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HYPONORMAL MEASURABLE OPERATORS AFFILIATED TO A SEMIFINITE VON NEUMANN ALGEBRA

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Abstract—Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$. We study the cases when a hyponormal τ-measurable operator (or a restriction of it) is normal. We obtain a criterion for the hyponormality of a τ-measurable operator in terms of its singular value function. The set of all τ-measurable hyponormal operators is closed in the topology of τ-local convergence in measure. This assertion is a generalization of Problem 226 from the book "Halmos P.R., A Hilbert Space Problem Book, Second edition, Springer, New York (1982)" to the setting of unbounded operators. The set of all τ-measurable cohyponormal operators is closed in the topology of τ-local convergence in measure if and only if the von Neumann algebra $\mathcal{M}$ is finite.

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1. Introduction

Bounded hyponormal operators on a Hilbert space are the contents of articles by many researchers (see, for instance, [1–7] and the references therein). In the context of semifinite von Neumann algebras, the author published the articles [8–13] on the properties of (unbounded) τ-measurable hyponormal operators (also see [14]). Suppose that a von Neumann operator algebra $\mathcal{M}$ acts in a Hilbert space $\mathcal{H}$; $\mathcal{M}^{\mathrm{pr}}$ is the lattice of projections in $\mathcal{M}$; τ is a faithful normal semifinite trace on $\mathcal{M}$; $S(\mathcal{M},\tau)$ is the $*$-algebra of all τ-measurable operators; $S(\mathcal{M},\tau)^{\mathrm{h}}$ is the Hermitian part of $S(\mathcal{M},\tau)$; $\mu(t;X)$ is the singular value function of an operator $X \in S(\mathcal{M},\tau)$. We list the main results of our article; some of them are new even for the algebra $\mathcal{M} = \mathcal{B}(\mathcal{H})$ with $\tau = \mathrm{tr}$. If $T \in S(\mathcal{M},\tau)^{\mathrm{h}}$, $P \in \mathcal{M}^{\mathrm{pr}}$, and the operator $A := PT$ is hyponormal then $TP = PT$ and $A = PTP \in S(\mathcal{M},\tau)^{\mathrm{h}}$ (Theorem 2). If an operator $T \in S(\mathcal{M},\tau)$ is hyponormal, $P \in \mathcal{M}^{\mathrm{pr}}$, and $TP = \lambda P$ for some $\lambda \in \mathbb{C}$ then $TP = PT$ and the operator $T|_{P\mathcal{H}}$ is normal (Theorem 3). An operator $T \in S(\mathcal{M},\tau)$ is hyponormal (cohyponormal) if and only if $\mu(t;TP) \geq \mu(t;T^*P)$ (respectively, $\mu(t;T^*P) \geq \mu(t;TP)$) for all $t > 0$ and $P \in \mathcal{M}^{\mathrm{pr}}$ with $\tau(P) < +\infty$ (Theorem 6). In particular, an operator $T \in S(\mathcal{M},\tau)$ is normal if and only if $\mu(t;TP) = \mu(t;T^*P)$ for all $t > 0$ and $P \in \mathcal{M}^{\mathrm{pr}}$ with $\tau(P) < +\infty$ (Corollary 3). The set of all τ-measurable hyponormal operators is $t_{\tau l}$-closed (Theorem 7). This assertion is a generalization of [15, Problem 226] to unbounded operators. The set of all τ-measurable cohyponormal operators is $t_{\tau l}$-closed if and only if the von Neumann algebra $\mathcal{M}$ is finite (Corollary 4).

2. Definitions and Notation

Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$, let $\mathcal{M}^{\mathrm{pr}}$ be the lattice of projections ($P = P^2 = P^*$) in $\mathcal{M}$, let I be the unity of $\mathcal{M}$, and let $P^{\perp} = I - P$ for $P \in \mathcal{M}^{\mathrm{pr}}$. Let $\mathcal{M}^+$ be the cone of positive elements in $\mathcal{M}$, let $\|\cdot\|$ be the C^*-norm on $\mathcal{M}$, and let $\mathcal{M}_1 = \{X \in \mathcal{M} : \|X\| \leq 1\}$ be the unit ball of $\mathcal{M}$. Given $P, Q \in \mathcal{M}^{\mathrm{pr}}$, write $P \sim Q$ (the Murray–von Neumann equivalence) if $P = U^*U$ and $Q = UU^*$ for some $U \in \mathcal{M}$; say that $\mathcal{M}$ is *finite* if I is equivalent to no projection $P \in \mathcal{M} \setminus \{I\}$. The notation $P \preceq Q$ means that $P \sim R$ for some $R \in \mathcal{M}^{\mathrm{pr}}$ with $R \leq Q$.

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A mapping $\varphi : \mathcal{M}^+ \to [0, +\infty]$ is a *trace* if $\varphi(X+Y) = \varphi(X) + \varphi(Y)$ and $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{M}^+$, $\lambda \geq 0$ (here $0 \cdot (+\infty) \equiv 0$) and $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{M}$. A trace φ is called

- *faithful* if $\varphi(X) > 0$ for all $X \in \mathcal{M}^+$, $X \neq 0$;
- *normal* if $X_i \nearrow X$ ($X_i, X \in \mathcal{M}^+$) implies $\varphi(X) = \sup \varphi(X_i)$;
- *finite* if $\varphi(I) < +\infty$;
- *semifinite* if $\varphi(X) = \sup\{\varphi(Y) : Y \in \mathcal{M}^+,\ Y \leq X,\ \varphi(Y) < +\infty\}$ for every $X \in \mathcal{M}^+$ (see [16, Chapter V, Section 2; 17, Chapter 1, Section 1.15]).

An operator on $\mathcal{H}$, not necessarily bounded or densely defined, is *affiliated* to a von Neumann algebra $\mathcal{M}$ whenever it commutes with all unitary operators in the commutant $\mathcal{M}'$ of $\mathcal{M}$. Henceforth, τ stands for a faithful normal semifinite trace on $\mathcal{M}$; $\mathcal{M}_\tau^{\mathrm{pr}} = \{P \in \mathcal{M}^{\mathrm{pr}} : \tau(P) < +\infty\}$. A closed operator X affiliated to $\mathcal{M}$ whose domain $\mathcal{D}(X)$ is dense in $\mathcal{H}$ is *τ-measurable* whenever, given $\varepsilon > 0$, there exists $P \in \mathcal{M}^{\mathrm{pr}}$ such that $P\mathcal{H} \subset \mathcal{D}(X)$ and $\tau(P^\perp) < \varepsilon$. The set $S(\mathcal{M}, \tau)$ of all τ-measurable operators is a $*$-algebra under the taking of adjoint operators, the multiplication by scalars, and the strong addition and multiplication obtained as the closure of the ordinary operations (see [18, Chapter IX; 17, Chapter 2, Section 2.3]). Given a family $\mathcal{L} \subset S(\mathcal{M}, \tau)$, denote by $\mathcal{L}^+$ and $\mathcal{L}^{\mathrm{h}}$ the positive and Hermitian parts of $\mathcal{L}$. Denote by $\leq$ the partial order on $S(\mathcal{M}, \tau)^{\mathrm{h}}$ generated by the proper cone $S(\mathcal{M}, \tau)^+$. If $X \in S(\mathcal{M}, \tau)$ and $X = U|X|$ is the polar decomposition of X then $U \in \mathcal{M}$ and $|X| = \sqrt{X^*X} \in S(\mathcal{M}, \tau)^+$. An operator $A \in S(\mathcal{M}, \tau)$ is *hyponormal* whenever $A^*A \geq AA^*$; and A is *cohyponormal* whenever A^* is hyponormal.

The $*$-algebra $S(\mathcal{M}, \tau)$ is endowed with the topology t_τ of convergence in measure (see [18, Chapter IX, Section 2; 17, Chapter 2, Section 2.5]) with a fundamental system of neighborhoods of zero constituted by the sets

$$U(\varepsilon, \delta) = \{X \in S(\mathcal{M}, \tau) : \exists\, P \in \mathcal{M}^{\mathrm{pr}}\ (\|XP\| \leq \varepsilon \text{ and } \tau(P^\perp) \leq \delta)\}, \quad \varepsilon > 0,\ \delta > 0.$$

It is known that $(S(\mathcal{M}, \tau), t_\tau)$ is a complete metrizable topological $*$-algebra [17, Chapter 2, Sections 2.3 and 2.5] and $\mathcal{M}$ is complete in $(S(\mathcal{M}, \tau),\ t_\tau)$ [17, Chapter 2, Section 2.5]. For the convergence of a net $\{X_j\}_{j \in J} \subset S(\mathcal{M}, \tau)$ to $X \in S(\mathcal{M}, \tau)$ in the topology t_τ, the notation $X_j \xrightarrow{\tau} X$ is used and $\{X_j\}_{j \in J}$ is said to *converge to X in the measure τ*.

Denote by $\mu(t; X)$ the *singular value function* of $X \in S(\mathcal{M}, \tau)$, meaning the nonincreasing right-continuous function $\mu(\cdot\,; X) : (0, +\infty) \to [0, +\infty)$ defined by

$$\mu(t; X) = \inf\{\|XP\| : P \in \mathcal{M}^{\mathrm{pr}},\ \tau(P^\perp) \leq t\}, \quad t > 0.$$

Lemma 1 [19]. *Suppose that $X, Y \in S(\mathcal{M}, \tau)$ and $A, B \in \mathcal{M}$. Then*

(i) $\mu(t; X) = \mu(t; |X|) = \mu(t; X^*)$ *for all $t > 0$;*

(ii) *if $|X| \leq |Y|$ then $\mu(t; X) \leq \mu(t; Y)$ for all $t > 0$;*

(iii) $\mu(t; AXB) \leq \|A\|\|B\|\mu(t; X)$ *for all $t > 0$;*

(iv) $\mu(s + t; X + Y) \leq \mu(s; X) + \mu(t; Y)$ *for all $s, t > 0$;*

(v) $\mu(t; f(|X|)) = f(\mu(t; X))$ *for all continuous functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f(0) = 0$ and $t > 0$.*

The topology t_τ of convergence in measure can be localized as follows: Given $\varepsilon, \delta > 0$ and $P \in \mathcal{M}_\tau^{\mathrm{pr}}$, define the sets

$$\mathcal{V}(\varepsilon, \delta, P) = \{X \in S(\mathcal{M}, \tau) : \exists\, Q \in \mathcal{M}^{\mathrm{pr}}\ (Q \leq P,\ \|XQ\| \leq \varepsilon, \text{ and } \tau(P - Q) \leq \delta)\}.$$

The space $S(\mathcal{M}, \tau)$ becomes a topological vector space with respect to the topology $t_{\tau l}$ of *τ-local convergence in measure* with a neighborhood of zero $\Theta = \{\mathcal{V}(\varepsilon, \delta, P)\}_{\varepsilon, \delta > 0; P \in \mathcal{M}_\tau^{\mathrm{pr}}}$. We use the symbol $X_i \xrightarrow{\tau l} X$ for denoting $t_{\tau l}$-convergence. Employing the standard technique of reducing von Neumann algebras, we can show (also see [20, 21]) that $X_i \xrightarrow{\tau l} X$ if and only if $X_i P \xrightarrow{\tau} XP$ for all $P \in \mathcal{M}_\tau^{\mathrm{pr}}$, cf. [22, p. 114]; clearly, $t_{\tau l} \leq t_\tau$. See [20, 21; 23–25] for the properties of the topology $t_{\tau l}$. If the trace τ is finite then $t_\tau = t_{\tau l}$ is the minimal metrizable topology coordinated with the ring structure in $S(\mathcal{M}, \tau)$ [26].

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ is the $*$-algebra of all bounded linear operators on $\mathcal{H}$ and $\tau = \mathrm{tr}$ is the canonical trace then $S(\mathcal{M},\tau)$ coincides with $\mathcal{B}(\mathcal{H})$; the topology t_τ coincides with the topology of the norm $\|\cdot\|$; $t_{\tau l}$ coincides with the topology of the strong operator convergence. We have

$$\mu(t;X) = \sum_{n=1}^{\infty} s_n(X)\chi_{[n-1,n)}(t), \quad t > 0,$$

where $\{s_n(X)\}_{n=1}^{\infty}$ is the sequence of the s-numbers of a compact operator X; χ_A is the indicator of a set $A \subset \mathbb{R}$ [27, Chapter II].

If $\mathcal{M}$ is abelian (i.e., commutative) then $\mathcal{M} \simeq L^\infty(\Omega,\Sigma,\nu)$ and $\tau(f) = \int_\Omega f\,d\nu$, where (Ω,Σ,ν) is a localizable measure space; the $*$-algebra $S(\mathcal{M},\tau)$ coincides with the algebra of all measurable complex functions f on (Ω,Σ,ν) that are bounded everywhere outside a set of finite measure (with functions equal almost everywhere identified). The function $\mu(t;f)$ coincides with the nonincreasing rearrangement of the function $|f|$; see [28] for the properties of rearrangements.

3. The Main Results

Lemma 2 [29, Lemma 2]. *If $T \in S(\mathcal{M},\tau)^+$, $P \in \mathcal{M}^{\mathrm{pr}}$, and $PT + TP \geq 0$ then $TP = PT$.*

Theorem 1. *Suppose that $T \in S(\mathcal{M},\tau)^+$ and $P \in \mathcal{M}^{\mathrm{pr}}$. The following hold:*

(i) *if $U = 2P - I$ and $T - UTU \geq 0$ then $TP = PT$;*

(ii) *if $Q \in \mathcal{M}^{\mathrm{pr}}$ with $PQ = 0$ and*

$$T(P - tQ) + (P - tQ)T \geq 0 \tag{1}$$

for some $t > 0$ then $TP = PT$ and $TQ = QT = 0$.

Proof. (i): From the inequality $T - (2P - I)T(2P - I) \geq 0$ we obtain

$$PT + TP - 2PTP \geq 0.$$

Since $PTP \geq 0$, we have $PT + TP \geq 0$. By Lemma 2, $TP = PT$.

(ii): Multiplying both sides of (1) from the left and from the right by the projection Q, we conclude that $-2QTQ \geq 0$. Since $QTQ \geq 0$, we have $QTQ = |T^{1/2}Q|^2 = 0$ and $T^{1/2}Q = 0$. Therefore, $TQ = T^{1/2} \cdot T^{1/2}Q = 0$ and $QT = (TQ)^* = 0$. Now, (1) implies $PT + TP \geq 0$; hence, $TP = PT$ by Lemma 2. The theorem is proved. $\square$

Example 1. The positivity of T is essential in item (ii) of Theorem 1. In the algebra $\mathbb{M}_2(\mathbb{C})$, for the projections $P := \mathrm{diag}(0,1)$, $Q := \mathrm{diag}(1,0)$, and the Hermitian matrix

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we have $T(P - Q) + (P - Q)T = 0$ (cf. (1)), but $TP \neq PT$ and $TQ \neq 0 \neq QT$.

Theorem 2. *Suppose that $T \in S(\mathcal{M},\tau)^{\mathrm{h}}$, $P \in \mathcal{M}^{\mathrm{pr}}$, and the operator $A := PT$ is hyponormal. Then $TP = PT$ and $A = PTP \in S(\mathcal{M},\tau)^{\mathrm{h}}$.*

Proof. Since $PT^2P = AA^* \leq A^*A = TPT$, multiplying both sides of this inequality from the left and from the right by P, we infer

$$0 \leq PT^2P \leq PTPTP = (PTP)^2 = |PTP|^2.$$

Hence, with the use of the inequality $PTPTP \leq PTITP = PT^2P$, due to the operator monotonicity of the function $f(t) = \sqrt{t}$ $(t \geq 0)$, we have

$$\sqrt{PT^2P} \leq |PTP| = \sqrt{PTPTP} \leq \sqrt{PT^2P},$$

i.e., $|PTP| = \sqrt{PT^2P}$. Squaring both sides of the last equality, we get $PTPTP = PT^2P (= PTPTP + PTP^\perp TP)$. Therefore,

$$PTP^\perp TP = |P^\perp TP|^2 = 0$$

and $P^\perp TP = 0$. Thus, $TP = PTP = (PTP)^* = (TP)^* = PT$ and $A = PTP \in S(\mathcal{M},\tau)^{\mathrm{h}}$. The theorem is proved. $\square$

Corollary 1. *If an operator $A = A^2 \in S(\mathcal{M}, \tau)$ is hyponormal (or cohyponormal) then $A \in \mathcal{M}^{\mathrm{pr}}$.*

PROOF. By [30, Theorem 2.21], every operator $A = A^2 \in S(\mathcal{M}, \tau)$ is representable as the product $A = PT$, where $P = A(A + A^* - I)^{-1} \in \mathcal{M}^{\mathrm{pr}}$ and the operator $T = A + A^* - I$ invertible in $S(\mathcal{M}, \tau)$ belongs to $S(\mathcal{M}, \tau)^{\mathrm{h}}$. Now the claim follows from the spectral theorem. If the operator $A = A^2 \in S(\mathcal{M}, \tau)$ is cohyponormal then $A^* = A^{*2}$ is hyponormal. $\square$

Theorem 3. *Suppose that an operator $T \in S(\mathcal{M}, \tau)$ is hyponormal, $P \in \mathcal{M}^{\mathrm{pr}}$, and $TP = \lambda P$ for some $\lambda \in \mathbb{C}$. Then $PT = TP$ and the operator $T|_{P\mathcal{H}}$ is hyponormal.*

PROOF. Since $PTT^*P \le PT^*TP$, we have

$$0 \le (PT - \lambda P)(PT - \lambda P)^* = PTT^*P - \overline{\lambda}PTP - \lambda PT^*P + |\lambda|^2 P$$
$$\le PT^*TP - |\lambda|^2 P - \lambda PT^*P + |\lambda|^2 P = PT^*\lambda P - \lambda PT^*P = 0.$$

Consequently, $T^*P - \overline{\lambda}P = (PT - \lambda P)^* = 0$ and $PT = \lambda P = TP$. We have $T|_{P\mathcal{H}} = PTP$. It is easy to see that

$$(PTP)^*PTP = PT^*PTP = PT^*PPT = PT^*P\lambda P = \lambda PT^*P,$$
$$PTP(PTP)^* = PTPT^*P = PPTT^*P = PTT^*P = \lambda PT^*P.$$

The theorem is proved. $\square$

Corollary 2. *Suppose that an operator $T \in S(\mathcal{M}, \tau)$ is hyponormal, $P_1, P_2 \in \mathcal{M}^{\mathrm{pr}}$, and $TP_1 = \lambda_1 P_1$, $TP_2 = \lambda_2 P_2$ for some nonzero $\lambda_1, \lambda_2 \in \mathbb{C}$, $\lambda_1 \ne \lambda_2$. Then $P_1P_2 = 0$.*

PROOF. For $\lambda_1 \notin \{0, \lambda_2\}$, we have $P_1T = \lambda_1 P_1$ by Theorem 3, and

$$\lambda_1\lambda_2 P_1P_2 = TP_1TP_2 = TP_1TP_2 = T\lambda_1 P_1P_2 = \lambda_1 TP_1P_2 = \lambda_1\lambda_1 P_1P_2 = \lambda_1^2 P_1P_2.$$

Therefore, $\lambda_1(\lambda_2 - \lambda_1)P_1P_2 = 0$ and $P_1P_2 = 0$.

Given $\lambda_2 \notin \{0, \lambda_1\}$, we consider the product $\lambda_1\lambda_2 P_2P_1$ and likewise obtain

$$P_2P_1 = 0 = 0^* = (P_2P_1)^* = P_1P_2. \quad \square$$

Theorem 4. *Suppose that an operator $T \in S(\mathcal{M}, \tau)$ is hyponormal and $P \in \mathcal{M}^{\mathrm{pr}}$.*

(i) *If $0 \le TP \le P$ then $TP = PT$.*

(ii) *Let $TP = PTP$. Then the operator $T|_{P\mathcal{H}}$ is hyponormal; if $T|_{P\mathcal{H}}$ is normal then $TP = PT$.*

PROOF. (i): For $A := TP$ we have $0 \le A \le P$; therefore, $AP = PA = A$ by [31, Chapter 2, Section 2.17]. Likewise, from $0 \le A^2 \le P$ we obtain $A^2P = PA^2 = A^2$. Note that $PT^* = (TP)^* = A$ and $PTT^*P \le PT^*TP$. Then

$$0 \le (PT - TP)(PT - TP)^* = PTT^*P - PTTP - TPT^*P + TPPT^*$$
$$\le PT^*TP - PTA - AT^*P + A^2 = 2A^2 - PTA - AT^*P$$
$$= 2A^2 - PTPA - APT^*P = 2A^2 - PTPA - APT^*P$$
$$= 2A^2 - PA^2 - A^2P = 2A^2 - A^2 - A^2 = 0$$

and $TP = PT$.

(ii): We have $PT^*TP \ge PTT^*P$ and $T|_{P\mathcal{H}} = PTP$. Let $TP = PTP$. It is easy to see that

$$(PTP)^*PTP = (TP)^*TP = PT^*PT \ge PTT^*P \ge PTPT^*P = PTP(PTP)^*,$$

i.e., the operator $T|_{P\mathcal{H}}$ is hyponormal. Let $T|_{P\mathcal{H}}$ be normal. Then $PTP(PTP)^* = (PTP)^*PTP$ and $P^{\perp}TP = 0$; therefore,

$$0 \le (TP - PT)(TP - PT)^* = TPT^*P - TPT^*P - PTPT^* + PTT^*P$$
$$= TPT^* - TPT^*P - TPT^* + PTT^*P = -TPT^*P + PTT^*P$$
$$= -PTPPT^*P + PTT^*P \le -PTPPT^*P + PT^*TP = PT^*P^{\perp}TP = 0$$

and $TP = PT$. The theorem is proved. $\square$

Theorem 5. *Suppose that $T \in \mathcal{M}_1$ and $P \in \mathcal{M}^{\mathrm{pr}}$.*

(i) *If $TP \geq P$ then $TP = PT = P$.*

(ii) *If T is hyponormal and $TT^*P = P$ (or $\tau(TT^*P) = \tau(P) < +\infty$) then $T^*TP = P$.*

Proof. (i): Since $P \leq TP \leq I$, we have $0 \leq TP - P \leq P^{\perp} \in \mathcal{M}^{\mathrm{pr}}$. Therefore,

$$0 = (TP - P)P^{\perp} = P^{\perp}(TP - P) = TP - P$$

by [31, Chapter 2, Section 2.17] and $TP = P = PT^*$. Owing to $0 \leq TT^* \leq I$, we have

$$0 \leq (PT - P)(PT - P)^* = PTT^*P - PTP - PT^*P + P \leq P - P = 0$$

and $PT = P$.

(ii): If $TT^*P = P$ then $0 \leq (T^*T)^2 \leq T^*T \leq I$ and $P = PTT^*P \leq PT^*TP \leq P$, i.e., $PTT^*P = PT^*TP = P$. (If $\tau(TT^*P) = \tau(P) < +\infty$ then from the estimates $PTT^*P \leq PT^*TP \leq P$ and

$$\tau(P) = \tau(TT^*P) = \tau(TT^*PP) = \tau(PTT^*P) \leq \tau(PT^*TP) \leq \tau(P)$$

by the finiteness of the trace τ, we obtain $PTT^*P = PT^*TP = P$.) Thus,

$$0 \leq (T^*TP - P)^*(T^*TP - P) = P(T^*T)^2P - 2PT^*TP + P = P(T^*T)^2P - P \leq P - P = 0$$

and $T^*TP = P$. The theorem is proved. □

Theorem 6. *An operator $T \in S(\mathcal{M}, \tau)$ is hyponormal (cohyponormal) if and only if $\mu(t; TP) \geq \mu(t; T^*P)$ ($(\mu(t; T^*P) \geq \mu(t; TP))$ for all $t > 0$ and $P \in \mathcal{M}_{\tau}^{\mathrm{pr}}$.*

Proof. ($\Rightarrow$): Since $PT^*TP \geq PTT^*P$ for T hyponormal, by items (i), (ii), and (v) of Lemma 1, for all $t > 0$ and $P \in \mathcal{M}^{\mathrm{pr}}$ we have the estimate

$$\mu(t; TP) = \mu(t; (PT^*TP)^{1/2}) = \mu(t; PT^*TP)^{1/2} \geq \mu(t; PTT^*P)^{1/2} = \mu(t; (PT^*TP)^{1/2}) = \mu(t; T^*P).$$

($\Leftarrow$): Let $\mu(t; TP) \geq \mu(t; T^*P)$ for all $t > 0$ and $P \in \mathcal{M}_{\tau}^{\mathrm{pr}}$. Suppose that $A_- \neq 0$ in the Jordan decomposition

$$T^*T - TT^* = A_+ - A_-,$$

where $A_+, A_- \in S(\mathcal{M}, \tau)^+$ with $A_+A_- = 0$. Let a real $\varepsilon > 0$ be such that the spectral projection $E^{A_-}(\varepsilon, +\infty) = P$ is nonzero. Passing to a subprojection if necessary, we assume that $\tau(P) < +\infty$. We have

$$PT^*TP - PTT^*P = -PA_-P \leq -\varepsilon P,$$

i.e.,

$$PT^*TP + \varepsilon P \leq PTT^*P. \tag{2}$$

Consider the reduced von Neumann algebra $\mathcal{M}_P = P\mathcal{M}P$ with unity P and the reduced faithful normal trace $\tau_P = \tau(P \cdot P)$ on $\mathcal{M}_P$. Then

$$S(\mathcal{M}_P, \tau_P) = PS(\mathcal{M}, \tau)P.$$

Let $\mu_P(t; \cdot)$ be the singular value function calculated according the trace τ_P; then by using (2), the well-known representation

$$\mu(t; X) = \inf\{s > 0 : d_X(s) \leq t\}, \quad t > 0$$

(see [19, Proposition 2.2], where $d_X(s) = \tau(E^{|X|}(s, +\infty))$, $s > 0$, is the distribution function of the operator $X \in S(\mathcal{M}, \tau)$ and $E^{|X|}(s, +\infty)$ is the spectral projection of $|X|$ corresponding to the interval $(s, +\infty)$), and employing items (i) and (v) of Lemma 1, we obtain

$$\begin{aligned}\mu(t; T^*P)^2 = \mu(t; |T^*P|^2) = \mu(t; PTT^*P) = \mu_P(t; PTT^*P) \geq \mu_P(t; PT^*TP + \varepsilon P)\\ = \mu_P(t; PT^*TP) + \varepsilon = \mu(t; PT^*TP) + \varepsilon = \mu(t; |TP|^2) + \varepsilon = \mu(t; TP)^2 + \varepsilon;\end{aligned}$$

a contradiction. The theorem is proved. □

Corollary 3. *An operator $T \in S(\mathcal{M}, \tau)$ is normal if and only if $\mu(t; TP) = \mu(t; T^*P)$ for all $t > 0$ and $P \in \mathcal{M}_\tau^{\mathrm{pr}}$.*

The topology $t_{\tau l}$ can also be defined in terms of singular value functions. Namely, the family $\widetilde{\Theta} = \{\widetilde{\mathcal{V}}(\varepsilon, \delta, P)\}_{\varepsilon,\delta>0; P\in\mathcal{M}_\tau^{\mathrm{pr}}}$, where

$$\widetilde{\mathcal{V}}(\varepsilon, \delta, P) = \{X \in S(\mathcal{M}, \tau) : \mu(\delta; XP) < \varepsilon\},$$

defines a neighborhood basis of zero for $t_{\tau l}$.

Theorem 7. *The set of all τ-measurable hyponormal operators is $t_{\tau l}$-closed.*

Proof. Each base neighborhood of an operator $B \in S(\mathcal{M}, \tau)$ in the topology $t_{\tau l}$ has the form

$$V_{\varepsilon,\delta,P}(B) := \{A \in S(\mathcal{M}, \tau) : \mu(\delta; AP - BP) < \varepsilon\},$$

where $\varepsilon, \delta > 0$ and $P \in \mathcal{M}_\tau^{\mathrm{pr}}$ (also see [20, Section 2; 21, Section 2; 25, Section 3]). Our theorem states that if each such neighborhood of B contains a hyponormal operator then

$$\mu(t; B^*P) \le \mu(t; BP)$$

for all $t > 0$ and $P \in \mathcal{M}_\tau^{\mathrm{pr}}$ (see Theorem 6). For a fixed $P \in \mathcal{M}_\tau^{\mathrm{pr}}$, consider the projection

$$Q := P \vee \mathrm{s}_l(B^*P).$$

Since $\mathrm{s}_l(B^*P) \preceq P$, we have $\tau(Q) \le \tau(P) + \tau(\mathrm{s}_l(B^*)) \le 2\tau(P) < +\infty$ and $Q \in \mathcal{M}_\tau^{\mathrm{pr}}$.

To better understand the idea of using the neighborhood assumption, assume temporarily that a stronger assumption holds; namely, that there exists a hyponormal operator $A \in S(\mathcal{M}, \tau)$ such that

$$AQ = BQ.$$

In this case, passing to the adjoint operators, we have $QA^* = QB^*$, which implies

$$\begin{aligned}\mu(t; B^*P) &= \mu(t; \mathrm{s}_l(B^*P)QB^*P) \le \|\mathrm{s}_l(B^*P)\|\mu(t; QB^*P) = \mu(t; QA^*P)\\ &\le \|Q\|\mu(t; A^*P) = \mu(t; A^*P) \le \mu(t; AP) = \mu(t; AQP) = \mu(t; BQP) = \mu(t; BP)\end{aligned}$$

for all $t > 0$ by item (iii) of Lemma 1 and Theorem 6 for the operator A.

The proof is actually done again by carrying ε and δ along the above chain of reasoning using Lemma 1 on the properties of singular value functions. For arbitrary $t > 2\delta > 0$, we infer

$$\begin{aligned}\mu(t; B^*P) &= \mu(t; \mathrm{s}_l(B^*P)QB^*P) \le \|\mathrm{s}_l(B^*P)\|\mu(t; QB^*P) = \mu(t; QB^*P)\\ &= \mu(t; QB^*P - QA^*P + QA^*P) \le \mu(\delta; QB^*P - QA^*P) + \mu(t-\delta; QA^*P)\\ &\le \|P\|\mu(\delta; QB^* - QA^*) + \mu(t-\delta; QA^*P) = \mu(\delta; BQ - AQ) + \mu(t-\delta; QA^*P)\\ &\le \varepsilon + \mu(t-\delta; QA^*P) \le \varepsilon + \|Q\|\mu(t-\delta; A^*P) \le \varepsilon + \mu(t-\delta; AP) = \varepsilon + \mu(t-\delta; AQP)\\ &= \varepsilon + \mu(t-\delta; AQP - BQP + BQP) \le \varepsilon + \mu(\delta; AQP - BQP) + \mu(t-2\delta; BQP)\\ &\le 2\varepsilon + \mu(t-2\delta; BQP) = 2\varepsilon + \mu(t-2\delta; BP).\end{aligned}$$

If t is a continuity point of $\mu(\cdot\,; BP)$ then, due to the smallness of ε and δ, we obtain $\mu(t; B^*P) \le \mu(t; BP)$ and the operator B is hyponormal by Theorem 6. Finally, recall that the singular value function $\mu(\cdot\,; X)$ ($X \in S(\mathcal{M}, \tau)$) is right continuous on $\mathbb{R}^+$ and has at most countably many discontinuity points. The theorem is proved. □

Corollary 4. *The set of all τ-measurable cohyponormal operators is $t_{\tau l}$-closed if and only if the von Neumann algebra $\mathcal{M}$ is finite.*

Proof. Sufficiency: Suppose that the set of all cohyponormal operators in $S(\mathcal{M},\tau)$ is $t_{\tau l}$-closed. Then the set of all τ-measurable normal operators is $t_{\tau l}$-closed as the intersection of two $t_{\tau l}$-closed sets in $S(\mathcal{M},\tau)$. Recall that the set $\mathcal{M}^{\text{iso}}$ of all isometries ($U^*U = I$) is a $t_{\tau l}$-closed set in $S(\mathcal{M},\tau)$ [20, Lemma 3.7(3)]. Consequently, the set $\mathcal{M}^{\text{u}}$ of all unitary operators ($U^*U = UU^* = I$) is a $t_{\tau l}$-closed set in $S(\mathcal{M},\tau)$. But $\mathcal{M}^{\text{u}}$ is $t_{\tau l}$-closed if and only if the von Neumann algebra $\mathcal{M}$ is finite [24, Theorem 1(i)].

Necessity: A von Neumann algebra $\mathcal{M}$ is finite if and only if the involution $A \mapsto A^*$ is $t_{\tau l}$-continuous from $S(\mathcal{M},\tau)$ into $S(\mathcal{M},\tau)$ [20, Theorem 4.1(5)]. Moreover, if a net $\{X_j\}_{j\in J} \subset (\mathcal{M},\tau)$ of cohyponormal operators $t_{\tau l}$-converges to an operator $X \in S(\mathcal{M},\tau)$ then $X_j^* \xrightarrow{\tau l} X^*$. Since the operators X_j^*, $j \in J$, are hyponormal, X^* is hyponormal; therefore, X is cohyponormal. $\square$

Remark 1. In [32], Muratov and Chilin considered another topology of local convergence in measure on algebras of locally measurable operators, different from $t_{\tau l}$. The question of the closedness of the set of all locally measurable hyponormal operators in that topology remains open.

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CONFLICT OF INTEREST

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