ISSN 0955-792X (PRINT)
ISSN 1465-363X (ONLINE)

Journal of Logic and Computation

Volume 22 Number 4 August 2012

www.logcom.oxfordjournals.org

Special Issue: Computability at Logic Colloquium 2009

OXFORD UNIVERSITY PRESS

THE INTERNATIONAL FEDERATION FOR COMPUTATIONAL LOGIC IFCoLoG

Algorithmic reducibilities of algebraic structures

ISKANDER KALIMULLIN, *Department of Mathematics and Mechanics, Kazan Federal University, Kazan, 420008, Kremlevskaya str. 18, Russia.*
E-mail: iskander.kalimullin@ksu.ru

Abstract

We describe all possible relations between certain reducibilities of algebraic structures which are based on the mass problems of structure presentability.

Keywords: Algebraic structure, constructivizible (computable) structure, Turing degrees, e-degrees (enumeration degrees), degree spectra of models.

1 Introduction

We say that a countable algebraic structure $\mathfrak{A}$ is *weakly reducible* to a countable algebraic structure $\mathfrak{B}$ ($\mathfrak{A} \leq_w \mathfrak{B}$), if for every isomorphic copy $\mathfrak{B}'$ of the structure $\mathfrak{B}$, the universe $|\mathfrak{B}'|$ of which consists from natural numbers, there is an isomorphic copy $\mathfrak{A}'$ of the structure $\mathfrak{A}$, the universe $|\mathfrak{A}'|$ of which also consists from natural numbers, such that the atomic diagram $D(\mathfrak{A}')$ is Turing reducible to the atomic diagram $D(\mathfrak{B}')$. If the reducibility of atomic diagrams above is realized by a fixed Turing operator, then we say that the structure $\mathfrak{A}$ is *strongly reducible* to the structure $\mathfrak{B}$ ($\mathfrak{A} \leq_s \mathfrak{B}$). Replacing Turing reducibility of atomic diagrams by enumeration reducibility, we get a notion of *weak e-reducibility* ($\mathfrak{A} \leq_{we} \mathfrak{B}$) and, in the case of fixed enumeration operator, the notion of *strong e-reducibility* ($\mathfrak{A} \leq_{se} \mathfrak{B}$).

It is clear, that strong (e-)reducibility implies weak (e-)reducibility. Stukachev [14] has shown that weak (strong) e-reducibility implies weak (strong) reducibility, and also that strong e-reducibility of a structure $\mathfrak{A}$ to a structure $\mathfrak{B}$ holds, if the structure $\mathfrak{A}$ is Σ-definable without parameters in the hereditarily finite superstructure $\mathbb{HF}(\mathfrak{B})$ (see [1] and [3] for the definition of $\mathbb{HF}(\mathfrak{B})$). Furthermore, he noted in [14] that if a structure has a degree (that is the class of Turing degrees of its isomorphic copies has the least element, which is called the *degree* of the structure [11]) then all mentioned reducibilities to this structure (including the Σ-definability) are equivalent up to a finite constant enrichment.

The goal of the article is to prove the following theorem, which says that there are no other relationships between mentioned above reducibilities. This theorem was announced in [9].

THEOREM 1
There are countable algebraic structures $\mathfrak{A}_k$, $1 \leq k \leq 5$, and a countable algebraic structure $\mathfrak{B}$, such that the following statements hold:

(1) $\mathfrak{B} \leq_{se} \mathfrak{A}_1$ and $\mathfrak{B}$ is not Σ-definable in $\mathbb{HF}(\mathfrak{A}_1)$;
(2) $\mathfrak{A}_2 \leq_s \mathfrak{B}$ and $\mathfrak{A}_2 \not\leq_{we} \mathfrak{B}$;
(3) $\mathfrak{A}_3 \leq_{we} \mathfrak{B}$ and $\mathfrak{A}_3 \not\leq_s (\mathfrak{B}, \overrightarrow{b})$;
(4) $\mathfrak{A}_4 \leq_w \mathfrak{B}$, $\mathfrak{A}_4 \not\leq_{we} \mathfrak{B}$ and $\mathfrak{A}_4 \not\leq_s (\mathfrak{B}, \overrightarrow{b})$;
(5) $\mathfrak{A}_5 \leq_{we} \mathfrak{B}$, $\mathfrak{A}_5 \leq_s \mathfrak{B}$ and $\mathfrak{A}_5 \not\leq_{se} (\mathfrak{B}, \overrightarrow{b})$;

Vol. 22 No. 4,
Published online September 10, 2010 doi:10.1093/logcom/exq046

In (3)–(5) $\vec{b}$ is an arbitrary finite array of elements of $\mathfrak{B}$.

For the basics of computability theory, and of enumeration reducibility and the notation used in this article, see [12], [2] and [13].

2 Preliminaries

The proof of the theorem uses the method of coding of families of subsets of ω into algebraic structures (see [4]). For example, we can code a countable family $\mathcal{V} \subseteq 2^{\omega}$ into the graph $\mathfrak{EF}(\mathcal{V})$ with the vertices A, $\mathsf{B}_{i,j,S}$ (where $i,j \in \omega$, $S \in \mathcal{V}$), $\mathsf{C}_{i,j,S}$ (where $i \in \omega$, $j \in S \in \mathcal{V}$) and the edges $\{\mathsf{A}, \mathsf{B}_{i,0,S}\}$ (where $i \in \omega$, $S \in \mathcal{V}$), $\{\mathsf{B}_{i,j,S}, \mathsf{B}_{i,j+1,S}\}$ (where $i,j \in \omega$, $S \in \mathcal{V}$), $\{\mathsf{B}_{i,j,S}, \mathsf{C}_{i,j,S}\}$ (where $i \in \omega$, $j \in S \in \mathcal{V}$).

The reducibilities between graphs of the form $\mathfrak{EF}(\mathcal{V})$ can be easily described using the notion of the enumeration of a family.

Given a family $\mathcal{V}$, we say that a set R is *an enumeration* of $\mathcal{V}$ if

$$\mathcal{V} \cup \{\emptyset\} = \{R^{(m)} \mid m \in \omega\}, \text{ where } R^{(m)} = \{x \mid \langle m,x \rangle \in R\}.$$

A set S is an *enumerator* of $\mathcal{V}$ if the set

$$\mathrm{Pr}_1(S) = \{x \mid (\exists y)[\langle x,y \rangle \in S]\}$$

is an enumeration of $\mathcal{V}$.

It is easy to see that a family $\mathcal{V}$ has a computably enumerable (c.e.) enumeration if and only if it has a computable enumerator. In this case, we say that the family $\mathcal{V}$ is c.e.

It is easy to note that $\mathfrak{EF}(\mathcal{U}) \leq_w \mathfrak{EF}(\mathcal{V})$ if and only if for every enumerator R of $\mathcal{V}$, there is an enumerator S of $\mathcal{U}$ such that $S \leq_T R$. Here, we have $\mathfrak{EF}(\mathcal{U}) \leq_s \mathfrak{EF}(\mathcal{V})$ if and only if there is a fixed Turing operator ensuring that $S \leq_T R$ for each enumerator R of V.

Also, $\mathfrak{EF}(\mathcal{U}) \leq_{we} \mathfrak{EF}(\mathcal{V})$ if and only if for every enumeration R of $\mathcal{V}$ there is an enumeration S of $\mathcal{U}$ such that $S \leq_e R$. Analogously, we have $\mathfrak{EF}(\mathcal{U}) \leq_{se} \mathfrak{EF}(\mathcal{V})$ if and only if there is a fixed enumeration operator ensuring that $S \leq_e R$ for each enumeration R of V.

The relation '$\mathfrak{EF}(\mathcal{U})$ is Σ-definable in $\mathbb{HF}(\mathfrak{EF}(\mathcal{V}))$' considered on families $\mathcal{U}$ and $\mathcal{V}$ was studied in [10]. In particular, we will use the following result from [10], which completely describes the relation '$\mathfrak{EF}(\mathcal{U})$ is Σ-definable in $\mathbb{HF}(\mathfrak{EF}(\mathcal{E}^{\infty}))$', where $\mathcal{E}^{\infty}$ is the family of all infinite c.e. sets.

PROPOSITION 1 [10]
For a family $\mathcal{U}$ the structure $\mathfrak{EF}(\mathcal{U})$ is Σ-definable in $\mathbb{HF}(\mathfrak{EF}(\mathcal{E}^{\infty}))$ if and only if

(i) the family $\mathcal{U}$ consists of c.e. sets;
(ii) the index set $I(\mathcal{U}) = \{n \mid W_n \in \mathcal{U}\}$ is Σ^0_3; and
(iii) there is a c.e. cover of $\mathcal{U}$ (a **cover** of a family $\mathcal{U}$ is a family $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{D}(\mathcal{V}) = \mathcal{D}(\mathcal{U})$, where $\mathcal{D}(\mathcal{X})$ denotes the family of all finite subsets of elements of $\mathcal{X}$).

Note that if a family $\mathcal{U}$ consists of c.e. sets, then the graph $\mathfrak{EF}(\mathcal{U})$ is *se*-equivalent to any of its expansions by finitely many constants $(\mathfrak{EF}(\mathcal{U}), \vec{b})$. Moreover, $(\mathfrak{EF}(\mathcal{U}), \vec{b})$ is always Σ-definable in $\mathbb{HF}(\mathfrak{EF}(\mathcal{U}))$.

The structures $\mathfrak{A}_k$, $1 \leq k \leq 5$, and the structure $\mathfrak{B}$ from Theorem 1 will be constructed in the form $\mathfrak{EF}(\mathcal{U})$, where $\mathcal{U}$ is a family of c.e. sets. Hence, **it is enough to prove Theorem 1 assuming that the array $\vec{b}$ is empty**.

We will also use families of the form

$$\mathcal{W}(\mathcal{R},\nu)=\{\{n\}\oplus X \mid n\in\omega \,\&\, X\in\mathcal{R} \,\&\, X\neq\nu(n)\},$$

where $\mathcal{R}$ is a c.e. family (e.g. the family of all finite sets, the family of all c.e. sets, etc.) and ν is a numbering of sets, i.e. a function from ω to 2^{ω}. This is a modification of the family, obtained by Wehner [15]. Namely, Wehner's family can be considered as $\mathcal{W}(\mathcal{F},\varepsilon)$, where $\mathcal{F}$ is the family of all finite sets, and ε is the standard numbering $\varepsilon(n)=W_n$ of all c.e. sets.

In [7], the families $\mathcal{W}(\mathcal{F},\nu)$ were studied for arbitrary Δ^0_2 numberings.

PROPOSITION 2 [7]
Let $\mathcal{R}\supseteq\mathcal{F}$ be a c.e. family of sets, and ν be a Δ^0_2 numbering (i.e. the predicate '$m\in\nu(n)$' is Δ^0_2). For a set X, the conditions (i), (ii), (iii) and (iv) are equivalent:

(i) there is an X-c.e. enumeration of $\mathcal{W}(\mathcal{R},\nu)$;
(ii) there is an X-c.e. cover of $\mathcal{W}(\mathcal{R},\nu)$, that is an X-c.e. family $\mathcal{V}\subseteq\mathcal{W}(\mathcal{R},\nu)$ such that

$$\mathcal{D}(\mathcal{V})=\mathcal{D}(\mathcal{W}(\mathcal{R},\nu))=\{\{n\}\oplus F \mid n\in\omega \,\&\, F\in\mathcal{F}\};$$

(iii) there is a computable function f such that for all $m,n\in\omega$ we have $W^X_{f(m,n)}\in\mathcal{R}$ and

$$\{z\in\omega \mid z<m\}\subseteq W^X_{f(m,n)}\neq\nu(n);$$

(iv) there is a computable function f such that for all $m,n\in\omega$ we have $W^X_{f(m,n)}\in\mathcal{F}$ and

$$\{z\in\omega \mid z<m\}\subseteq W^X_{f(m,n)}\neq\nu(n).$$

Furthermore, the conditions (i)–(iv) follow from condition (v) below:

(v) there is an X-c.e. set Z such that the symmetric difference $Z\,\triangle\,\nu(n)$ is infinite for all n.

If $\nu=\varepsilon^A$, where $\varepsilon^A(n)=W^A_n$, $n\in\omega$, for some fixed set A of low Turing degree, then the conditions (i)-(v) are equivalent to the condition (vi) below:

(vi) there is an X-c.e. set Z such that the symmetric difference $Z\,\triangle\,\nu(n)$ is not empty for all n,

which simply means that $X\not\leq_T A$.

Note that (iii) is just a reformulation of (ii). In [7], the proposition was proved only for the family of all finite sets $\mathcal{R}=\mathcal{F}$. In fact, the proof from [7] allows us to replace the family of all finite sets $\mathcal{F}$ by an arbitrary c.e. family $\mathcal{R}\supseteq\mathcal{F}$. Moreover, the uniformity of the proof allows us to apply Proposition 2 not only for the weak reducibility $\leq_w$, but also for the strong reducibility $\leq_s$ between structures of the form $\mathfrak{E}\mathfrak{F}(\mathcal{W}(\mathcal{R},\nu))$. In particular,

$$\mathfrak{E}\mathfrak{F}(\mathcal{W}(\mathcal{F},\nu))\equiv_s\mathfrak{E}\mathfrak{F}(\mathcal{W}(\mathcal{E},\nu)),$$

where $\mathcal{F}$ is the family of all finite sets, $\mathcal{E}$ is the family of all c.e. sets and ν is an arbitrary Δ^0_2 numbering.

For the reducibilties $\leq_{se}$ and $\leq_{we}$, we have the following result about the family $\mathcal{W}(\mathcal{E},\varepsilon)$, where $\mathcal{E}$ is the family of all c.e. sets and

$$\varepsilon(n)=W_n, n\in\omega.$$

Proposition 3 [8]
A set X is not c.e. if and only if there is an enumeration $R \leq_e X$ of the family $\mathcal{W}(\mathcal{E},\varepsilon)$. Furthermore, there is a fixed enumeration operator Φ such that $\Phi(X)$ is an enumeration of $\mathcal{W}(\mathcal{E},\varepsilon)$ for any $X \notin \Delta^0_2$.

Consider now the family $\mathcal{E}^\infty$ of all infinite c.e. sets. As is shown in the Appendix every enumeration R of $\mathcal{E}^\infty$ has a high e-degree, i.e. $J(J(\emptyset)) \leq_e J(R)$, where $J(X) = K(X) \oplus \overline{K(X)}$ and $K(X) = \{x \in \omega \mid x \in \Phi_x(X)\}$ for the standard listing of all enumeration operators $\{\Phi_x\}_{x\in\omega}$. Hence, for every enumeration R of $\mathcal{E}^\infty$ the set $\Phi(K(R))$ is always an enumeration of $\mathcal{W}(\mathcal{E},\varepsilon)$. Thus, $\mathfrak{EF}(\mathcal{W}(\mathcal{E},\varepsilon)) \leq_{se} \mathfrak{EF}(\mathcal{E}^\infty)$.

On the other hand, if $\mathfrak{EF}(\mathcal{W}(\mathcal{E},\varepsilon))$ is Σ-definable in $\mathbb{HF}(\mathfrak{EF}(\mathcal{E}^\infty))$ then by Proposition 1 there is a c.e. cover of $\mathcal{W}(\mathcal{E},\varepsilon)$, and hence by Proposition 2 [(ii) $\Longrightarrow$ (i)] the family $\mathcal{W}(\mathcal{E},\varepsilon)$ is c.e. contradicting Proposition 2 [(i) $\Longrightarrow$ (vi)].

Thus, the statement (1) of Theorem 1 follows.

Conclusion 1
For the structures $\mathfrak{B} = \mathfrak{EF}(\mathcal{W}(\mathcal{E},\varepsilon))$ and $\mathfrak{A}_1 = \mathfrak{EF}(\mathcal{E}^\infty)$ we have $\mathfrak{B} \leq_{se} \mathfrak{A}_1$ but $\mathfrak{B}$ is not Σ-definable in $\mathbb{HF}(\mathfrak{A}_1)$.

The following result says that we cannot replace $\mathcal{E}$ by the family $\mathcal{F}$ in Proposition 3.

Proposition 4 [8]
If a non-c.e. family $\mathcal{V}$ consists of finite sets, then there is a non-c.e. set X such that there is no enumeration $R \leq_e X$ of the family $\mathcal{V}$.

As a corollary, we can immediately obtain the statement (2) of Theorem 1.

Conclusion 2
For the structures $\mathfrak{B} = \mathfrak{EF}(\mathcal{W}(\mathcal{E},\varepsilon))$ and $\mathfrak{A}_2 = \mathfrak{EF}(\mathcal{W}(\mathcal{F},\varepsilon))$ we have $\mathfrak{A}_2 \leq_s \mathfrak{B}$ and $\mathfrak{A}_2 \not\leq_{we} \mathfrak{B}$.

Below, we prove the remaining statements of Theorem 1.

The proof of statements (3) and (4). We start with a series of lemmas which can be viewed as a weak analogue of Proposition 2 for enumeration reducibility. In fact, taken together, they generalize Proposition 3.

Lemma 1
Let ν be a Δ^0_2 numbering, and a set $X \in \Delta^0_2$ be such that $X \triangle \nu(n)$ is infinite for every $n \in \omega$. Then there is a computable function b such that for every $m,n \in \omega$ the set $\Phi_{b(m,n)}(X)$ is finite and

$$\{z \in \omega \mid z < m\} \subseteq \Phi_{b(m,n)}(X) \neq \nu(n).$$

Proof. Let $X = \lim_s X_s$ and $\nu(n) = \lim_s \nu_s(n)$ for every $n \in \omega$, where $\{X_s\}_{s\in\omega}$ and $\{\nu_s(n)\}_{n,s\in\omega}$ are strong arrays of finite sets. We construct each enumeration operator $\Phi_{b(m,n)} = \Theta_{m,n}$, $m,n \in \omega$, as the union of e-operators $\Theta_{m,n,s}$, $s \in \omega$, where $\Theta_{m,n,s} \subseteq \Theta_{m,n,s+1}$.

Construction.

Stage $s = 0$. $\Theta_{n,0} = \{\langle z, \emptyset\rangle \mid z < m\}$.

Stage $s+1$. The enumeration operator $\Theta_{m,n,s+1}$ is produced from $\Theta_{m,n,s}$ by adding all axioms $\langle x, \{x\}\rangle$, such that $x < s$ and for each $y < x$

$$y \in \nu_s(n) \Longleftrightarrow y \in \Theta_{n,s}(X_s).$$

End of the construction

Set $\Phi_{b(m,n)} = \Theta_{m,n} = \cup_s \Theta_{m,n,s}$ for every $m,n \in \omega$. Because $\langle x, F\rangle \in \Theta_{m,n}$ implies $F \subseteq \{x\}$, we have $\Theta_{m,n}(X) = \lim_s \Theta_{m,n,s}(X_s)$. Then $\Theta_{m,n}(X) = \nu(n)$ if and only if the e-operator $\Theta_{m,n}$ is infinite as a set of axioms.

Suppose for a contradiction that $\Theta_{m,n}(X) = \nu(n)$. Then $\Theta_{m,n}$ contains all axioms $\langle x, \{x\}\rangle$, $x \in \omega$, so that $X \cup \{z \mid z < m\} = \Theta_{m,n}(X) = \nu(n)$, contrary to the hypothesis. Thus, $\Phi_{b(m,n)}(X) = \Theta_{m,n}(X) \neq \nu(n)$ and $\Phi_{b(m,n)}(X) = \Theta_{m,n}(X)$ is finite. ■

Lemma 2

Let ν be a Δ^0_2 numbering. Then there is a computable function b such that for every $m,n \in \omega$ and $X \notin \Delta^0_2$ the set $\Phi_{b(m,n)}(X)$ is computable and

$$\{z \in \omega \mid z < m\} \subseteq \Phi_{b(m,n)}(X) \neq \nu(n).$$

Proof. Let $\nu(n) = \lim_s \nu_s(n)$ for all $n \in \omega$, where $\{\nu_s(n)\}_{n,s\in\omega}$ is a strong array of finite sets.

We construct each e-operator $\Phi_{b(m,n)} = \Theta_{m,n}$, $n \in \omega$, as the union of e-operators $\Theta_{m,n,s}$, $s \in \omega$, where $\Theta_{m,n,s} \subseteq \Theta_{m,n,s+1}$.

Construction.

Stage $s = 0$. $\Theta_{n,0} = \{\langle z, \emptyset\rangle \in \omega \mid z < m\}$.

Stage $s+1$. We say that an axiom $\langle\langle x,y\rangle, F\rangle$ is *an axiom of type I at stage* $s+1$, if $x < s, y < s$, $\nu_s(n) \restriction x \subseteq \Theta_{m,n,s}(F)$ and $\Theta_{m,n,s}(F) \restriction x - \nu_s(n) \neq \emptyset$. We say also that an axiom $\langle\langle x, 2y+1\rangle, F\rangle$ is *an axiom of type II at stage* $s+1$, if $x < s, y < s$, $\nu_s(n) \restriction x \subseteq \Theta_{m,n,s}(F)$ and $x \in F$.

The enumeration operator $\Theta_{m,n,s+1}$ is produced from $\Theta_{m,n,s}$ by adding all axioms of types I and II at stage $s+1$.

End of the construction

Finally, set $\Phi_{b(m,n)} = \Theta_{m,n} = \cup_s \Theta_{m,n,s}$ for every $m,n \in \omega$.

For a set $R \subseteq \omega$ and $x \in \omega$ let

$$R^{[x]} = \{\langle x,y\rangle \in R \mid y \in \omega\} \text{ and } R^{[>x]} = \{\langle z,y\rangle \in R \mid z > x \,\&\, y \in \omega\}.$$

(In contrast with the notation $R^{(x)} = \{y \mid \langle x,y\rangle \in R\}$ used often in the article, $R^{[x]}$ is a subset of R, that allows to obtain $R^{[>x]}$ as the disjoint union of $R^{[z]}$, $z > x$.)

Fix $m,n \in \omega$. It is easy to see that if an axiom $\langle\langle x,y\rangle, F\rangle$ is an axiom of type I at a stage s, then $\langle\langle x,y'\rangle, F\rangle$ is also an axiom of type I at stage s for each $y' < y$. Analogously, if an axiom $\langle\langle x, 2y+1\rangle, F\rangle$ is an axiom of type II at a stage s, then $\langle\langle x, 2y'+1\rangle, F\rangle$ is also an axiom of type II at stage s for each $y' < y$. It follows that for a set X and an integer x we have one of the following possibilities:

Case 1

$(\Theta_{m,n}(X))^{[x]} = \{\langle x,y\rangle \mid y \in \omega\}$;

Case 2

$(\Theta_{m,n}(X))^{[x]} = \{\langle x, 2y+1\rangle \mid y \in \omega\} \cup \{\langle x,y\rangle \mid y < t\}$ for some t;

Case 3

$(\Theta_{m,n}(X))^{[x]} = \{\langle x, 2y+1\rangle \mid y < s\} \cup \{\langle x,y\rangle \mid y < t\}$ for some s, t.

Thus, for each $x \in \omega$ and $X \subseteq \omega$ the set $(\Theta_{m,n}(X))^{[x]}$ is computable.

Note that Case 1 holds if and only if there are infinitely many y such that for some finite $F \subseteq X$, the axiom $\langle\langle x,y\rangle,F\rangle$ is an axiom of type I at some stage, which is equivalent to $\Theta_{m,n}(X)\restriction x-\nu(n)\neq\emptyset$ and $\nu(n)\restriction x\subseteq\Theta_{m,n}(X)$.

If Case 1 does not hold, then Case 2 holds if and only if there are infinitely many y such that for some finite $F\subseteq X$ the axiom $\langle\langle x,2y+1\rangle,F\rangle$ is an axiom of type II at some stage, which is equivalent to $x\in X$ and $\nu(n)\restriction x\subseteq\Theta_{m,n}(X)$.

In Case 3, we also note that if $x\notin X$ then there are no axioms of type II $\langle\langle x,2y+1\rangle,F\rangle$ such that $F\subseteq X$, and hence $(\Theta_{m,n}(X))^{[x]}=\{\langle x,y\rangle\mid y<t\}$ for some t.

Fix now an arbitrary set X which is not Δ^0_2. Suppose that $\Theta_{m,n}(X)=\nu(n)$. Then for each $x\in\omega$ Case 2 holds if $x\in X$, otherwise (if $x\notin X$) Case 3 holds, that is, $(\Theta_{m,n}(X))^{[x]}=\{\langle x,y\rangle\mid y<t\}$ for some t. Thus,

$$x\in X\iff\langle x,(\mu y)[\langle x,y\rangle\notin\nu(n)]+1\rangle\in\nu(n)$$

for each $x\in\omega$, so that $X\in\Delta^0_2$. This is a contradiction.

Thus, $\Theta_{m,n}(X)\neq\nu(n)$. To show that $\Theta_{m,n}(X)$ is computable, suppose at first that $\nu(n)\subseteq\Theta_{m,n}(X)$. Then $x_0\in\Theta_{m,n}(X)-\nu(n)$ for some x_0, and hence for all $x>x_0$ we have Case 1, that is, $(\Theta_{m,n}(X))^{[>x_0]}=\omega^{[>x_0]}$ is computable.

Suppose now that $\nu(n)-\Theta_n(X)\neq\emptyset$. Fix $t\in\omega$ such that

$$\nu_t(n)\restriction t-\Theta_{m,n}(X)\neq\emptyset.$$

Then for each $x>t$ and $s>t$ there are no axioms $\langle\langle x,y\rangle,F\rangle$ of type I or II at stage s such that $F\subseteq X$. Also, for all axioms $\langle\langle x,y\rangle,F\rangle$ of type I or II at stages $s\leq t$ we have necessarily $x<s$. Therefore, $(\Theta_{m,n}(X))^{[>t]}=\emptyset$.

Thus, in any case there is an $x_0\in\omega$ such that $(\Theta_{m,n}(X))^{[>x_0]}$ is computable. Hence $\Theta_{m,n}(X)$ is computable, since for each x the set $(\Theta_{m,n}(X))^{[x]}$ is also computable. ■

PROPOSITION 5

Let ν be a Δ^0_2 numbering, and a set X be such that $X\,\triangle\,\nu(n)$ is infinite for every $n\in\omega$. Then there is an enumeration $R\leq_e X$ of the family $\mathcal{W}(\mathcal{E},\nu)$, where $\mathcal{E}$ is the family of all c.e. sets.

PROOF. By Lemmas 1 and 2, there is a computable function b such that for every $m,n\in\omega$ the set $\Phi_{b(m,n)}(X)$ is computable and

$$\{z\in\omega\mid z<m\}\subseteq\Phi_{b(m,n)}(X)\neq\nu(n).$$

Again, let $\nu(n)=\lim_s\nu_s(n)$ for all $n\in\omega$, where $\{\nu_s(n)\}_{n,s\in\omega}$ is a strong array of finite sets.

Let R be such that for each $n,k,x,t\in\omega$

$R^{(\langle n,k,x,t\rangle)}=\{n\}\oplus\emptyset$, if $x\notin W_k\cup\cup_{s\geq t}\nu_s(n)$;

$R^{(\langle n,k,x,t\rangle)}=\{n\}\oplus W_k$, if $x\in W_k-\cup_{s\geq t}\nu_s(n)$ or $x\in\cap_{s\geq t}\nu_s(n)-W_k$;

$R^{(\langle n,k,x,t\rangle)}=\{n\}\oplus\Phi_{b(r,n)}(X)$, if $x\in W_k\cup\cup_{s\geq t}\nu_s(n)$ and $r\geq t$ is the least integer such that

$$x\in\overline{W_{k,r}}\cup\nu_r(n)\text{ and }x\in\overline{\nu_r(n)}\cup W_{k,r}.$$

Then $R\leq_e X$ and R is an enumeration of $\mathcal{W}(\mathcal{E},\nu)$. ■

We now are ready to prove statements (3) and (4) of Theorem 1.

Let α be a numbering of non-computable sets such that the numbering of Turing jumps $\alpha(n)'$ is Δ^0_2 (so α is a numbering of uniformly low sets) and for every $n_1 \neq n_2$ and $Z \subseteq \omega$

$$Z \text{ is c.e. in } \alpha(n_1) \,\&\, Z \text{ is c.e. in } \alpha(n_2) \Longrightarrow Z \text{ is c.e.}$$

(α can be chosen from the c.e. sets by a remark of Sorbi [13]).

Let ε^α be a numbering such that $\varepsilon^\alpha(n) = W_n^{\alpha(n)}$ for each $n \in \omega$. Then $\varepsilon^\alpha(n)$ is c.e. in $\alpha(n)$ uniformly in n, and hence the numbering ε^α is Δ^0_2.

We prove that $\mathfrak{EF}(\mathcal{W}(\mathcal{E},\varepsilon^\alpha)) \leq_{we} \mathfrak{EF}(\mathcal{W}(\mathcal{E},\varepsilon))$. Let R be an enumeration of $\mathcal{W}(\mathcal{E},\varepsilon)$. Then R is not c.e. Suppose that the symmetric differences

$$(R \oplus \emptyset) \triangle \varepsilon^\alpha(n_1) \text{ and } (\emptyset \oplus R) \triangle \varepsilon^\alpha(n_2)$$

are finite for some n_1 and n_2. Since R is infinite we have $n_1 \neq n_2$. Furthermore, R is c.e. in both of $\alpha(n_1)$ and $\alpha(n_2)$, contradicting the choice of α. Then, there must be a set $X \in \{R \oplus \emptyset, \emptyset \oplus R\}$ such that the symmetric difference $X \triangle \varepsilon^\alpha(n)$ is infinite for every $n \in \omega$. By Proposition 5, there is an enumeration S of $\mathcal{W}(\mathcal{E},\varepsilon^\alpha)$ such that $S \leq_e X \leq_e R$.

Suppose that $\mathfrak{EF}(\mathcal{W}(\mathcal{E},\varepsilon^\alpha)) \leq_s \mathfrak{EF}(\mathcal{W}(\mathcal{E},\varepsilon))$. Since each $\alpha(n)$ is not computable, then by the uniformity of Proposition 2 [(vi)$\Longrightarrow$(i)] there is a computable function h such that $W_{h(n)}^{\alpha(n)}$ is an enumeration of $\mathcal{W}(\mathcal{E},\varepsilon)$ for every $n \in \omega$. If $\mathfrak{EF}(\mathcal{W}(\mathcal{E},\varepsilon^\alpha)) \leq_s \mathfrak{EF}(\mathcal{W}(\mathcal{E},\varepsilon))$ then there is another computable function g such that $W_{g(n)}^{\alpha(n)}$ is an enumeration of $\mathcal{W}(\mathcal{E},\varepsilon^\alpha)$ for every $n \in \omega$. Again by the uniformity of Proposition 2 [(i)$\Longrightarrow$(iii)] there must be a computable function f such that $W_{f(n)}^{\alpha(n)} \neq \varepsilon^\alpha(n) = W_n^{\alpha(n)}$ for every $n \in \omega$. This contradicts the Recursion Theorem.

CONCLUSION 3
For the structures $\mathfrak{B} = \mathfrak{EF}(\mathcal{W}(\mathcal{E},\varepsilon))$ and $\mathfrak{A}_3 = \mathfrak{EF}(\mathcal{W}(\mathcal{E},\varepsilon^\alpha))$ we have $\mathfrak{A}_3 \leq_{we} \mathfrak{B}$ and $\mathfrak{A}_3 \not\leq_s \mathfrak{B}$.

Note that by Proposition 2 we have

$$\mathfrak{EF}(\mathcal{W}(\mathcal{F},\varepsilon^\alpha)) \equiv_s \mathfrak{EF}(\mathcal{W}(\mathcal{E},\varepsilon^\alpha)).$$

Hence, we already proved that

$$\mathfrak{EF}(\mathcal{W}(\mathcal{F},\varepsilon^\alpha)) \leq_w \mathfrak{EF}(\mathcal{W}(\mathcal{E},\varepsilon))$$

and

$$\mathfrak{EF}(\mathcal{W}(\mathcal{F},\varepsilon^\alpha)) \not\leq_s \mathfrak{EF}(\mathcal{W}(\mathcal{E},\varepsilon))$$

On the other hand, by Propositions 3 and 4, we have

$$\mathfrak{EF}(\mathcal{W}(\mathcal{F},\varepsilon^\alpha)) \not\leq_{we} \mathfrak{EF}(\mathcal{W}(\mathcal{E},\varepsilon)),$$

as $\mathcal{W}(\mathcal{F},\varepsilon^\alpha)$ consists of finite sets.

CONCLUSION 4
For the structures $\mathfrak{B} = \mathfrak{EF}(\mathcal{W}(\mathcal{E},\varepsilon))$ and $\mathfrak{A}_4 = \mathfrak{EF}(\mathcal{W}(\mathcal{F},\varepsilon^\alpha))$ we have $\mathfrak{A}_4 \leq_w \mathfrak{B}$, $\mathfrak{A}_4 \not\leq_{we} \mathfrak{B}$ and $\mathfrak{A}_4 \not\leq_s \mathfrak{B}$.

The proof of statement (5). The idea of the proof is similar to the previous proof. The difference here is that we need to take enumeration oracles instead of the Turing oracles $\alpha(n)$.

For example, we can try to do the following. Fix a numbering α of non-c.e. sets such that

(1) the numbering of the enumeration jumps $J(\alpha(n))$ is Δ_2^0;

(2) for all n and $Z \subseteq \omega$

$$Z \oplus \overline{Z} \leq_e \alpha(n) \Longrightarrow Z \text{ is computable;}$$

(3) for all $n_1 \neq n_2$ and $Z \subseteq \omega$

$$Z \leq_e \alpha(n_1) \,\&\, z \leq_e \alpha(n_2) \Longrightarrow Z \text{ is c.e.}$$

Then we try to define $\mathfrak{A}_5 = \mathfrak{E}\mathfrak{F}(\mathcal{W}(\mathcal{E},\beta))$, where $\beta(n) = \Phi_n(\nu(n))$ for every $n \in \omega$.

Then the reducibility $\mathfrak{A}_5 \leq_{we} \mathfrak{B} = \mathfrak{E}\mathfrak{F}(\mathcal{W}(\mathcal{E},\varepsilon))$ would follow from Lemmas 1 and 2 by the same arguments, as for $\mathfrak{A}_3 \leq_{we} \mathfrak{B}$. The reducibility $\mathfrak{A}_5 \leq_s B$ would follow from condition (v) of Theorem 2, because if R is an enumerator of $\mathcal{W}(\mathcal{E},\varepsilon)$ then R is not computable so that the symmetric difference $(R \oplus \overline{R}) \triangle \alpha(n)$ is infinite for every $n \in \omega$ (otherwise $R \oplus \overline{R} \leq_e \alpha(n)$ contradicting the choice of α).

However, we can not get $\mathfrak{A}_5 \not\leq_{se} \mathfrak{B}$ similarly with $\mathfrak{A}_3 \not\leq_s \mathfrak{B}$ since we have no (even nonuniform!) version of the implication [(i) $\Longrightarrow$ (iii)] of Proposition 2 for enumeration reducibility.

In this reason, we need to construct step-by-step a sequence $\nu(n)$, satisfying the conditions 1)-3) in such a way that we will have $\mathfrak{A}_5 \not\leq_{se} \mathfrak{B}$ for a structure $\mathfrak{A}_5$ of the form $\mathfrak{E}\mathfrak{F}(\mathcal{W}(\mathcal{E},\beta))$, where β is derived by a direct diagonalization as is in Wehner's original proof [15] instead of using the Recursion Theorem. Namely,

$$\beta(n) = \begin{cases} \emptyset, & \text{if } \psi(n) \text{ is undefined,} \\ \Phi_{\psi(n)}(\nu(n)), & \text{if } \psi(n) \text{ is defined,} \end{cases}$$

for some partial computable function ψ.

To define the function ψ we consider the following c.e. set:

$$A_0 = \{\langle n,k \rangle \mid \langle k,2n \rangle \in \Phi_n(\omega)\}.$$

Note that the condition $\langle k,2n \rangle \in \Phi_n(\omega)$ means simply that $\{n\} \oplus \emptyset \subseteq (\Phi_n(\omega))^{(k)}$. By the Uniformization Theorem, there is a partial computable function κ such that for every $n \in \omega$

$$\kappa(n) \text{ is defined } \Longleftrightarrow \langle n,\kappa(n) \rangle \in A_0 \Longleftrightarrow (\exists k)[\langle n,k \rangle \in A_0].$$

If $\kappa(n)$ is defined we will denote by F_n a finite set such that $\langle k,2n \rangle \in \Phi_n(F_n)$.

By the *s-m-n* Theorem, we can define also a partial computable function ψ such that $\mathrm{dom}(\psi) = \mathrm{dom}(\kappa)$, and for every $n \in \mathrm{dom}(\psi)$ and $X \subseteq \omega$, we have

$$\Phi_{\psi(n)}(X) = \{x \mid \langle \kappa(n), 2x+1 \rangle \in \Phi_n(X)\}.$$

In the uniform step-by-step construction below the sets $\nu(n) = X_n$ will be defined in such a way that for every $n \in \omega$ we will have $X_n = \lim_s X_n[s]$, where $X_n[s]$ is the approximation of X_n after stage s. The numbering $\nu(n)$ will be Δ_2^0.

Furthermore, we meet the requirements:

$G:(\forall n)(\forall m\neq n)(\forall s)(\forall x\in X_n[s])(\forall y\in X_m[s])[x\in X_n$ or $y\in X_m]$.

$P_n:\psi(n)$ is defined $\Longrightarrow F_n\subseteq X_n$;

$R_{\langle n,u,v\rangle}:\{z\,|\,z<u\}\subseteq X_n^{(\langle u,v\rangle)}\,\&\,X_n^{(\langle u,v\rangle)}\neq W_v$.

Here the variables n,m,s,x,y,u,v range over the set on non-negative integers ω.

It follows from the requirements $R_{\langle n,u,v\rangle}$ and the proof of Proposition 5 that there exists an e-operator Θ such that $\Theta(X_n)$ is an enumeration of $\mathcal{W}(\mathcal{E},\varepsilon)$ for every $n\in\omega$ (setting $\Phi_{b(u,v)}(X)=X^{(\langle u,v\rangle)}$). In particular, each $X_n=\nu(n)$, $n\in\omega$, is not c.e.

By the Theorem 13 from [6], each pair X_n and X_m, $n\neq m$, is e-ideal, that is for the c.e. set $A_{n,m}=\cup_s(X_n[s]\times Y_n[s])$ we have $X_n\times X_m\subseteq A_{n,m}$ and $\overline{X}_n\times\overline{X}_m\subseteq\overline{A_{n,m}}$. It follows from this article that the numbering $\nu(n)=X_n$ satisfies the conditions (1)–(3). We have the condition (3) by Theorem 8 [(I)$\Longrightarrow$ (V)] of [6] (e-ideal pairs are minimal pairs in the e-degrees). We have the condition (2) by Remark 3 of [6] (the e-degrees of e-ideal pairs of non-c.e. sets are quasi-minimal). Finally, we have the condition (1) by Remark 5 of [6] (the e-degrees of e-ideal pairs of non-c.e. Σ^0_2 sets are low).

It follows from conditions 1)-3) that for $\mathfrak{B}=\mathfrak{EF}(\mathcal{W}(\mathcal{E},\varepsilon))$ and $\mathfrak{A}_5=\mathfrak{EF}(\mathcal{W}(\mathcal{E},\beta))$, where

$$\beta(n)=\begin{cases}\emptyset, & \psi(n)\text{ is not defined,}\\ \Phi_{\psi(n)}(\nu(n)), & \psi(n)\text{ is defined,}\end{cases}$$

we still have reducibilities $\mathfrak{A}_5\leq_{we}\mathfrak{B}$ and $\mathfrak{A}_5\leq_s\mathfrak{B}$.

Indeed, by the condition 1) on ν the numbering β must be Δ^0_2. The reducibility $\mathfrak{A}_5\leq_s\mathfrak{B}$ follows from Proposition 2 [(v)$\Longrightarrow$(i)], since every enumerator R of $\mathcal{W}(\mathcal{E},\varepsilon)$ is not computable, and so by the condition 2) on ν, the symmetric difference $(R\oplus\overline{R})\,\triangle\,\beta(n)$ must be infinite for every $n\in\omega$. To see that $\mathfrak{A}_5\leq_{we}\mathfrak{B}$ it suffices to use Lemmas 1 and 2 exactly as in Proposition 5.

We prove that $\mathfrak{A}_5\not\leq_{se}\mathfrak{B}$. By the proof of Proposition 5, the requiremrents $R_{\langle n,u,v\rangle}$ imply that there is an enumeration operator Θ such that $\Theta(\nu(n))$ is an enumeration of $\mathcal{W}(\mathcal{E},\varepsilon)$ for every n. Suppose that $\mathfrak{A}_5\leq_{se}\mathfrak{B}$. Then there is an enumeration operator Ψ such that $\Psi(R)$ is an enumeration of $\mathcal{W}(\mathcal{E},\beta)$ for every enumeration of $\mathcal{W}(\mathcal{E},\varepsilon)$. Let n_0 be an index such that $\Phi_{n_0}(X)=\Psi(\Theta(X))$ for every $X\subseteq\omega$. We have now that

$$\mathcal{W}(\mathcal{E},\beta)\cup\{\emptyset\}=\{(\Phi_{n_0}(\nu(n)))^{(k)}\,|\,k\in\omega\}$$

for every $n\in\omega$. Since the family $\mathcal{W}(\mathcal{E},\beta)$ contains some sets of the form $\{n_0\}\oplus U$, the values $\kappa(n_0)$, $\psi(n_0)$ and F_n are defined. By the requirement P_{n_0}, the set $\nu(n_0)$ contains the finite set F_{n_0}. Hence, the set $\Phi_{n_0}(\nu(n_0))^{(\kappa(n_0))}$ is not empty and contains the subset $\{n_0\}\oplus\emptyset=\{2n_0\}$. But the non-empty set $\Phi_{n_0}(\nu(n_0))^{(\kappa(n_0))}$ belongs to the family $\mathcal{W}(\mathcal{E},\beta)$, and hence, must have the form $\{n_1\}\oplus U$ for some n_1 such that $U\neq\beta(n_1)$. This is possible only if $n_1=n_0$ so that $\Phi_{n_0}(\nu(n_0))^{(\kappa(n_0))}=\{n_0\}\oplus U$. The definition of the function ψ ensures that $U=\Phi_{\psi(n_0)}(\nu(n_0))=\beta(n)$. This is a contradiction.

It remains to construct uniformly the sequence of sets $\nu(n)=X_n=\lim_s X_n[s], n\in\omega$, satisfying the requirements G, P_n and $R_{\langle n,u,v\rangle}$ for all $n,u,v\in\omega$. The priority ranking of the requirements is:

$$G<P_0<P_1<P_2<\cdots R_0<R_1<R_2<\cdots.$$

Each requirement P_n can injure a requirement $R_{\langle n,u,v\rangle}$, $u,v\in\omega$, only once. Requirement G will not injure other requirements, but it will force P-requirements to injure another P-requirements of lower priority.

Let ψ_s be the finite part of ψ, computed by stage s of its computation.

Construction.

The initialization (injury) of requirement $R_{\langle n,u,v\rangle}$ means that the special parameter $x_{\langle n,u,v\rangle}$ (if defined) has been included into X_n. After that, $x_{\langle n,u,v\rangle}$ becomes undefined.

Stage $s=0$. The sets X_n, $n\in\omega$, are empty from the beginning. The parameter $x_{\langle n,u,v\rangle}$ is not defined for every $n,u,v\in\omega$.

Stage $s+1=2\langle n,z\rangle$. (To satisfy the requirement P_n).

If the requirement P_n has not received attention and the value $\psi_s(n)$ is already defined, then add the finite set F_n into X_n. Initialize the requirements $R_{n,u,v}$ $(u,v\in\omega)$. We say that the requirement P_n *received attention* at stage $s+1$.

Otherwise do nothing.

Stage $s+1=2\langle n,u,v,z\rangle+2$. (To satisfy requirement $R_{\langle n,u,v\rangle}$).

If the parameter $x_{\langle n,u,v\rangle}$ is not defined, then define $x_{\langle n,u,v\rangle}$ to be equal to the least integer such that $\langle\langle u,v\rangle,x_{\langle n,u,v\rangle}\rangle\notin X_n$ and $x_{\langle n,u,v\rangle}>u$. Add the finite set $\{z\,|\,z<u\}\cup\{x_{\langle n,u,v\rangle}\}$ to X_n.

If the parameter $x_{\langle n,u,v\rangle}$ is defined and $x_{\langle n,u,v\rangle}\in W_{u,s}$, then remove $x_{\langle n,u,v\rangle}$ from X_n and initialize the requirement $P_{\langle n',u',v'\rangle}$ for every $\langle n',u',v'\rangle>\langle n,u,v\rangle$.

Otherwise, do nothing.

End of the construction.

It is clear that $X_n=\lim_s X_n[s]$ exists for each $n\in\omega$ (membership of each integer in X_n can be changed at most three times). Furthermore, the satisfaction of the requirements P_n, $n\in\omega$, is also obvious. By the standard finite injury arguments each requirement $R_{\langle n,u,v\rangle}$, $n,u,v\in\omega$ can be injured only finitely many times, and hence each requirement $R_{\langle n,u,v\rangle}$, $n,u,v\in\omega$, is satisfied.

We prove that the requirement G is also satisfied. Suppose that for some $s\in\omega$ there are integers $x\in X_n[s]-X_n$ and $y\in X_m[s]-X_m$ for $n\neq m$. Then x and y must be determined via current parameters of the requirements $R_p, p\in\omega$, at the stage s. Namely, at stage s, we must have $x=\langle\langle u,v\rangle,x_{\langle n,u,v\rangle}\rangle$ and $y=\langle\langle u'v'\rangle,x_{\langle m,u',v'\rangle}\rangle$ for some $u,v,u'v'\in\omega$ (otherwise x and y can not be later removed from X_n and X_m). Without loss of generality, we can assume $\langle n,u,v\rangle<\langle m,u',v'\rangle$. Since $x\notin X_n$ the requirement $R_{\langle n,u,v\rangle}$ cannot be initialized at the stages $s'>s$. Also, at some stage $t>s$ the integer x must be removed from X_n. But then the requirement $R_{\langle m,u',v'\rangle}$ must be initialized at the stage t, contradicting the fact that $y\notin X_m$.

Thus, all requirements are satisfied so that for the numbering

$$\beta(n)=\begin{cases}\emptyset, & \psi(n)\text{ is not defined,}\\ \Phi_{\psi(n)}(\nu(n)), & \psi(n)\text{ is defined,}\end{cases}$$

where $\nu(n)=X_n$, $n\in\omega$, we have the following conclusion.

Conclusion 5

For the structures $\mathfrak{B}=\mathfrak{E}\mathfrak{F}(\mathcal{W}(\mathcal{E},\varepsilon))$ and $\mathfrak{A}_5=\mathfrak{E}\mathfrak{F}(\mathcal{W}(\mathcal{F},\beta))$ we have $\mathfrak{A}_5\leq_{we}\mathfrak{B}$, $\mathfrak{A}_5\leq_s B$ and $\mathfrak{A}_5\not\leq_{se}\mathfrak{B}$.

Theorem 1 follows now from Conclusions 1–5.

Acknowledgements

Supported by RFBR grants 09-01-97010, 10-01-00399 and by the President of the Russian Federation grant MK-1784.2010.1 and by the Russian Ministry of Science and Education (FCP).

References

[1] J. Barwise. *Admissible Sets and Structures*, Springer, 1975.
[2] S. B. Cooper. *Computability Theory*. Chapman & Hall/CRC, 2004.
[3] Y. L. Ershov. Definability and Computability. *Siberian School of Algebra and Logic* [in Russian], 1996.
[4] S. S. Goncharov, V. S. Harizanov, J. F. Knight, C. McCoy, R. G. Miller, and R. Solomon. Enumerations in computable structure theory. *Annals of Pure and Applied Logic*, **136**, 219–246, 2005.
[5] C. G. Jockusch, Jr. Degrees in which the recursive sets are uniformly recursive. *Canadian Journal of Mathematics*, **24**, 1092–1099, 1972.
[6] I. Sh. Kalimulllin. Definability of the jump operator in the enumeration degrees. *Journal of Mathematical Logic*, **3**, 257–267, 2003.
[7] I. Sh. Kalimullin. Spectra of degrees of some structures. *Algebra and Logic*, **46**, 399–408, 2007.
[8] I. Sh. Kalimullin. Enumeration degrees and enumerability of familes. *Journal of Logic and Computation*, **19**, 151–158, 2009.
[9] I. Sh. Kalimullin. Relations between algorithmic reducibilities of algebraic systems. *Russian Mathematics (Iz VUZ)*, **53**, 58–59, 2009.
[10] I. Sh. Kalimullin and V. G. Puzarenko. Reducibility on families. *Algebra and Logic*, **48**, 20–32, 2009.
[11] L. Richter. Degrees of structures. *Journal of Symbolic Logic*, **46**, 723–731, 1981.
[12] R. I. Soare. *Recursively Enumerable Sets and Degrees*, Springer, 1987.
[13] A. Sorbi. The enumeration degrees of Σ_2^0 sets. In *Complexity, Logic, and Recursion Theory*, A. Sorbi, ed. pp. 303–330. Dekker, 1997.
[14] A. I. Stukachev. Degrees of presentability of structures. I. *Algebra and Logic*, **46**, 419–432, 2007.
[15] S. Wehner. Enumerations, countable structures, and Turing degrees. *Proceedings of American Mathematical Society*, **126**, 2131–2139, 1998.

Received 2 November 2009

Appendix

Here, we prove that for every enumeration $\mathcal{R}$ of the family $\mathcal{E}^\infty$ of all infinite c.e. sets we have $J(J(\emptyset)) \leq_e J(R)$. To do so, we need the following analogue of Limit Lemma for enumeration reducibility.

LEMMA 3
The following are equivalent for sets $Y, X \subseteq \omega$:

(i) $Y \leq_e J(X)$;

(ii) There is $L \leq_e X$ such that

$$Y = \bigcup_{x=0}^{\infty} (L^{(2x)} - L^{(2x+1)}).$$

We can assume also that $L^{(2x+1)} \in \{\emptyset, \omega\}$ for each $x \in \omega$.

PROOF. (i)$\Longrightarrow$(ii). Let $Y \leq_e J(X)$ and let W be a c.e. set of axioms (i.e. an enumeration operator) such that

$$y \in Y \iff (\exists \text{ finite } F)[\langle y, F\rangle \in W \,\&\, F \subseteq J(X)]$$

for each $y \in \omega$. For each finite set $F \subseteq \omega$, let F_1 and F_2 be such that

$$F = F_1 \oplus F_2.$$

Since $J(X) = K(X) \oplus \overline{K(X)}$, we have

$$y \in Y \iff (\exists \text{ finite } F)[\langle y, F\rangle \in W \,\&\, F_1 \subseteq K(X) \,\&\, F_2 \subseteq \overline{K(X)}]$$

for all $y \in \omega$. Define finally

$$L = \{\langle 2u, y\rangle \mid \langle y, D_u\rangle \in W \,\&\, (D_u)_1 \subseteq K(X)\} \cup \\ \cup \{\langle 2u+1, y\rangle : y \in \omega \,\&\, (D_u)_2 \cap K(X) \neq \emptyset\},$$

where D_u is the finite set with the canonical index u.
Then $L \leq_e X$,

$$Y = \bigcup_{u=0}^{\infty} (L^{(2u)} - L^{(2u+1)}),$$

and also $L^{(2u+1)} \in \{\emptyset, \omega\}$ for all $u \in \omega$.

(ii)$\Longrightarrow$(i). Suppose that for some $L \leq_e X$ we have

$$Y = \bigcup_{x=0}^{\infty} (L^{(2x)} - L^{(2x+1)}).$$

Since $L \leq_1 K(X)$, we can fix a computable function f such that

$$z \in L \iff f(z) \in K(X)$$

for all $z \in \omega$. Then for all $y \in \omega$ we have

$$y \in Y \iff (\exists x)[f(\langle 2x, y\rangle) \in K(X) \,\&\, f(\langle 2x+1, y\rangle) \notin K(X)],$$

and consequently $Y \leq_e J(X) = K(X) \oplus \overline{K(X)}$. ■

The following result can be viewed as an analogue of the well-known result of Jockusch [5]. Namely, Jockusch [5] proved that the family of all (graphs of) computable functions is c.e. in a Turing degree **x** iff $\mathbf{0}'' \leq \mathbf{x}'$. The proof, in fact, shows that the family $\mathcal{E}^{\infty}$ of all infinite c.e. sets has the same property. Now, we can extend the last version for enumeration degrees (although the original version can not be so extended).

PROPOSITION 6
The following conditions are equivalent for a set X:

(i) there is an enumeration $R \leq_e X$ of $\mathcal{E}^\infty$;
(ii) $J(J(\emptyset)) \leq_e J(X)$.

PROOF. (i) $\Longrightarrow$ (ii). Let $R \leq_e X$ be an enumeration of the family $\mathcal{E}^\infty$, i.e.

$$\mathcal{E}^\infty \cup \{\emptyset\} = \{R^{(x)} \mid x \in \omega\}.$$

For each $i \in \omega$ consider the set

$$M_i = \{s \mid W_{i,s+1} - W_{i,s} \neq \emptyset\}.$$

It is clear that

$$W_i \in \mathcal{E}^\infty \Longleftrightarrow M_i \in \mathcal{E}^\infty,$$

and also the predicate '$s \in M_i$' is computable in i and s. Define

$$L = \{\langle 2x, i\rangle \mid (\exists s)[\langle x, s\rangle \in R]\} \cup \{\langle 2x+1, i\rangle \mid (\exists s)[\langle x, s\rangle \in R \,\&\, s \notin M_i]\}.$$

It is easy to see that $L \leq_e X$. Moreover,

$$i \in \bigcup_{x=0}^{\infty} (L^{(2x)} - L^{(2x+1)}) \Longleftrightarrow (\exists x)[R^{(x)} \neq \emptyset \,\&\, R^{(x)} \subseteq M_i].$$

Hence

$$I(\mathcal{E}^\infty) = \{i \mid W_i \in \mathcal{E}^\infty\} = \bigcup_{x=0}^{\infty} (L^{(2x)} - L^{(2x+1)}).$$

By Lemma 3, we have $I(\mathcal{E}^\infty) \leq_e J(X)$. But $J(J(\emptyset)) \equiv_e I(\mathcal{E}^\infty) \oplus \overline{I(\mathcal{E}^\infty)} \equiv_e I(\mathcal{E}^\infty)$. It now follows that $J(J(\emptyset)) \leq_e J(X)$.

(ii) $\Longrightarrow$ (i). Suppose $J(J(\emptyset)) \leq_e J(X)$. Then by Lemma 3

$$I(\mathcal{E}^\infty) = \bigcup_{x=0}^{\infty} (L^{(2x)} - L^{(2x+1)}).$$

for some $L \leq_e X$. Without loss of generality, we can assume that $L^{(2x)} = L^{(2x+1)} = \emptyset$ for some $x \in \omega$. Define

$$R = \{\langle\langle x, i\rangle, y\rangle \mid \langle 2x+1, i\rangle \in L \vee \langle 2x, i\rangle \in L \,\&\, y \in W_i\}.$$

It is clear that $R \leq_e X$ and

$$R^{(\langle x, i\rangle)} = \begin{cases} \emptyset & \text{if } i \notin L^{(2x)} \cup L^{(2x+1)}, \\ W_i & \text{if } i \in L^{(2x)} - L^{(2x+1)}, \\ \omega & \text{if } i \in L^{(2x+1)}. \end{cases}$$

It follows that

$$\mathcal{E}^\infty \cup \{\emptyset\} = \{R^{(n)} \mid n \in \omega\},$$

so that $R \leq_e X$ is an enumeration of $\mathcal{E}^\infty$. ■

Journal of Logic and Computation

Volume 22 Number 4 August 2012

Contents

OXFORD
UNIVERSITY PRESS