ISSN 0001-4346, Mathematical Notes, 2012, Vol. 92, No. 3, pp. 412–421. © Pleiades Publishing, Ltd., 2012.
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Integral Properties of the Classical Warping Function of a Simply Connected Domain

R. G. Salakhudinov*

Lobachevsky Research Institute of Mathematics and Mechanics, Kazan Federal University

Received October 23, 2009

Abstract—Let $u(z,G)$ be the classical warping function of a simply connected domain G. We prove that the L^p-norms of the warping function with different exponents are related by a sharp isoperimetric inequality, including the functional $u(G)=\sup_{x\in G}u(x,G)$. A particular case of our result is the classical Payne inequality for the torsional rigidity of a domain. On the basis of the warping function, we construct a new physical functional possessing the isoperimetric monotonicity property. For a class of integrals depending on the warping function, we also obtain a priori estimates in terms of the L^p-norms of the warping function as well as the functional $u(G)$. In the proof, we use the estimation technique on level lines proposed by Payne.

DOI: 10.1134/S0001434612090143

Keywords: *warping function, isoperimetric inequality, isoperimetric monotonicity, torsional rigidity, Payne inequality, level lines, Schwartz symmetrization.*

1. INTRODUCTION

Let G be a simply connected domain in the plane. By $u(x,G)$ we denote the warping function of the domain, i.e., the solution of the following boundary-value problem:

$$\begin{aligned} \Delta u &= -2, \qquad x\in G,\\ u &= 0, \qquad x\in\partial G. \end{aligned} \tag{1}$$

The existence and uniqueness of the warping function is one of the classical results of the theory of partial differential equations.

Note that another interpretation of the boundary-value problem (1) is related to the theory of flow of a viscous fluid in a tube with fixed cross-section (see, for example, [1]).

One of the important physical functionals of a domain in the torsion theory of rods with given cross-section is the torsional rigidity defined by the Saint-Venant formula [2]

$$P(G) := 2\int_G u(x,G)\,dx. \tag{2}$$

Another important characteristic of the function $u(x,G)$ in the mechanics of liquid flow is related to the volume of underground water in a porous medium owing to the infiltration flow to a drainage basin. Up to a constant, the volume of water is given by the functional

$$H(G) := \int_G \sqrt{u(x,G)}\,dx.$$

Another important example of a physical functional [3] involving the warping function is

$$u(G) := \sup_{x\in G} u(x,G). \tag{3}$$

*E-mail: Rustem.Salahudinov@ksu.ru