, vol. 112, no. 3, 360–368 (2022).
14. Akemann C.A., Anderson J., and Pedersen G.K., "Triangle inequalities in operator algebras," Linear Multilinear Algebra, vol. 11, no. 2, 167–178 (1982).
15. Ber A.F., Levitina G.B., and Chilin V.I., "Derivations with values in quasi-normed bimodules of locally measurable operators," Siberian Adv. Math., vol. 25, no. 3, 169–178 (2015).

16. Takesaki M., *Theory of Operator Algebras. I*. Operator Algebras and Non-Commutative Geometry V, Springer, Berlin (2002) (Encyclopaedia of Mathematical Sciences; vol. 124).
17. Takesaki M., *Theory of Operator Algebras. II*. Operator Algebras and Non-Commutative Geometry VI, Springer, Berlin (2003) (Encyclopaedia of Mathematical Sciences; vol. 125).
18. Fack T. and Kosaki H., "Generalized s-numbers of τ-measurable operators," Pacific J. Math., vol. 123, no. 2, 269–300 (1986).
19. Stroh A. and West G. P., "τ-Compact operators affiliated to a semifinite von Neumann algebra," Proc. Roy. Irish Acad. Sect. A, vol. 93, no. 1, 73–86 (1993).
20. Edwards R.D., *Functional Analysis. Theory and Applications*, Holt Rinehart and Winston, New York, Chicago, San Francisco, Toronto, and London (1965).
21. Conradie J. and Crowther Ch., "The algebra of τ-measurable operators," in: *Operator algebras, Operator Theory and Applications*, Birkhäuser, Basel (2010), 103–119 (Oper. Theory Adv. Appl.; vol. 195).
22. Bikchentaev A.M., "Minimality of convergence in measure topologies on finite von Neumann algebras," Math. Notes, vol. 75, no. 3, 315–321 (2004).
23. Bikchentaev A.M., "Ideal spaces of measurable operators affiliated to a semifinite von Neumann algebra," Sib. Math. J., vol. 59, no. 2, 243–251 (2018).
24. Bikchentaev A.M., "Renormalizations of measurable operator ideal spaces affiliated to semi-finite von Neumann algebra," Ufa Math. J., vol. 11, no. 3, 3–10 (2019).
25. Yeadon F.J., "Convergence of measurable operators," Proc. Cambridge Phil. Soc., vol. 74, no. 2, 257–268 (1973).
26. Tembo I.D., "Invertibility in the algebra of τ-measurable operators," in: *Operator Algebras, Operator Theory and Applications*, Birkhäuser, Basel (2010), 245–256 (Oper. Theory Adv. Appl.; vol. 195).
27. Bikchentaev A.M., "Essentially invertible measurable operators affiliated to a semifinite von Neumann algebra and commutators," Sib. Math. J., vol. 63, no. 2, 224–232 (2022).
28. Rolewicz S., *Metric Linear Spaces*. Second Edition, PWN–Polish Sci. and Reidel, Warsaw and Dordrecht (1984).
29. Rudin W., *Functional Analysis*, McGraw-Hill, New York (1991).
30. Bourbaki N., *General Topology. Fundamental Structures*, Nauka, Moscow (1968) [Russian translation].
31. Banach S., *Theory of Linear Operators*, North-Holland, Amsterdam etc. (1987) (North-Holland Math. Library; vol. 38).

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