Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 47, No. 3, pp. 75–77, 2003 *UDC 517.95*

EXISTENCE OF EIGENVALUES OF SPECTRAL PROBLEM OF THE THEORY OF DIELECTRIC WAVEGUIDES

Ye.M. Karchevskiĭ and S.I. Solov'yev

1. The problem on proper surface waves of a cylindrical dielectric weakly directing waveguide is considered (see [1], [2]). The mathematical statement of this problem (see, e. g., [1], [2]) can be reduced to search of values of the parameter λ, $\lambda \neq q_2$, with which decreasing at infinity nontrivial solutions u of the differential problem

$$\Delta u + (q_i - \lambda)u = 0, \quad x \in \Omega_i, \quad i = 1, 2, \quad [u]_x = 0, \quad \left[\frac{\partial u}{\partial \nu}\right]_x = 0, \quad x \in \Gamma_1 \tag{1}$$

exist. Here $q_i = k_0^2 n_i^2$, $i = 1, 2$, $k_0 > 0$, $n_1 > n_2 > 0$, $k_0^2 = \omega^2 \varepsilon_0 \mu_0$, where n_1 and n_2 are refraction ratio of the waveguide and an isotropic ambient, respectively, ε_0 is the electric constant, μ_0 is the magnetic constant, ω is a given frequency of electromagnetic oscillations, $\lambda = \beta^2$, β is the longitudinal constant of propagation, $\partial u/\partial \nu$ is the derivative by exterior normal to the boundary Γ_1 of the domain Ω_1 on the plane $\mathbb{R}^2$, $\Omega_2 = \mathbb{R}^2 \setminus \overline{\Omega}_1$, $[v]_x$ is the jump of the function v at the point $x \in \Gamma_1$ under the transition over the contour Γ_1, Δ stands for the Laplace operator.

In the previous papers by the authors (see [3], [4]) it was established that eigenvalues λ of the differential problem can exist only on the interval (q_2, q_1). It was also proved that this problem has at least one simple real eigenvalue to which a positive eigenfunction corresponds.

In this article the initial differential problem is formulated in an equivalent form as a variational problem on eigenvalues within the interval (q_2, q_1). By means of the variational statement we obtain necessary and sufficient conditions for the existence of at least m, $m \geq 1$, eigenvalues of the initial problem (Theorem 1), establish a result on the existence of exactly m eigenvalues of the problem (Theorem 2). In the study of the variational problem for eigenvalues we use technique developed in [5]. In the capacity of consequences of the rather general sufficient conditions of Theorems 1 and 2 we suggest simplified sufficient conditions which ensure the existence of a given number of eigenvalues of the initial problem (Corollaries 1 and 2), which can be rather easily used in practice for estimation of the quantity of eigenvalues.

2. In what follows, as usual, we denote by L_2 and W_2^1 the real Lebesgue and Sobolev spaces, respectively.

Let Ω_1 be a bounded domain on the plane $\mathbb{R}^2$ with a Lipschitz-continuous boundary Γ_1, $\mathbb{R}$ be the numerical axis. We denote $V = W_2^1(\mathbb{R}^2)$, $H = L_2(\Omega_1)$, $\Lambda = (q_2, q_1)$ and define the mappings $a : \Lambda \times V \times V \to \mathbb{R}$ and $b : H \times H \to \mathbb{R}$ by the formulas

$$a(\mu, u, v) = \sum_{i=1}^{2} \int_{\mathbb{R}^2} \partial_i u \partial_i v \, dx + (\mu - q_2) \int_{\mathbb{R}^2} uv \, dx, \quad u, b \in V, \quad \mu \in \Lambda,$$

$$b(u, v) = (q_1 - q_2) \int_{\Omega_1} uv \, dx, \quad u, v \in H.$$