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On Axiomatics of Symmetric and Asymmetric Concrete Logics

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Abstract—We refined the axiomatics of asymmetric logics. For logics $X(km, k)$ of family subsets of the km-element set X, which cardinal numbers are multiples of k we completely described the cases in which $X(km, k)$ a) is symmetric or b) is asymmetric. For an infinite set Ω and a natural number $n \geq 2$ we constructed the concrete logics $\mathcal{E}_\Omega^n$ and completely described the cases in which these logics are asymmetric. For asymmetric logics $\mathcal{E}$ we determine when both the set $A \in \mathcal{E}$ and its complement A^c are atoms of the logic $\mathcal{E}$. Let a symmetric logic $\mathcal{E}$ of a finite set Ω be not a Boolean algebra, and let $\mathcal{A}$ be an algebra of subsets from Ω, and assume that $\mathcal{E} \subset \mathcal{A}$. Then there exists a measure on $\mathcal{E}$, that does not admit an extension to a measure on $\mathcal{A}$.

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1. INTRODUCTION

Let Ω be a non-empty set. Denote by 2^Ω the set of all subsets of the set Ω. A family $\mathcal{E} \subseteq 2^\Omega$ is called a *set logic* on Ω (see [1–3]), if the following conditions hold:

(i) $\Omega \in \mathcal{E}$;

(ii) $A \in \mathcal{E} \Rightarrow A^c := \Omega \setminus A \in \mathcal{E}$;

(iii) $A \cup B \in \mathcal{E}$ for all $A, B \in \mathcal{E}$ with $A \cap B = \emptyset$.

A set logic $\mathcal{E}$ is called a σ-*class*, if $\{A_n\}_{n=1}^\infty \subset \mathcal{E}$, $A_n \cap A_m = \emptyset\ (n \neq m) \Rightarrow \cup_{n=1}^\infty A_n \in \mathcal{E}$. A *charge* on set logic $\mathcal{E}$ is a mapping $\nu : \mathcal{E} \to \mathbb{R}$, such that

$$A, B \in \mathcal{E}, A \cap B = \emptyset \Rightarrow \nu(A \cup B) = \nu(A) + \nu(B).$$

A *measure* on $\mathcal{E}$ is a charge ν such that $\nu(A) \geq 0$ for all $A \in \mathcal{E}$. If $\nu(\Omega) = 1$, then the measure ν is called a *state* (or a *probability measure*).

We study a σ-classes, and also charges and measures on them. This is related to “the generalized measure theory” [2, 3], which can be considered as nearest to the classical (here “ the classical” means on “σ-algebras of sets”) version of measure theory on quantum logics [1, 2]. On the quantum logic approach in the axiomatics of physical systems see [4, Chap. VI, §5]. If $\mathcal{E}$ is a set logic (i.e., a concrete logic), then the family $\mathcal{S}$ of all states on $\mathcal{E}$ is complete and the pair $(\mathcal{E}, \mathcal{S})$ satisfies all the requirements of a physical systems model [4, Chap. VI, §6].

We continue the investigations of [5–15], pay particular attention to classes of a) symmetric and b) asymmetric set logics. In Corollary 3, we refined the axiomatics of asymmetric logics. For logics $X(km, k)$ of family subsets of the km-element set X, whose cardinal numbers are multiples of k we

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completely describe the cases in which $X(km,k)$ a) is symmetric (Proposition 3) or b) is asymmetric (Proposition 4). For an infinite set Ω and a natural number $n \geq 2$ we construct the concrete logics $\mathcal{E}_\Omega^n$ (Lemma 3) and completely describe the cases in which these logics are asymmetric (Theorem 1). For asymmetric logics $\mathcal{E}$ we determine when both the set $A \in \mathcal{E}$ and its complement A^c are atoms of the logic $\mathcal{E}$ (Theorem 2). Also we study charges and measures on set logics.

2. NOTATION AND DEFINITIONS

Put $\Omega_n = \{1, 2, \ldots, n\}$ for every number $n \in \mathbb{N}$. The following statement is well-known.

Lemma 1. *If $\mathcal{E}$ is a set logic, then the following condition holds:*

(iv) $B \setminus A \in \mathcal{E}$ for all $A, B \in \mathcal{E}$ with $A \subset B$.

Indeed, from $B^c \subset A^c$ we have $A \cap B^c = \emptyset$. Hence, $A \cup B^c \in \mathcal{E}$ by (iii) and

$$(A \cup B^c)^c = A^c \cap B = B \setminus A \in \mathcal{E}.$$

A family $\mathcal{E} \subset 2^\Omega$ is a set logic if and only if it satisfies conditions (i) and (iv). Let us verify sufficiency (i.e., fulfilment of (ii) and (iii)).

(ii) If $A \in \mathcal{E}$, then $A \subset \Omega \in \mathcal{E}$, hence, $\Omega \setminus A = A^c \in \mathcal{E}$.

(iii) If $A, B \in \mathcal{E}$ and $A \cap B = \emptyset$, then $A \subset B^c$ and $B^c \setminus A = B^c \cap A^c \in \mathcal{E}$; therefore, $A \cup B = (B^c \cap A^c)^c \in \mathcal{E}$.

Example 1. Let $\mathcal{E} \subset 2^\Omega$ be a set logic and $T \in \mathcal{E} \setminus \{\emptyset\}$. Then, the family

$$\mathcal{E}_T = \{A \in \mathcal{E} \mid A \subset T\}$$

is the set logic with the maximal element T. Since $T \in \mathcal{E}_T$, we should verify (iv). If $A, B \in \mathcal{E}_T$, $A \subset B$, then $A, B \in \mathcal{E}$ and $A \subset B \subset T$. Hence, $B \setminus A \subset T$ and by Lemma 1 we have $B \setminus A \in \mathcal{E}$, therefore, $B \setminus A \in \mathcal{E}_T$.

By definition, an *atom* in a set logic $\mathcal{E}$ is a minimal with respect to inclusion element of the set $\mathcal{E} \setminus \{\emptyset\}$. The set of all atoms in $\mathcal{E}$ we denote by $\alpha(\mathcal{E})$. It is easy to see that $\mathcal{E}$ is the set of all sums of elements of the set $\alpha(\mathcal{E})$ (a sum is the union of a family of sets, any two of which have the empty intersection). For $A \in \mathcal{E}$ put $\tilde{\mathcal{E}}(A) = \mathcal{E} \setminus \{\emptyset, \Omega, A, A^c\}$.

A state m_x on a logic $\mathcal{E}$ of subsets of Ω is called *concentrated at a point* $x \in \Omega$ if for all $A \in \mathcal{E}$

$$m_x(A) = \begin{cases} 1, & \text{if} \quad x \in A; \\ 0, & \text{if} \quad x \notin A. \end{cases}$$

For $A, B \subset \Omega$ define their symmetric difference

$$A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cap B^c) \cup (B \cap A^c) = (A \cup B) \setminus (A \cap B).$$

Then, $A^c \triangle B = A \triangle B^c = (A \triangle B)^c$ and $A^c \triangle B^c = A \triangle B$.

3. SYMMETRIC AND ASYMMETRIC CONCRETE LOGICS

Proposition 1. *Let $\mathcal{E}$ be a set logic and $A, B \in \mathcal{E}$. Then,*

$$A \cap B \in \mathcal{E} \iff A \cup B \in \mathcal{E}.$$

Proof. "$\Leftarrow$". By Lemma 1 we have $(A \cup B) \setminus A = B \setminus A \in \mathcal{E}$. Therefore, $B \setminus (B \setminus A) = A \cap B \in \mathcal{E}$ by Lemma 1.

"$\Rightarrow$". Since $A^c \cup B^c = (A \cap B)^c \in \mathcal{E}$, by the above proved $A^c \cap B^c \in \mathcal{E}$ and $A \cup B = (A^c \cap B^c)^c \in \mathcal{E}$. □

Corollary 1. *Let $\mathcal{E}$ be a set logic and $A, B \in \mathcal{E}$. If $A \cap B \in \mathcal{E}$, then $A \triangle B \in \mathcal{E}$.*

Proof. Since $A \cup B \in \mathcal{E}$, we have $A \triangle B = (A \cup B) \setminus (A \cap B) \in \mathcal{E}$ by Lemma 1. □

Corollary 2. *Let $\mathcal{E}$ be a set logic on Ω and $B \subset \Omega$. Then,*

$$B \in \mathcal{E} \iff \exists A \in \mathcal{E}\ (A \cap B, A \cup B \in \mathcal{E}).$$

Proof. "⇒". We choose $A \in \{B, B^c\}$.

"⇐". For $C = A \cap B \in \mathcal{E}$ we have $A \cap C = C \in \mathcal{E}$ and by Corollary 1 we obtain $A \triangle C = A \setminus B \in \mathcal{E}$. Now $B = (A \cup B) \setminus (A \setminus B) \in \mathcal{E}$ by Lemma 1. □

Definition 1. *A set logic $\mathcal{E}$ is called a symmetric logic, if it meets condition*

(v) $A \triangle B \in \mathcal{E}$ *for all* $A, B \in \mathcal{E}$.

A family $\mathcal{E} \subset 2^\Omega$ is a symmetric logic if and only if it satisfies conditions (i) and (v) [9, Proposition 1].

Definition 2 [11]. *A set logic $\mathcal{E}$ is called an asymmetric logic, if it meets condition*

(vi) $A \cap B \in \mathcal{E} \iff A \triangle B \in \mathcal{E}$ *for all* $A, B \in \mathcal{E}$.

From Corollary 1 follows

Corollary 3. *For a set logic $\mathcal{E}$ the following conditions are equivalent:*

(vii) *if $A, B \in \mathcal{E}$ and $A \triangle B \in \mathcal{E}$, then $A \cap B \in \mathcal{E}$;*

(viii) *$\mathcal{E}$ is an asymmetric logic.*

A set logic $\mathcal{E}$ is an algebra of sets if and only if $\mathcal{E}$ is symmetric and asymmetric [11, Proposition 4.5].

Lemma 2. *Let $\mathcal{E}$ be a symmetric logic and a mapping $\nu : \mathcal{E} \to \mathbb{R}$ meet condition*

(ix) $\nu(A \triangle B) \le \nu(A) + \nu(B)$ *for all* $A, B \in \mathcal{E}$.

Then, $\nu(A) \ge 0$ for all $A \in \mathcal{E}$.

Proof. For $A = B = \emptyset$ from (ix) we obtain $\nu(\emptyset) = \nu(\emptyset \triangle \emptyset) \le 2\nu(\emptyset)$, hence $\nu(\emptyset) \ge 0$. Now for every $A \in \mathcal{E}$ by (ix) we have $0 \le \nu(\emptyset) = \nu(A \triangle A) \le 2\nu(A)$. □

A measure ν on a symmetric logic $\mathcal{E}$ is called $\triangle$-*subadditive*, if it meets condition (ix). From Lemma 2 follows that every charge ν on a symmetric logic $\mathcal{E}$, which satisfies condition (ix), is a $\triangle$-subadditive measure. The following assertion is known (see [9, Lemma 1]); here we present its new proof.

Proposition 2. *For a measure ν on a symmetric logic $\mathcal{E}$ the following conditions are equivalent:*

(x) *ν is $\triangle$-subadditive;*

(xi) $\nu(A \triangle B) \le \nu(A \triangle C) + \nu(C \triangle B)$ for all $A, B, C \in \mathcal{E}$.

Proof. (x)⇒(xi). By associativity and commutativity of the operation $\triangle$ of symmetrical difference, we have

$$\nu(A \triangle B) = \nu(A \triangle B \triangle (C \triangle C)) = \nu((A \triangle C) \triangle (B \triangle C)) \le \nu(A \triangle C) + \nu(C \triangle B)$$

for all sets $A, B, C \in \mathcal{E}$.

(xi)⇒(x). Suppose that condition (xi) holds, but (x) does not hold. Then, there exist sets $A, B \in \mathcal{E}$ such that $\nu(A \triangle B) > \nu(A) + \nu(B)$. Therefore,

$$\nu(\Omega) - \nu(A \triangle B) < \nu(\Omega) - \nu(A) - \nu(B).$$

Since $A^c \triangle B = (A \triangle B)^c$ and $\Omega \setminus A = A^c$, we have

$$\begin{aligned} \nu(A^c \triangle B) + \nu(B) < \nu(A^c) &= \nu(A^c \triangle (C \triangle C)) = \nu((A^c \triangle C) \triangle C) \\ &\le \nu((A^c \triangle C) \triangle B) + \nu(B \triangle C) \\ &= \{\text{for} \quad C = A^c\} = \nu(B) + \nu(B \triangle A^c). \end{aligned}$$

We obtain a contradiction. Proposition is proved. □

Corollary 4. *For a measure ν on a symmetric logic $\mathcal{E}$ the mapping*

$$(A, B) \mapsto d(A, B) := \nu(A \triangle B) \ (\mathcal{E} \times \mathcal{E} \to \mathbb{R}^+)$$

defines the pseudometric d on $\mathcal{E}$ if and only if ν is $\triangle$-subadditive.

Example 2. If in conditions of Example 1 the set logic $\mathcal{E}$ is symmetric (respectively, asymmetric), then the logic $\mathcal{E}_T$ also is symmetric (respectively, asymmetric).

Example 3 ([11, Example 4.2]). Let $\Omega = \{z_n\}_{n=1}^\infty$ be a sequence of complex numbers and $\Omega \in \ell_1$, i.e., the series $\sum_{n=1}^\infty z_n$ is absolutely convergent. Let $\Lambda \in \{\mathbb{Q}, \mathbb{R}\}$ and $z = \sum_{n=1}^\infty z_n$. Recall that every permutation of the sequence $\{z_n\}_{n=1}^\infty$ retain absolute convergence and the sum z. A family

$$\mathcal{E}_{\Lambda,\Omega} = \left\{ I \subset \Omega \mid \sum_{x \in I} x = \lambda z \quad \text{for some} \quad \lambda \in \Lambda \right\}$$

is an asymmetric logic (the sum of the empty sequence by definition is equal to zero, hence, $\emptyset \in \mathcal{E}_{\Lambda,\Omega}$). Moreover, $\mathcal{E}_{\mathbb{R},\Omega}$ is a σ-class and $\mathcal{E}_{\mathbb{Q},\Omega}$ is its sublogic.

Example 4 ([11, Example 4.3]). Let $\mathcal{A}$ be the Lebesgue σ-algebra on $\Omega = [0, 1]$, μ be a linear Lebesgue measure with $\mu(\Omega) = 1$. Then, $\mathcal{E}_{\mathbb{Q},\mu} = \{A \in \mathcal{A} \mid \mu(A) \in \mathbb{Q}\}$ is an asymmetric logic.

Example 5 ([13, Example 4.4]). Let Ω be an infinite set. Put

$$\mathcal{B} = \{A \subseteq \Omega \mid \text{card}(A) \quad \text{is finite or} \quad \text{card}(\Omega \setminus A) \quad \text{is finite}\},$$

$$\mathcal{E}_{\Omega}^{\text{even}} = \{A \subseteq \Omega \mid \text{card}(A) \quad \text{is even or} \quad \text{card}(\Omega \setminus A) \quad \text{is even}\} \subset \mathcal{B}.$$

Then, $\mathcal{B}$ is an algebra of subsets of Ω and $\mathcal{E}_{\Omega}^{\text{even}}$ is a symmetric logic.

Recall that on $\mathcal{E}_{\Omega}^{\text{even}}$ every state is $\triangle$-subadditive [13, Proposition 4.4].

Definition 3 [16]. *Consider numbers $k, m \in \mathbb{N}$ and let $X = \{x_1, x_2, \ldots, x_{km}\}$. Denote by $X(km, k)$ the family of all subsets of X such that their cardinalities are multiples of k:*

$$X(km, k) = \{A \subset X \mid \text{card}(A) = ik, i = 0, 1, 2, \ldots, m\}.$$

Then, $\mathcal{E} = X(km, k)$ is a set logic with $\alpha(\mathcal{E}) = \{A \subset X \mid \text{card}(A) = k\}$.

Every function $f : X \to \mathbb{R}$ defines a charge ν_f on a set logic $X(km, k)$ by the formula

$$\nu_f(A) = \sum_{x \in A} f(x), \quad A \in X(km, k).$$

Such charges are called *regular*. It was shown in [16] that every measure on a set logic $X(km, k)$ admits the unique extension to a charge on the algebra 2^X. The proof of this fact is based on an interesting combinatorial lemma, asserting that $km - 1$ of some k-element sets can be choosen as generators of the logic $X(km, k)$.

In [17], the author presented a direct proof of this fact; he also described the extreme points of the state space of the logic $X(km, k)$ and the automorphisms of this logic. He also showed that for any charge ν on the set logic $X(km, k)$ for $m \geq 3$ there exists the unique function $f : X \to \mathbb{R}$ such that $\nu = \nu_f$.

Proposition 3. *The set logic $X(km, k)$ on X is a symmetric logic if and only if a) $m = 1$ and $k \in \mathbb{N}$ is arbitrary or b) $k \in \{1, 2\}$ and $m \in \mathbb{N}$ is arbitrary.*

Proof. For condition a) we have $X = \{x_1, \ldots, x_k\}$ and the set logic $X(km, k) = \{\emptyset, X\}$ is symmetric. For condition b) we consider separately cases $k = 1$ and $k = 2$.

Case I. Let $k = 1$ and $m \in \mathbb{N}$ be arbitrary. Then, $X = \{x_1, \ldots, x_m\}$, and the set logic $X(km, k) = 2^X$, clearly, is symmetric.

Case II. Let $k = 2$ and $m \in \mathbb{N}$ be arbitrary (by already analyzed case a), we assume that $m \geq 2$). Then, $X = \{x_1, x_2, \ldots, x_{2m}\}$ and the logic $X(km, k)$ is isomorphic to the well-known symmetric logic $\mathcal{E} = \{A \subset \Omega_{2m} \mid \text{card}(A) \text{ is even}\}$.

Now we show that the set logic $X(km, k)$ is not symmetric in the case of $k \geq 3, m > 1$.

Since $m \geq 2$ we have $\text{card}(X) \geq 2k$ and the logic $X(km, k)$ possesses two non-intersecting (i.e., disjoint) atoms $A_1 = \{a_1, a_2, a_3, \ldots, a_k\}$ and $B_1 = \{b_1, b_2, b_3, \ldots, b_k\}$. Put $A = \{x, a_2, a_3, \ldots, a_k\}$ and $B = \{x, b_2, b_3, \ldots, b_k\}$. Then,

$$A \triangle B = (A \backslash (A \cap B)) \cup (B \backslash (A \cap B)) = \{a_2, a_3, \ldots, a_k, b_2, b_3, \ldots, b_k\}$$

and $\text{card}(A \triangle B) = 2k - 2$. Since $k \geq 3$, we have $k < \text{card}(A \triangle B) = 2k - 2 < 2k$. It means that k is not a divisor of the natural number $2k - 2$. Therefore, $A \triangle B \notin X(km, k)$. □

Corollary 5. *There is a non-$\triangle$-subadditive state on the symmetric logic $X(2m, 2)$ $(m \geq 2)$.*

Proof. Let $\mathcal{E}$ be a finite symmetric logic with the property: "every state on $\mathcal{E}$, which is an affine combination of concentrated states, is $\triangle$-subadditive". Then, $\mathcal{E}$ is a Boolean algebra [11, Theorem 4.17]. In particular, if on a finite symmetric logic $\mathcal{E}$ every state is $\triangle$-subadditive, then $\mathcal{E}$ is a Boolean algebra [13, Theorem 4.3]. But the logic $X(2m, 2)$ $(m \geq 2)$ is not a Boolean algebra. □

If a symmetric logic is not a Boolean algebra, then it contains a sublogic isomorphic to $X(4, 2)$ [11, Corollary 4.6]. From Lemma 1 it follows that if ν is a measure on an asymmetric logic $\mathcal{E}$, then $\nu(A \triangle B) \leq \nu(A) + \nu(B)$ for all $A, B \in \mathcal{E}$ with $A \triangle B \in \mathcal{E}$.

Corollary 6. *Let a symmetric logic $\mathcal{E}$ of subsets of a finite set Ω be no Boolean algebra, let $\mathcal{A}$ be an algebra of subsets of Ω and $\mathcal{E} \subset \mathcal{A}$. Then, there exists a measure on $\mathcal{E}$, that does not admit an extension to a measure on $\mathcal{A}$.*

Proof. Every measure on a Boolean set algebra is $\triangle$-subadditive. □

Proposition 4. *A set logic $X(km,k)$ on X is asymmetric if and only if a) $m = 1$ and $k \in \mathbb{N}$ is arbitrary or b) k is odd and $m \in \mathbb{N}$ is arbitrary.*

Proof. For condition a) we have $X = \{x_1, \dots, x_k\}$, and the set logic $X(km,k) = \{\emptyset, X\}$, obviously, is asymmetric.

Let us consider condition b).

Case I. If k is odd, then we show that the set logic $X(km,k)$ is asymmetric. By Corollary 3 it suffices to prove that for arbitrary $A, B \in X(km,k)$ such that $A\triangle B \in X(km,k)$, its intersection $A \cap B$ also lies in $X(km,k)$. Since $A, B, A\triangle B \in X(km,k)$, we have $\mathrm{card}(A) = s_1 k$, $\mathrm{card}(B) = s_2 k$, $\mathrm{card}(A\triangle B) = s_3 k$, where $s_i \in \mathbb{N}, 0 \le s_i \le m$, $i = 1,2,3$. Then,

$$\begin{aligned}\mathrm{card}(A\triangle B) &= \mathrm{card}[(A\backslash(A\cap B)) \cup (B\backslash(A\cap B))] = \mathrm{card}(A\backslash(A\cap B)) + \mathrm{card}(B\backslash(A\cap B))\\ &= \mathrm{card}(A) - \mathrm{card}(A\cap B) + \mathrm{card}(B) - \mathrm{card}(A\cap B)\\ &= \mathrm{card}(A) + \mathrm{card}(B) - 2\mathrm{card}(A\cap B),\end{aligned}$$

hence, $s_3 k = s_1 k + s_2 - 2\mathrm{card}(A\cap B)$ and $\mathrm{card}(A \cap B) = \dfrac{(s_1+s_2-s_3)k}{2} = \dfrac{s_4 k}{2}$, where $s_4 \in \mathbb{N}$, $s_4 = s_1 + s_2 - s_3$. Let $\mathrm{card}(A\cap B) = n$, where $n \in \mathbb{N} \cup \{0\}$. If $n = 0$, then $A \cap B = \emptyset \in X(km,k)$. If $n > 0$, then $\mathrm{card}(A\cap B) = \dfrac{s_4 k}{2} = n$, hence, $s_4 k = 2n$. It means that $s_4 k$ is an even number. Since a number k is odd, s_4 is even. Then, $s_4 = 2j$ and $\dfrac{s_4}{2} = j$, where $j \in \mathbb{N}$. Therefore, $\mathrm{card}(A\cap B) = \dfrac{s_4}{2}k = jk$, $A \cap B \in X(km,k)$ and the logic $X(km,k)$ is asymmetric.

Case II. If k is even, then $k = 2t$ with $t \in \mathbb{N}$. We show that the set logic $X(km,k)$ is not asymmetric (by already analyzed case a), we assume that $m \ge 2$).

Since $m \ge 2$ we have $\mathrm{card}(X) \ge 2k$ and the logic $X(km,k)$ possesses two non-intersecting (i.e., disjoint) atoms $A_1 = \{a_1, a_2, a_3, \dots, a_{2t}\}$ and $B_1 = \{b_1, b_2, b_3, \dots, b_{2t}\}$. Put

$$A = \{x_1, \dots, x_t, a_{t+1}, \dots, a_{2t}\}, \quad B = \{x_1, \dots, x_t, b_{t+1}, \dots, b_{2t}\}, \quad \text{then}$$

$$A\triangle B = (A\backslash(A\cap B)) \cup (B\backslash(A\cap B)) = \{a_{t+1}, \dots, a_{2t}, b_{t+1}, \dots, b_{2t}\},$$

$\mathrm{card}(A\triangle B) = 2t = k$ and $A\triangle B \in X(km,k)$. But $A \cap B = \{x_1, \dots, x_t\}$ and $\mathrm{card}(A\cap B) = t = \dfrac{1}{2}k$. Therefore, $A \cap B \notin X(km,k)$ and the logic $X(km,k)$ is not asymmetric. □

Lemma 3. *Let Ω be an infinite set and a natural number $n \ge 2$. Then, the family*

$$\mathcal{E}^n_\Omega = \{A \subset \Omega : \mathrm{card}(A) = ns \text{ or } \mathrm{card}(A^c) = ns, \quad \text{where} \quad s \in \mathbb{N}\cup\{0\}\}$$

is a set logic.

Proof. Obviously, $\Omega \in \mathcal{E}^n_\Omega$ and $A \in \mathcal{E}^n_\Omega \iff A^c \in \mathcal{E}^n_\Omega$, i.e., conditions (i) and (ii) of definition of the set logics hold true. Let us verify (iv): for $A, B \in \mathcal{E}^n_\Omega$ with $A \subset B$ we check that $B \setminus A \in \mathcal{E}^n_\Omega$. Three cases are possible: a) $\mathrm{card}(A) = ns_1, \mathrm{card}(B) = ns_2$ with $s_1, s_2 \in \mathbb{N}\cup\{0\}$; then $s_1 \le s_2$ and $\mathrm{card}(B\setminus A) = ns_2 - ns_1 = n(s_2 - s_1)$, hence $B \setminus A \in \mathcal{E}^n_\Omega$; b) $\mathrm{card}(A) = ns_1$, $\mathrm{card}(B) = +\infty$ with $s_1 \in \mathbb{N}\cup\{0\}$; c) $\mathrm{card}(A) = \mathrm{card}(B) = +\infty$.

For b) we have $\mathrm{card}(B^c) = ns_2$ and $(B\setminus A)^c = (B\cap A^c)^c = B^c \cup A$, and also $B^c \cap A = \emptyset$. Since $\mathrm{card}((B\setminus A)^c) = \mathrm{card}(B^c) + \mathrm{card}(A) = n(s_1+s_2)$ and $B \setminus A \in \mathcal{E}^n_\Omega$.

For c) we have $\mathrm{card}(A^c) = ns_1$, $\mathrm{card}(B^c) = ns_2$ and $s_1 \ge s_2$ by the inclusion $B^c \subset A^c$. Since

$$B\setminus A = B\triangle A = B^c\triangle A^c = A^c \setminus B^c,$$

we have $\mathrm{card}(B\setminus A) = \mathrm{card}(A^c\setminus B^c) = n(s_1 - s_2)$ and $B \setminus A \in \mathcal{E}^n_\Omega$. □

Clearly, the logic $\mathcal{E}^n_\Omega$ lies in algebra $\mathcal{B}$ of subsets Ω from Example 5.

Theorem 1. *Let Ω be an infinite set and a natural number $n \geq 3$ be odd. Then, $\mathcal{E}_\Omega^n$ is an asymmetric logic.*

Proof. Let $A, B, A\triangle B \in \mathcal{E}_\Omega^n$. By Corollary 3 it is necessary to verify that the set $A \cap B$ lies in $\mathcal{E}_\Omega^n$. Assume that $A \cap B \neq \emptyset$ and $A \neq B$. Three cases are possible:

a) $\mathrm{card}(A) = ns_1, \mathrm{card}(B) = ns_2$ with $s_1, s_2 \in \mathbb{N}$;

b) $\mathrm{card}(A) = ns_1, \mathrm{card}(B) = +\infty$ with $s_1 \in \mathbb{N}$;

c) $\mathrm{card}(A) = \mathrm{card}(B) = +\infty$.

For a) we have $\mathrm{card}(A\triangle B) < +\infty$ and $\mathrm{card}(A \cap B) < +\infty$. Since $A\triangle B \in \mathcal{E}_\Omega^n$, there exists $s_3 \in \mathbb{N} \cup \{0\}$ such that $\mathrm{card}(A\triangle B) = ns_3$. Then, $\mathrm{card}A + \mathrm{card}B - 2\mathrm{card}(A \cap B) = ns_3$, and, hence,

$$\mathrm{card}(A \cap B) = \frac{\mathrm{card}(A) + \mathrm{card}(B) - ns_3}{2} = \frac{n(s_1 + s_2 - s_3)}{2} = \frac{ns^*}{2} \in \mathbb{N}.$$

Since the greatest common divisor $(2, n) = 1$, the number s^* is divisible by 2. $\frac{s^*}{2} = s^{**}$, then $\mathrm{card}(A \cap B) = ns^{**}$, and $A \cap B \in \mathcal{E}_\Omega^n$.

For b) we have $\mathrm{card}(A) = ns_1$, $\mathrm{card}(B^c) = ns_2$, $\mathrm{card}((A\triangle B)^c) = \mathrm{card}(A\triangle B^c) < +\infty$ and $\mathrm{card}(A \cap B) < +\infty$. Let $\mathrm{card}((A\triangle B)^c) = \mathrm{card}(A\triangle B^c) = ns_3$, then

$$\mathrm{card}(A) + \mathrm{card}(B^c) - 2\mathrm{card}(A \cap B^c) = ns_3, \quad \mathrm{card}(A \cap B^c) = \frac{n(s_1 + s_2 - s_3)}{2} = \frac{ns^*}{2} = ns^{**},$$

since the greatest common divisor $(2, n) = 1$. Then, we obtain

$$\mathrm{card}(A \cap B) = \mathrm{card}(A \setminus (A \cap B^c)) = \mathrm{card}(A) - \mathrm{card}(A \cap B^c) = 3s_1 - 3s^{**} = 3(s_1 - s^{**}) = 3s_4,$$

and $A \cap B \in \mathcal{E}_\Omega^n$.

For c) we have $\mathrm{card}(A^c) = ns_1$, $\mathrm{card}(B^c) = ns_2$. Then,

$$\mathrm{card}((A \cap B)^c) = \mathrm{card}(A^c \cup B^c) \leqslant \mathrm{card}(A^c) + \mathrm{card}(B^c) < +\infty.$$

Since $\mathrm{card}(A\triangle B) = \mathrm{card}(A^c\triangle B^c) = ns_3$, where $s_3 \in \mathbb{N}$, we have

$$\mathrm{card}(A^c) + \mathrm{card}(B^c) - 2\mathrm{card}(A^c \cap B^c) = ns_3, \quad \mathrm{card}(A^c \cap B^c) = \frac{n(s_1 + s_2 - s_3)}{2} = \frac{ns^*}{2} = ns^{**},$$

since the greatest common divisor $(2, n) = 1$. Then,

$$\begin{aligned}\mathrm{card}((A \cap B)^c) &= \mathrm{card}(A^c \cup B^c) = \mathrm{card}(A^c\triangle B^c) + \mathrm{card}(A^c \cap B^c) \\ &= ns_3 + ns^{**} = n(s_3 + s^{**}) = ns_4.\end{aligned}$$

Therefore, $A \cap B \in \mathcal{E}_\Omega^n$.

If a natural number $n \geq 2$ is even, then the logic $\mathcal{E}_\Omega^n$ is not asymmetric (it can be verified analogously to Case II in the proof of Proposition 4). □

Theorem 2. *Let $\mathcal{E}$ be a set logic and $A \in \mathcal{E}$. If $A\triangle B \notin \mathcal{E}$ for all $B \in \tilde{\mathcal{E}}(A)$, then $A, A^c \in \alpha(\mathcal{E})$. The inverse assertion holds for an asymmetric logic $\mathcal{E}$.*

Proof. Since

$$A \in \mathcal{E} \Leftrightarrow A^c \in \mathcal{E} \quad \text{and} \quad A\triangle B \in \mathcal{E} \Leftrightarrow A^c\triangle B = (A\triangle B)^c \in \mathcal{E},$$

it is suffices to show that $A^c \in \alpha(\mathcal{E})$. If $A\triangle B \notin \mathcal{E}$ for all $B \in \tilde{\mathcal{E}}(A)$, then the set $(A\triangle B)^c = A^c\triangle B$ does not lie in the logic $\mathcal{E}$ for all $B \in \tilde{\mathcal{E}}(A)$. In particular, there is no set $B \in \tilde{\mathcal{E}}(A)$ with $B \subset A^c$ (otherwise $A^c\triangle B = A^c \setminus B \in \mathcal{E}$ by Lemma 1). Therefore, $A^c \in \alpha(\mathcal{E})$.

Let now $\mathcal{E}$ be an asymmetric logic and $A, A^c \in \alpha(\mathcal{E})$. There does not exist any set $B \in \mathcal{E} \setminus \{\emptyset\}$ with $B \subset A^c, B \neq A^c$ (since A^c is an atom), i.e., for every set $B \in \tilde{\mathcal{E}}(A)$ exactly one of the following conditions holds true:

1) $B \cap A^c = A^c$ or 2) $B \cap A^c \notin \mathcal{E}$ and $B \cap A^c \neq A^c$.

In case 1), $B \supset A^c$ and $B \neq A^c$, hence, $A \supset B^c$ and $A \neq B^c \neq \emptyset$. Since A is an atom, we have $A = B^c$. Therefore, $B = A^c$, a contradiction.

In case 2), via asymmetry of the logic $\mathcal{E}$, we have $B\triangle A^c \notin \mathcal{E}$. Therefore, $A\triangle B = (B\triangle A^c)^c \notin \mathcal{E}$. □

Example 6. The set logic $\mathcal{E} = X(8,4)$ is not either symmetric or asymmetric (see Propositions 3 and 4). Let $X = \Omega_8$ and $A = \{1,2,3,5\}$. Then, $A, A^c \in \alpha(\mathcal{E})$, and also for $B = \{1,2,7,8\} \in \tilde{\mathcal{E}}(A)$ we have $A\triangle B \in \mathcal{E}$.

Example 7. For the symmetric logic $\mathcal{E} = X(4,2)$ we have $A, A^c \in \alpha(\mathcal{E})$ for all sets $A \in \mathcal{E} \setminus \{\emptyset, X\}$.

Proposition 5. *If $\mathcal{E}$ is a finite symmetric logic, then $card(\mathcal{E}) = 2^n$ for some $n \in \mathbb{N}$. In particular, $card(X(2m,2)) = 2^{2m-1}$ for all $m \in \mathbb{N}$.*

Proof. We call elements $A, B, \ldots$ of a symmetric logic $\mathcal{E}$ vectors and define the sum of vectors A and B as $A\triangle B$. Let us also define the multiplication operation of vectors by elements of the field $\mathbb{Z}_2 = \{0,1\}$ by the formulas $0 \times A = \emptyset$, $1 \times A = A$ $(A \in \mathcal{E})$. All of the axioms of a linear space over field $\mathbb{Z}_2$ clearly hold in this case. By assumption our space is finite-dimensional; we choose in this space some basis $e_1, \ldots, e_n$. Then, all vectors of the space are all possible linear combinations of $e_1, \ldots, e_n$, of cardinality 2^n.

For every $k \in \mathbb{N}$ by binomial theorem we obtain

$$\sum_{i=0}^{k} \binom{k}{i} = (1+1)^k = 2^k, \quad \sum_{i=0}^{k} (-1)^i \binom{k}{i} = (1-1)^k = 0.$$

Therefore, $\mathrm{card}(X(2m,2)) = \sum_{i=0}^{2m} \binom{2m}{2i} = 2^{2m-1},\ m \in \mathbb{N}$. □

Proposition 6. *Let $\mathcal{E}$ be a set logic on Ω, $B \subset \Omega$, and $A \in \mathcal{E}$ with $A\triangle B \in \mathcal{E}$. If a) $A = \{x\}$ or b) $\mathcal{E}$ is asymmetric, then $B \in \mathcal{E}$.*

Proof. If $A \cap B = \emptyset$, then $A\triangle B = A \cup B \in \mathcal{E}$ and $B = (A \cup B) \setminus A \in \mathcal{E}$ by Lemma 1. Let now $A \cap B \neq \emptyset$. For condition a) we have $A \cap B = A$, i.e., $A \subset B$ and $x \in B$. By assumption $A\triangle B = B \setminus \{x\} \in \mathcal{E}$. Hence,

$$B = (B \setminus \{x\}) \cup \{x\} \in \mathcal{E}$$

by axiom (iii) of set logic.

For condition b) we have $A \cap B \in \mathcal{E}$ and $A \setminus (A \cap B) = A \cap B^c \in \mathcal{E}$ by Lemma 1. By assumption $A\triangle B \in \mathcal{E}$, hence $B \cap A^c = (A\triangle B) \setminus (A \cap B^c) \in \mathcal{E}$ by Lemma 1. Therefore,

$$B = (B \cap A^c) \cup (B \cap A) \in \mathcal{E}$$

by axiom (iii) of set logic. □

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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