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Strict Superharmonicity of Mityuk's Function for Countably Connected Domains of Simple Structure

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Abstract—Strict superharmonicity of generalized reduced module as a function of a point (we call it Mityuk's function) is established for the subclass of countably connected domains with unique limit point boundary component. The function just mentioned was first studied in detail by I.P. Mityuk and plays now an important role in the research of the exterior inverse boundary value problems of the theory of analytic functions in the multiply connected domains. At the heart of such a research one can see the fact that the critical points of Mityuk's function are only maxima, saddles or semi-saddles of corresponding surface. This fact is followed from the above strict superharmonicity.

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To 85th anniversary of our teacher,
Professor L.A. Aksent'ev

1. INTRODUCTION AND PRELIMINARIES

Theory of the exterior inverse boundary value problems (IBVP) for simply connected domains has been worked out early in 1950th by M.T. Nuzhin and F.D. Gakhov (see, e.g., [1] and [2]). Essential break in the treating of the solvability problem for the exterior IBVP in finitely connected domains has been accomplished in 1983: an approach proposed by M.I. Kinder has led to the appearance of the paper [3]. The opened problems and perspectives were so attractive that the investigation project [4–12] which has formed on this way developed up to the middle of 1990th (see [13]).

One of the project activities dealt with the expansion of the obtained finitely connected results to the infinitely connected case. The start has been given to this activity by A.V. Kazantsev in his thesis [9] where he studied the simplest case—with the unique limit point boundary component.

It turns out that the multiply connected "pattern" proposed in [3] and [8] remains valid also in the case below where it will be built on by more or less complicated constructions concerned with the convergence of the function sequences generating by the countable connectivity of the domains in question. Nevertheless, the installation of the above constructions into a whole one will be begun with countably connected version of the strict superharmonicity property of Mityuk's function—multiply connected counterpart of the logarithm of the inner mapping radius. One can consider this property as a board on which the above "pattern" is installed.

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Central link of the "pattern" is presented by the connection of the exterior IBVP and Mityuk's function. Let us briefly describe this connection along the lines of [13].

Solution of the exterior IBVP in the multiply connected case D has the form

$$z(w) = \int f'(w)F^{-2}(w,a)dw, \quad w \in D.$$

Here $f(w)$ is the holomorphic function in D solving the interior IBVP with respect to the initial data that we have in the exterior problem. Function $F(w,a)$ maps the domain D conformally and univalently onto the unit disk with concentric circular slits such that the outer boundary component of the domain D corresponds to the unit circle, and $F(a,a)=0$. Pole $w=a$ is find from the Gakhov equation [3]

$$f''(w)/f'(w) = 2\phi_1'(w,w)/\phi(w,w), \tag{1}$$

where $\phi(w,a) = F(w,a)/(w-a)$ and $\phi_1'(w,w) = (\partial/\partial w)\phi(w,\omega)_{|\omega=w}$. The roots of the equation (1) are the critical points of the surface defining by the equation $M = M(w)$ where the function

$$M(w) = (2\pi)^{-1}\ln(|f'(w)|/|\phi(w,w)|) \tag{2}$$

has been introduced and was studied by I.P. Mityuk [14]. His posing recognizes the value $M(w)$ as the generalized reduced module of the domain $f(D)$ at the point $f(w)$ relative to the component of $\partial f(D)$ corresponding to the outer component of ∂D under the mapping f. Let us call (2) *Mityuk's function*, and we will say that *Mityuk's radius* is defined by

$$\Omega(w) = \exp[2\pi M(w)]. \tag{3}$$

Nuances in the definitions of quantities $M(w)$ and $\Omega(w)$ as the functions of a point or a domain were examined in [15] for the simply connected case. In the report [16] the quantity (2) was said to be Mityuk's functional. The function (3) has been called in [9] the modified inner mapping radius.

Under the strict superharmonicity of the function $M(w)$ in the domain D we means

$$\frac{\partial^2 M(w)}{\partial w \partial \bar{w}} < 0, \quad w \in D. \tag{4}$$

For the finitely connected D's an inequality (4) has been proved in [4] and played an important role in the study of the stationary points characteristics of the function $M(w)$ and in the evaluation of a number of the corresponding exterior IBVP solutions.

Existence of the conformal and univalent mapping of an infinitely connected domain onto a circular slit disk has been established by H. Grötzsch [17, 18].

Let's introduce the class of domains under consideration.

Definition. *Class $\mathfrak{D}$ is defined to be the set of domains $D \subset \mathbb{C}$ satisfying the following conditions:*

1) *D is bounded by countably many isolated boundary components L_n, $n \geq 0$, each of which is a closed analytic contour, and by the unique limit point boundary component $\{\alpha\}$, $\alpha \in int\, L_0$, so $\partial D = \left(\bigcup_{n\geq 0} L_n\right) \cup \{\alpha\}$;*

2) *there exists an exhaustion $\{D_n\}_{n\geq 1}$ of D by an increasing sequence of the domains $D_n = D \setminus \overline{int\, l_n}$, $n \geq 1$, where $l_n \subset D$ is an analytic contour that has the diameter d_n and contains all of L_k's beginning with $k = n$ in its interior; furthermore, $l_{n+1} \subset int\, l_n$, $n \geq 1$, and*

$$\lim_{n\to\infty} d_n = 0. \tag{5}$$

One can easily prove the following

Proposition 1. *If D is a domain of the class $\mathfrak{D}$, then for any $\varepsilon > 0$ there is a number N such that for all $k \geq N$ the inclusion $L_k \subset K_\varepsilon(\alpha) := \{w \in \mathbb{C} : |w-\alpha| < \varepsilon\}$ takes place.*

In what follows, we will assume that D is in the class $\mathfrak{D}$, and the domains D_n, $n \geq 1$, constitute an exhaustion of D according to the condition 2) in the above definition.

Let for any fixed point $\omega \in D$ the function $F(w,\omega)$ maps the domain D conformally and univalently onto the circular slit unit disk such that L_0 corresponds to the unit circle, and, moreover, $F(\omega,\omega) = 0$

and $F'_w(\omega,\omega) > 0$. In the same way, let for an arbitrary number $n \geq 1$ and for any fixed point $\omega \in D_n$ the function $F_n(w,\omega)$ maps the domain D_n conformally and univalently onto the unit disk with n concentric circular slits where $F_n(\omega,\omega) = 0$, $(F_n)'_w(\omega,\omega) > 0$, and L_0 corresponds to the unit circle again. The results of [19] yield the following

Proposition 2. *For any fixed* $\omega \in D$ *the convergence* $\lim_{n\to\infty} F_n(w,\omega) = F(w,\omega)$ *is uniform on each compact subset of points* $w \in D$.

Assertion just stated enables us to transfer the properties of the mapping functions from the finitely connected case to the infinitely connected one. So, for example, we have the symmetry $|F(w,\omega)| = |F(\omega,w)|$, $w,\omega \in D$, inherited by the function $F(w,\omega)$ from the functions $F_n(w,\omega)$, $n \geq 1$.

Canonical representations $F(w,\omega) = (w-\omega)\phi(w,\omega)$ and $F_n(w,\omega) = (w-\omega)\phi_n(w,\omega)$, $n \geq 1$, extend the above symmetry to the functions $\phi(w,\omega)$ and $\phi_n(w,\omega)$, $n \geq 1$. Therefore, if the statement of Proposition 2 is transferred to the convergence $\lim_{n\to\infty} |\phi_n(w,\omega)| = |\phi(w,\omega)|$ (version of such a transferring will be given below), then it can be done in two forms—uniformly in w on compact subsets of D for any $\omega \in D$, and uniformly in ω on compact subsets of D for any $w \in D$.

Let now $n \geq 1$, and let $k_{0,n}(w,\bar\omega)$ be the Bergman kernel function of the first kind for the domain D_n with respect to the class $L^2_0(D_n)$ of all functions holomorphic with square integrable module and with single-valued primitive in D_n [20]. It is known that for such a function the following representation is carried out (see references in [5])

$$k_{0,n}(w,\bar\omega) = (2/\pi)\partial^2 \ln|\phi_n(w,\omega)|/\partial w\partial\bar\omega, \quad w,\omega \in D_n. \tag{6}$$

Let us consider the function

$$k_0(w,\bar\omega) = (2/\pi)\partial^2 \ln|\phi(w,\omega)|/\partial w\partial\bar\omega, \quad w,\omega \in D, \tag{7}$$

which positivity will be equivalent to the condition (4) of the strict superharmonicity of Mityuk's function (2).

Our main result is the following theorem.

Theorem 1. *Let D be the domain of the class $\mathfrak{D}$. Then*

$$k_0(w,\bar w) > 0, \quad w \in D. \tag{8}$$

Justification of this assertion exhausts the rest of the paper. Let's note only two following statements (as it is noted above, the first of them is the reformulation of the Theorem 1).

Theorem 2. *Mityuk's function (2) is superharmonic in a domain D of the class $\mathfrak{D}$.*

As in the finitely connected case, Theorem 2 is essentially used at the proof of the following fact which is traditionally formulated for Mityuk's radius (see [5]).

Corollary. *If D is a domain of the class $\mathfrak{D}$, and if $a \in D$ is a critical point of the function (3), then the index $\gamma(a)$ of this point as a singular one for the vector field* grad$\,\Omega$ *can assume only three values: -1, 0 and $+1$.*

Thus, the critical points of the functions (2) and (3) can be only saddles, semi-saddles, and local maxima of corresponding surfaces.

The last corollary form the background for the proof of the infinity of the critical points set of Mityuk's radius in the infinitely connected case.

2. PROOF OF THEOREM 1

Let us fix an arbitrary point $w_0 \in D$. Without loss of generality we assume that there exists a radius $\rho > 0$ such that the closed disk $\overline{K_\rho(w_0)} \subset D$. In fact, the role of the domain D in the last inclusion is played by an exhaustion element, say D_m, to which (and, therefore, to every D_n with $n \geq m$) the point w_0 belongs with some neighborhood. We are interested in the limits of the convergences of the function characteristics connected with D_n when $n \to \infty$. So, the initial elements of exhaustion $\{D_n\}_{n\geq 1}$, whether they contain w_0, or not, can be neglected, and we acquire the right to set $m = 1$.

We will divide the proof of the theorem into two parts: 1) we will establish the convergence

$$\lim_{n\to\infty} k_{0,n}(w_0,\bar w_0) = k_0(w_0,\bar w_0), \tag{9}$$

and then on its base 2) we will receive an inequality (8) where $w = w_0$.

1) By virtue of Proposition 2 the convergence $\lim_{n\to\infty} F_n(w,\omega) = F(w,\omega)$ is carried out uniformly in w on compact subsets of D for any fixed $\omega \in D$. It is convenient to narrow the range of varying for w and ω from a whole domain D to the disk $\overline{K_\rho(w_0)}$ and the disks of smaller radii centered at w_0. The use of Cauchy's integral formula shows that the convergence $\lim_{n\to\infty} \phi_n(w,\omega) = \phi(w,\omega)$ also takes place uniformly in w, but now on $K = \overline{K_{\rho/2}(w_0)}$ and for any fixed $\omega \in K$. It follows that, in turn,

$$\lim_{n\to\infty} \ln|\phi_n(w,\omega)| = \ln|\phi(w,\omega)| \tag{10}$$

uniformly in w on the disk K for any fixed $\omega \in K$. So, there exists a pointwise limit

$$\lim_{n\to\infty} \ln\frac{1}{|\phi_n(w,w)|} = \ln\frac{1}{|\phi(w,w)|}, \quad w \in K. \tag{11}$$

On the other hand, the function $F_n(w,\omega)$ solves the problem $\max|g'(\omega)|$ on the class $\Re(D_n,\omega,L_0)$ of functions $g(w)$ holomorphic and univalent in D_n with correspondence of the outer contour L_0 of the boundary ∂D_n and the circle $|z| = 1$ where $g(\omega) = 0$, $|g(\omega)| \le 1$, $\omega \in D_n$ (see [21], p. 644–645). Since $F_{n+1}(w,\omega) \in \Re(D_n,\omega,L_0)$, we have $|F'_{n+1}(\omega,\omega)| \le |F'_n(\omega,\omega)|$ for all $n \ge 1$ and $\omega \in D_n$. It means that the sequence of functions $1/|\phi_n(w,w)|$, $n \ge 1$, increases for any $w \in K$. This increase allows us to strengthen the pointwise convergence in (11) to uniform one due to the well-known Dini theorem.

It obviously follows from just established uniform convergence in (11) that

$$\lim_{n\to\infty} c_n = c \tag{12}$$

where

$$c_n = -\min_{w\in K} \ln|\phi_n(w,w)|, \quad n \ge 1, \quad c = -\min_{w\in K} \ln|\phi(w,w)|. \tag{13}$$

In what follows we will needed in the sequence of M. Schiffer's inequalities for the domains D_n when $n \ge 1$ [22]. We use them in the form

$$\ln\frac{1}{|\phi_n(w,\omega)|} \ge \frac{1}{2}\ln\frac{1}{|\phi_n(w,w)|} + \frac{1}{2}\ln\frac{1}{|\phi_n(\omega,\omega)|}, \quad n \ge 1, \tag{14}$$

when $w,\omega \in K$. Turning $n \to \infty$ in (14) we have

$$\ln\frac{1}{|\phi(w,\omega)|} \ge \frac{1}{2}\ln\frac{1}{|\phi(w,w)|} + \frac{1}{2}\ln\frac{1}{|\phi(\omega,\omega)|} \tag{15}$$

for $w,\omega \in K$. Inequalities (14) and (15) permit us to pass from the equalities (13) to the estimates

$$\ln\frac{1}{|\phi_n(w,\omega)|} \ge c_n, \quad n \ge 1; \quad \ln\frac{1}{|\phi(w,\omega)|} \ge c, \quad w,\omega \in K.$$

Thus, the functions $-\ln|\phi_n(w,\omega)| - c_n$, $n \ge 1$, and $-\ln|\phi(w,\omega)| - c$, harmonic in $\omega \in K$, are non-positive for any fixed $w \in K$, hence ([23], p. 37)

$$\left|\frac{\partial}{\partial\bar\omega}\ln\frac{1}{|\phi_n(w,\omega)|}\bigg|_{\omega=w_0}\right| \le \frac{2}{\rho}\left[\ln\frac{1}{|\phi_n(w,w_0)|} - c_n\right], \quad w \in K, \quad n \ge 1. \tag{16}$$

Due to the relations (10) and (12) the right-hand side of (16) tends to the function $-2[\ln|\phi(w,w_0)| + c]/\rho$ uniformly on K when $n \to \infty$, so it is uniformly bounded on K. Then the sequence of derivatives $-(\partial/\partial\bar\omega)\ln|\phi_n(w,w_0)|$ from the left-hand side (16) is also uniformly bounded on K. Furthermore, by Stoilov's theorem about the invariance of the uniform convergence (currently in ω) under the differentiation ([24], p. 37), which is applicable here due to (10) and to the symmetry of functions ϕ_n, $n \ge 1$, and ϕ in $w,\omega \in K$, the above sequence of derivatives pointwise converges in $w \in K$ to the derivative $-(\partial/\partial\bar\omega)\ln|\phi(w,w_0)|$. According to Golusin's theorem ([21], p. 20) the pointwise convergence just stated in w is uniform on the disk $\overline{K_{\rho/3}(w_0)}$. Applying Stoilov's theorem once again, but now to the latter convergence, we establish the convergence

$$\lim_{n\to\infty} \frac{\partial^2}{\partial w \partial \bar{\omega}} \ln \frac{1}{|\phi_n(w, w_0)|} = \frac{\partial^2}{\partial w \partial \bar{\omega}} \ln \frac{1}{|\phi(w, w_0)|}, \tag{17}$$

uniform on $\overline{K_{\rho/4}(w_0)}$. By virtue of (6) and (7) the relation (17) with $w = w_0$ is exactly the convergence (9).

2) Let us introduce the notations connected with the complements to the exhaustion domains $\{D_n\}_{n\geq 1}$. Let $B_0 = \mathbb{C}\backslash\overline{int\, L_0}$; $B_n = int\, L_n$, $b_n = int\, l_n$, and $\tilde{D}_n = \mathbb{C}\backslash\bar{D}_n$ for $n \geq 1$; $\tilde{D} = \mathbb{C}\backslash\bar{D}$. Then we consider the following functions [20]

$$\Gamma_n(w, \bar{w}) = \frac{1}{\pi^2} \iint\limits_{\tilde{D}_n} \frac{dxdy}{|z-w|^4} = \frac{1}{\pi^2} \left\{ \sum_{k=0}^{n-1} \iint\limits_{B_k} \frac{dxdy}{|z-w|^4} + \iint\limits_{b_n} \frac{dxdy}{|z-w|^4} \right\}, \quad n \geq 1,$$

$$\Gamma(w, \bar{w}) = \frac{1}{\pi^2} \iint_{\tilde{D}} \frac{dxdy}{|z-w|^4} = \frac{1}{\pi^2} \sum_{k=0}^{\infty} \iint\limits_{B_k} \frac{dxdy}{|z-w|^4}, \quad z = x + iy. \tag{18}$$

By the Proposition 1 the series in the right-hand side of (18) converges for any fixed $w \in D$, i. e. the function $\Gamma(w, \bar{w})$ is correctly defined. For any fixed $w = w_0 \in D$ we have

$$\lim_{n\to\infty} \Gamma_n(w_0, \bar{w}_0) = \Gamma(w_0, \bar{w}_0), \tag{19}$$

which is proved by the direct estimation of the difference $\Gamma_n(w_0, \bar{w}_0) - \Gamma(w_0, \bar{w}_0)$ owing to (5) and Proposition 1.

If we apply the convergences (9) and (19) to the following sequence of the Bergman–Schiffer inequalities [20]

$$k_{0,n}(w, \bar{w}) \geq \Gamma_n(w, \bar{w}), \quad w \in D_n, \quad n \geq 1,$$

with $w = w_0$, then we get the inequality $k_0(w_0, \bar{w}_0) \geq \Gamma(w_0, \bar{w}_0)$. The latter and the property $\Gamma(w_0, \bar{w}_0) > 0$, checked directly, imply the estimate (8) with $w = w_0$. In view of an arbitrariness of the choice of $w_0 \in D$ the Theorem 1 is proved.

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REFERENCES

1. M. T. Nuzhin, "On some inverse boundary value problems and their applications to the definition of sectional shape of twisted rods," Uch. Zap. Kazan. Univ. **109** (1), 97–120 (1949).
2. F. D. Gakhov, "On the inverse boundary problems," Dokl. Akad. Nauk SSSR **86** (4), 649–652 (1952).
3. L. A. Aksent'ev, M. I. Kinder, and S. B. Sagitova, "Solvability of the exterior inverse boundary value problem in the case of multiply connected domain," Tr. Semin. Kraev. Zadacham **20**, 22–34 (1983).
4. M. I. Kinder, "The number of solutions of F. D. Gakhov's equation in the case of a multiply connected domain," Izv. Vyssh. Uchebn. Zaved., Mat. **28** (8), 69–72 (1984).
5. M. I. Kinder, "Investigation of F. D. Gakhov's equation in the case of multiply connected domains," Tr. Semin. Kraev. Zadacham **22**, 104–116 (1985).
6. L. A. Aksent'ev, A. M. Elizarov, and M. I. Kinder, "Inverse boundary value problems for multiply connected domains on Riemann surfaces of genus zero," Tr. Semin. Kraev. Zadacham **21**, 19–32 (1984); Tr. Semin. Kraev. Zadacham **22**, 16–29 (1985); Tr. Semin. Kraev. Zadacham **23**, 25–36 (1987).
7. L. A. Aksent'ev, A. M. Elizarov, and M. I. Kinder, "Continuation of F.D. Gakhov's work in inverse boundary value problems," in *Proceedings of the Commemorative Seminar on Boundary Value Problems Dedicated to the 75th Birthday of Acad. F. D. Gakhov* (Minsk, 1981), pp. 139–142.

8. M. I. Kinder, "Exterior inverse boundary value problem in multiply connected regions and on Riemann surfaces," Cand. Sci. (Math.) Dissertation (Kazan State Univ., Kazan, 1984).
9. A. V. Kazantsev, "Extremal properties of the inner radius and their applications," Cand. Sci. (Math.) Dissertation (Kazan State Univ., Kazan, 1990).
10. L. A. Aksent'ev, A. V. Kazantsev, M. I. Kinder, and A. V. Kiselev, "Classes of uniqueness of an exterior inverse boundary value problem," Tr. Semin. Kraev. Zadacham **24**, 39–62 (1990).
11. L. A. Aksent'ev, A. V. Kazantsev, and M. I. Kinder, "On classes of uniqueness of an exterior inverse boundary value problem," in *Proceedings of the 2nd Mathematical Readings in Memory of M. Ya. Suslin, Sept. 23–28, Saratov, 1991,* p. 61.
12. A. V. Kiselev, "Geometric properties of solutions of the exterior inverse boundary value problem," Izv. Vyssh. Uchebn. Zaved., Mat. **36** (7), 20–25 (1992).
13. A. V. Kazantsev and M. I. Kinder, "Solvability of the exterior inverse boundary value problem in the case of multiply connected domains," in *Proceedings of the 11th International Conference on Algebra and Analysis, Dedicated to the 100th Anniversary of the Birth of N. G. Chebotarev, Kazan, June 5–11, 1994,* Vol. 2, pp. 65–66.
14. I. P. Mityuk, "A generalized reduced module and some of its applications," Izv. Vyssh. Uchebn. Zaved., Mat., No. 2, 110–119 (1964).
15. A. V. Kazantsev, "Gakhov set in the Hornich space under the Bloch restriction on pre-Schwarzians," Uch. Zap. Kazan. Univ., Ser. Fiz.-Mat. Nauki **155** (2), 65–82 (2013).
16. A. V. Kazantsev, "Zmorovich's method in the problem of investigation of Mityuk's functional," in *Proceedings of the 7th International Symposium on Fourier Series and Their Applications, Rostov, 2014,* pp. 79–80.
17. H. Grötzsch, "Über konforme Abbildung unendlich vielfach zusammenhängender schlichter Bereiche mit endlich vielen Häufungsrandkomponenten," Ber. Sächs. Akad. Wiss. Leipzig, Math.-Phys. Kl. **81**, 51–86 (1929).
18. H. Grötzsch, "Das Kreisbogenschlitztheorem der konformen Abbildung schlichter Bereiche," Ber. Sächs. Akad. Wiss. Leipzig, Math.-Phys. Kl. **83**, 238–253 (1931).
19. E. Reich and S. E. Warschawski, "On canonical conformal maps of regions of arbitrary connectivity," Pacif. J. Math. **10** (3), 965–989 (1960).
20. S. Bergman and M. Schiffer, "Kernel functions and conformal mapping," Compos. Math. **8**, 205–249 (1951).
21. G. M. Golusin, *Geometric Theory of Functions of a Complex Variable,* Vol. 26 of *Transl. Math. Monographs* (Am. Math. Soc., Providence, 1969).
22. M. Schiffer, "Hadamard's formula and variation of domain-functions," Am. J. Math. **68** (4), 417–448 (1946).
23. W. Hayman and P. Kennedy, *Subharmonic Functions*, (Academic, London, 1976), Vol. 1.
24. S. Stoilov, *The Theory of Functions of a Complex Variable* (Fizmatgiz, Moscow, 1962), Vol. 2.