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Marcinkiewicz exponents and integrals over non-rectifiable paths

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We introduce and study certain distributions generalizing the operation of curvilinear integration for the case where the path of integration is not rectifiable. Then we apply that distributions for solving of boundary value problems of Riemann—Hilbert type in domains with non-rectifiable boundaries.

Keywords: non-rectifiable path; integration; distribution; Cauchy integral; Riemann–Hilbert boundary value problem

1. Introduction

Let Γ be a directed path on the complex plane $\mathbb{C}$. The curvilinear integrals $\int_\Gamma f\,dz$, $\int_\Gamma f\,dz + g\,d\bar{z}$ have a great body of applications in various fields of applied and pure mathematics. Almost all mathematical models of the mechanics of solid media on the plane contain that integrals. In particular, the whole theory of the Riemann—Hilbert boundary value problem, which has numerous applications in various fields, bases on the Cauchy-type curvilinear integral (see books [1–4]). However, for non-rectifiable paths, these integrals are undefined. There arises necessity of generalization of those integrals in the case, where Γ is not rectifiable. Several authors offered approaches for the solving of this problem. Let us describe briefly some of them.

Stieltjes integral. Let $z : [0,1] \ni t \mapsto z(t) \in \mathbb{C}$ be homeomorphic mapping of the segment $[0,1]$ on the path Γ. Then we understand the integral

$$\int_\Gamma f\,dz + g\,d\bar{z} = \int_0^1 f(z(t))\,dz(t) + g(z(t))\,d\overline{z(t)}$$

in the Stieltjes' sense. This approach is described in the papers [5–7].

Approximation of the integrands. Let Γ be a Jordan arc. Then the main formula of calculus gives us intrinsic meaning of integral $\int_\Gamma p(z)dz$, where $p(z)$ is algebraic polynomial, even if Γ is not rectifiable. Then we define integral $\int_\Gamma f(z)\,dz$ as limit of integrals $\int_\Gamma p_n(z)dz$, where the sequence of polynomials $\{p_n\}$ uniformly converges to f on Γ. That approach is discussed in [8]. As shown in this paper, the limit exists under certain restrictions including inequality

$$\nu > \mathrm{dm}(\Gamma) - 1, \tag{1}$$

where ν and $\mathrm{dm}(\Gamma)$ are Hölder exponent of integrand f and Minkowskii dimension of path Γ relatively; see their definitions in the next section.

Approximation of the paths. Let $\Gamma_1, \Gamma_2, \dots, \Gamma_n, \dots,$ be a sequence of rectifiable paths (in particular, by polygons) such that $\lim_{n\to\infty} \Gamma_n = \Gamma$ in a certain sense. Let F and G be continuations of f and g onto the whole plane $\mathbb{C}$. Then the limit of $\int_{\Gamma_n} F\,dz + G\,d\bar{z}$, if it exists, can be considered as the desired integral $\int_\Gamma fdz + gd\bar{z}$. As a version, one can identify the integration over Γ_n with distribution $\omega \mapsto \int_{\Gamma_n} \omega(F\,dz + G\,d\bar{z})$, $\omega \in C^\infty(\mathbb{C})$ and study the limit of those distributions. In the following, we call these limits *sequential integrations*.

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A number of versions of this approach are studied in the papers [9–12]. In all these works, the existence of sequential integral is proved either under restriction (1) or under condition $\nu \geq \mathrm{dm}(\Gamma) - 1$ and certain additional requirements.

Stokes' formula. Let Γ be a closed Jordan curve bounding finite domain D and F and G (see previous item) have integrable partial derivatives in D. If Γ is rectifiable, then the Stokes formula implies

$$\int_\Gamma f dz = -\iint_D \frac{\partial F}{\partial \bar{z}} dz d\bar{z}, \quad \int_\Gamma g d\bar{z} = \iint_D \frac{\partial G}{\partial z} dz d\bar{z}.$$

If Γ is not rectifiable, then the right sides of these equalities define the left ones. Probably, this approach was proposed first in [13]. Then it was studied in a number of publications (see, for instance, [14] and surveys [15, 16]). The condition (1) arises in all these works. We consider also distributions

$$C^\infty(\mathbb{C}) \ni \omega \mapsto -\iint_D \frac{\partial F\omega}{\partial \bar{z}} dz\, d\bar{z}, \quad C^\infty(\mathbb{C}) \ni \omega \mapsto \iint_D \frac{\partial G\omega}{\partial z} dz\, d\bar{z},$$

and call them *Stokes integrations*.

In the present paper, we study the Stokes integrations and their applications. We simplify the definition of this kind of integrations, extend the class of paths, and improve the known criteria of integrability in terms of recently introduced [17, 18] new characteristics of non-rectifiable curves – so called Marcienkiewicz exponents. Then we apply these results for solving of certain boundary value problems for analytic functions in domains with non-rectifiable boundaries.

In the next section, we define the Marcinkiewicz exponents and describe other necessary concepts and auxiliary results. In Section 3, we introduce the Stokes integrations and study their existence, uniqueness, and connections with the sequential integrations. In Section 4, we study the Cauchy integral over non-rectifiable curves in the Stokes' sense and solve certain boundary value problems. Finally, we cite several examples of evaluation of the Marcienkiewicz exponents. These examples show that our results improve the known criteria of generalized integrability.

2. Preliminaries

In what follows, we need metric characteristics of non-rectifiable curves of dimensional and co-dimensional types. Seemingly, the oldest characteristic of that kind is the Minkowskii dimension ([19–22]; it is also called box-counting dimension, upper metric dimension upper metric dimension, and so on).

Definition 1
Let $E \subset \mathbb{C}$ be a compact set and $\mathcal{N}_r(E)$ is the least number of disks of radius r covering E. The limit

$$\mathrm{dm}(E) := \limsup_{r \to 0} \frac{\log \mathcal{N}_r(E)}{-\log r}$$

is a Minkowskii dimension of the set E.

As known, the Minkowskii dimension of any plane set E does not exceed 2, and $\mathrm{dm}(\Gamma) = 1$ for any rectifiable curve Γ. This dimension is fractional for known self-similar fractal sets; for instance, the Minkowskii dimension of von Koch snowflake is $\log_3 4$.

The next necessary characteristics are introduced in [17, 18] Marcinkiewicz exponents. They are characteristics of co-dimensional type; the name 'Marcinkiewicz exponents' is explained by Marcinkiewicz's results characterizing plane sets in terms of integrals over their complements (see, for instance, [23]). We define them first for a metric measure space $X = (X; d; \mu)$ equipped with a metric d and a Borel regular outer doubling measure μ such that $0 < \mu(B) < \infty$ for all balls $B = B(x; r) = \{y \in X : d(y; x) < r\}, x \in X, r > 0$.

Definition 2
Let E be a compact subset of a fixed open domain $Y \subset X$,

$$I_p(E, \mu) := \int_{Y \setminus E} \frac{d\mu(z)}{\mathrm{dist}^p(z, E)}.$$

The Marcinkiewicz exponent of set E with respect to measure μ is the least upper bound of set $\{p : I_p(E, \mu) < \infty\}$. We denote it $\mathfrak{m}(E, \mu)$.

If $\Gamma \subset \mathbb{C}$ is a closed curve bounding finite domain D and D' is a greater finite domain, that is, $\overline{D} \subset D'$, then the inner and outer Marcinkiewicz exponents of Γ are $\mathfrak{m}^\pm(\Gamma, \mu) := \mathfrak{m}(\Gamma, \mu^\pm)$, where $\mu^\pm$ are restrictions of μ on D and $D' \setminus D$ correspondingly.

We need a local version of this definition for curves on the plane $\mathbb{C}$.

Definition 3
Let t be a point of a directed path $\Gamma \subset \mathbb{C}$ such that for sufficiently small $r > 0$, the curve Γ divides $B(t, r)$ into left and right components $B^\pm(t, r)$. Then the left and right local Marcinkiewicz exponents of Γ at point t are the least upper bounds of sets $\{p : \lim_{r \to 0} I_p(E, t, r, \mu^\pm) < \infty\}$, where $\mu^\pm$ are restrictions of μ on the components $B^\pm(t, r)$. If the curve Γ is closed, then we call the left and right exponents inner and outer. In addition, we put $\mathfrak{m}^*(\Gamma; t; \mu) := \max\{\mathfrak{m}^+(\Gamma; t; \mu), \mathfrak{m}^-(\Gamma; t; \mu)\}$ and $\mathfrak{m}^*(\Gamma; \mu) := \inf\{\mathfrak{m}^*(\Gamma; t; \mu) : t \in \Gamma\}$.

Clearly, $\mathfrak{m}^*(\Gamma; t; \mu) \geq \mathfrak{m}(\Gamma; t; \mu)$ and $\mathfrak{m}^*(\Gamma; \mu) \geq \mathfrak{m}(\Gamma; \mu)$. The examples (see the following) show that these inequalities are strict for certain curves. We omit μ if it is the plane Lebesgue measure.

Lemma 1
For $X = \mathbb{C}$, we have $\mathfrak{m}^{\pm}(E) \geq 2 - \mathrm{dm}(E)$ and $\inf\{\mathfrak{m}^{\pm}(E; t) : t \in E\} = \mathfrak{m}^{\pm}(E)$. If $E \subset \mathbb{C}$ is a continuum, then $\mathfrak{m}(E, \mu) \leq 1$. The Marcinkiewicz exponents of rectifiable curves on the complex plane are equal to 1. For a closed curve Γ, we have $\inf\{\mathfrak{m}^{+}(\Gamma; t; \mu) : t \in \Gamma\} = \mathfrak{m}^{+}(\Gamma; \mu)$ and $\inf\{\mathfrak{m}^{-}(\Gamma; t; \mu) : t \in \Gamma\} = \mathfrak{m}^{-}(\Gamma; \mu)$.

The proof can be found in [17]. Obviously, Lemma 1 implies inequality $\mathfrak{m}^*(\Gamma; t) \geq 2 - \mathrm{dm}(\Gamma)$ for any $t \in \Gamma$.

Then we consider the Hölder condition and the Whitney extension.

The Hölder space $H_\nu(E)$, $0 < \nu \leq 1$ consists of defined on set $E \subset \mathbb{C}$ functions f satisfying condition

$$h_\nu(f, E) := \sup\left\{\frac{|f(t) - f(t')|}{|t - t'|^\nu} : t, t' \in E, t \neq t'\right\} < +\infty.$$

It is a Banach space with norm $h_\nu(f, E) + \|f\|_{C(E)}$. Let $\mathfrak{v} : E \mapsto (0, 1]$ be a fixed function. We refer a function f to class $H_{\mathfrak{v}}(E)$ if for any $t \in E$, there exists a value $r = r(t) > 0$ such that $f|_{E\cap B(t,r)} \in H_{\mathfrak{v}(t)}(E \cap B(t, r))$. In the following, we consider that $\inf\{\mathfrak{v}(t) : t \in E\} = \nu^* > 0$.

By virtue of the Whitney theorem (see, for instance, [23]), any continuous on a fixed compact $E \subset \mathbb{C}$ function f has an extension f^w on the whole complex plane satisfying the following conditions:

(a) $f^w|_E = f$;
(b) if $f \in H_\nu(E)$, then $f^w \in H_\nu(\mathbb{C})$;
(c) if $f \in H_\nu(E)$, then $f^w(z)$ is differentiable in $\mathbb{C} \setminus E$ and $|\nabla f(z)| \leq h_\nu(f, E)\mathrm{dist}^{\nu-1}(z, E)$ for $z \in \mathbb{C} \setminus E$.
We put without loss of generality that support of f^w is compact.

3. Stokes integrations

In the following, we consider integrations of differential form $f\,dz$ only. Clearly, the analogous results are valid for the form $g\,d\bar{z}$.

Let Γ be a closed Jordan curve bounding finite domain D, $\overline{D} \subset D'$, D' is finite domain, too, and both domains D and D' are measurable. Assume that F is differentiable in $D' \setminus \Gamma$ continuation of function f from Γ into $\mathbb{C}$ with compact support in D' and that its derivatives are integrable. If Γ is rectifiable and directed in a customary way, then we have by Stokes' formula

$$\int_\Gamma f\,dz = -\iint_D \frac{\partial F}{\partial \bar{z}}\,dz\,d\bar{z} = \iint_{D'\setminus D} \frac{\partial F}{\partial \bar{z}}\,dz\,d\bar{z}.$$

Moreover, if F has limit values $F^{\pm}$ on Γ from the left and from the right, respectively, then

$$\int_\Gamma (F^+(t) - F^-(t))dt = -\iint_{\mathbb{C}} \frac{\partial F}{\partial \bar{z}}\,dz\,d\bar{z}.$$

This equality is valid for non-closed Jordan arcs, too. But in this case, notation $F^{\pm}(t)$ loses the sense at end points of Γ. In what follows, Γ' stands for $\Gamma \setminus \{t_1, t_2\}$ if Γ is an arc with end points t_1, t_2, and $\Gamma' = \Gamma$ for closed curve.

Let us assume that a function f is defined on a directed path $\Gamma \subset \mathbb{C}$ (non-rectifiable, generally speaking, and with null plane measure).

Definition 4
If a continuous in $\mathbb{C} \setminus \Gamma$ and local integrable in $\mathbb{C}$ function F with local integrable derivative $F_{\bar{z}}$ has limit values $F^{\pm}$ on Γ' from the left and from the right, respectively, such that

$$F^+(t) - F^-(t) = f(t), \quad t \in \Gamma', \tag{2}$$

then we call F **Stokes integrator** of f and say that distribution

$$\int_\Gamma^{(S)} f \cdot dz : C_0^\infty \ni \omega \mapsto \int_\Gamma^{(S)} f\omega\,dz := -\iint_{\mathbb{C}} \frac{\partial F\omega}{\partial \bar{z}}\,dz\,d\bar{z} \tag{3}$$

is **Stokes integration** over Γ.

Clearly, support of the distribution belongs to Γ.

3.1. Existence

Theorem 1
Let Γ be a closed curve of null plane measure. A function $f \in H_{\mathfrak{v}}(\Gamma)$ is Stokes-integrable if

$$\mathfrak{v}(t) > 1 - \mathfrak{m}^*(\Gamma; t), \quad t \in \Gamma. \tag{4}$$

Proof

Let us fix a positive value $m(t) < \mathfrak{m}^*(\Gamma; t)$ such that $\mathfrak{v}(t) > 1 - m(t)$. For any $\tau \in \Gamma$, there exists radius $r = r(\tau) > 0$ such that at least one of values $\iota^{\pm} := I_m(\Gamma, \tau, r, \mu^{\pm})$ is finite and $f_j := f|_{\Gamma \cap B(\tau, r)} \in H_{\mathfrak{v}(\tau)}(\Gamma \cap B(\tau, r))$. The family of balls $\{B(\tau, r) : \tau \in \Gamma\}$ covers Γ. As set Γ is compact, because this family contains a finite covering $\{B_j = B(\tau_j, r(\tau_j)) : j = 1, 2, \ldots, n\}$.

Let $\psi_j \in C_0^{\infty}(\mathbb{C})$ be a non-negative function with support $\overline{B_j}$, $j = 1, 2, \ldots, n$. Then restriction $\sigma(t)$ of sum $\sum_{j=1}^{n} \psi_j$ on curve Γ is positive. We put $f_j(t) := f(t)\psi_j(t)\sigma^{-1}(t)$, $t \in \Gamma$. Obviously, $f_j \in H_{\mathfrak{v}(\tau_j)}(\Gamma)$. If $\iota^+ < \infty$, then we put $\varphi_j := f_j^w \chi^+$, and if ι^- is finite, then $\varphi_j := -f_j^w \chi^-$, where $\chi^{\pm}$ are characteristic functions of the bounded by Γ domain and its complement correspondingly. By virtue of property (c) of the Whitney extension under restriction (4), the sum $F = \sum_{j=1}^{n} \varphi_j$ has integrable derivative $F_{\overline{z}}$. Hence, it is the desired integrator.

As we have mentioned earlier, the generalized integrability was proved under restriction (1). If $\mathfrak{v}(t) = \nu$ is constant, then the restriction (4) turns into

$$\nu > 1 - \mathfrak{m}^*(\Gamma),$$

and $1 - \mathfrak{m}^*(\Gamma) \le \mathrm{dm}(\Gamma) - 1$ by Lemma 1. In the following, we will see that the last inequality is strict for some curves Γ, that is, Theorem 1 sharpens the known results.

Let Γ be a directed Jordan arc. We denote its beginning t_1 and its end point t_2. The arc does not divide neighborhoods of the points $z_{1,2}$, that is, the values $\mathfrak{m}^*(\Gamma; t_{1,2})$ are undefined. In this connection, we have to introduce certain additional restriction.

We say that a point $t \in \Gamma$ satisfies condition of smooth touch (ST-condition) if there exists a smooth arc λ such that the intersection $\lambda \cap \Gamma$ contains only the point t. One can show easily that the set $ST(\Gamma)$ of all points $t \in \Gamma$ satisfying the ST-condition is everywhere dense in Γ (see also [15]).

We assume that $t_{1,2} \in ST(\Gamma)$, that is, there exists a smooth arc Λ such that $\Gamma^* := \Gamma \cup \Lambda$ is a simple closed curve. We consider Whitney extension f^w of a function $f \in H_{\mathfrak{v}}(\Gamma) \subset H_{\nu^*}(\Gamma)$. If F^* is its Stokes integrator corresponding to closed curve Γ^*, then

$$F(z) = F^*(z) - \frac{1}{2\pi i} \int_{\Lambda} \frac{f^w(t)\, dt}{t - z}$$

is a Stokes integrator of f on Γ. Thus, it is valid □

Theorem 2

Let Γ be a directed Jordan arc of null plane measure such that $t_{1,2} \in ST(\Gamma)$. A function $f \in H_{\mathfrak{v}}(\Gamma)$ is Stokes-integrable if $\mathfrak{v}(t) > 1 - \mathfrak{m}^*(\Gamma^*; t)$ for $t \in \Gamma$.

The condition $t_{1,2} \in ST(\Gamma)$ means that Γ has no spiral curls at its end points. Let us construct an integration over a curve Γ with end curls. Clearly, the boundary of closed convex hull of Γ contains at least one point of Γ besides $t_{1,2}$. Consequently, set $ST(\Gamma)$ is everywhere dense in Γ and there exist points $t'_{1,2} \in ST(\Gamma)$ such that distances $|t'_{1,2} - t_{1,2}|$ are less than a prescribed positive ε. These points divide Γ into three arcs: arc Γ_0 beginning at t_1 and ending at t'_1, arc Γ_1 beginning at t'_1 and ending at t'_2, and arc Γ_2 beginning at t'_2 and ending at t_2. The arc Γ_1 satisfies assumptions of Theorem 2. Then we consider functions

$$k_\Gamma(z) = \frac{1}{2\pi i} \ln \frac{z - t_2}{z - t_1}, \quad k_0(z) = \frac{1}{2\pi i} \ln \frac{z - t'_1}{z - t_1}, \quad k_2(z) = \frac{1}{2\pi i} \ln \frac{z - t_2}{z - t'_2},$$

where the branches of logarithmic functions are determined by means of cuts along arcs Γ, Γ_0, and Γ_2 correspondingly, and condition $k_\Gamma(z) = k_0(\infty) = k_2(\infty) = 0$. These functions have unit jumps on the arcs Γ_0 and Γ_2 correspondingly, and consequently, the products $\phi_{0,2} := (f|_{\Gamma_{0,2}})^w k_{0,2}$ will be integrators for arcs $\Gamma_{0,2}$ and functions $f|_{\Gamma_{0,2}}$ if their derivatives $\dfrac{\partial \phi_{0,2}}{\partial \overline{z}}$ are locally integrable. In particular, the last requirement fulfils if

$$k_\Gamma \in L^p_{loc}, \quad ,p \ge 2, \tag{5}$$

and $\mathfrak{v}(t_{1,2}) > 1 - \frac{1}{2}\mathfrak{m}(\Gamma; t_{1,2})$. Thus, it is valid

Theorem 3

Let Γ be a directed Jordan arc of null plane measure satisfying conditions (5). A function $f \in H_{\mathfrak{v}}(\Gamma)$ is Stokes-integrable if $\mathfrak{v}(t) > 1 - \mathfrak{m}^*(\Gamma; t)$ for $t \in \Gamma'$ and $\mathfrak{v}(t_{1,2}) > 1 - \frac{1}{2}\mathfrak{m}(\Gamma; t_{1,2})$.

The condition (5) means that curls of Γ at its end points is rather slow. It is weaker than the assumption $t_{1,2} \in ST(\Gamma)$. For instance, arc $\Xi := \{z = r\exp(ir^{-p}) : 0 < r \le 1\}$ satisfies (5) for $p < 1$, but $0 \notin ST(\Xi)$ for any $p > 0$.

However, there exist integrability conditions for arcs with any orders of curls. Let $q(z)$ be a single-valued branch of function $q(z) = \sqrt{(z - t_1)(z - t_2)}$ determined by a cut along Γ and asymptotic condition $q(z) \sim z$ near the infinity point. Obviously, function q has jump $2q^+$ on Γ and vanishes at its end points. By means of the construction of integrator from the proofs of Theorems 1 and 3, we obtain

Theorem 4

Let Γ be a directed Jordan arc of null plane, $f \in H_{\mathfrak{v}}(\Gamma)$. A function fq^+ is Stokes-integrable if $\mathfrak{v}(t) > 1 - \mathfrak{m}^*(\Gamma; t)$ for $t \in \Gamma'$ and $\mathfrak{v}(t_{1,2}) > 1 - \frac{1}{2}\mathfrak{m}(\Gamma; t_{1,2})$.

3.2. Uniqueness

Clearly, the integrator cannot be unique. In general, even holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ Stokes integrator vanishing at ∞ is not unique. Let $\mathrm{dm}_{\mathrm{H}}(E)$ stands for Hausdorff dimension (see, for instance, [24]) of set E. We consider a curve Γ such that $\mathrm{dm}_{\mathrm{H}}(\Gamma) = \alpha > 1$ and $\mathfrak{h}_\alpha(\Gamma) > 0$, where $\mathfrak{h}_\alpha$ is the Hausdorff measure of order α. As proved by H. Cartan (see [24]), there exists a closed set $\gamma \subset \Gamma$ such that $0 < \mathfrak{h}_\alpha(\gamma) < \infty$. Let

$$F_H(z) := \int_\gamma \frac{d\mathfrak{h}_\alpha(\zeta)}{\zeta - z}.$$

This function is continuous in the whole complex plane, analytical in $\overline{\mathbb{C}} \setminus \gamma$, it vanishes at infinity, but it is not identical zero (see [25]). Clearly, $F_H^+(t) = F_H^-(t), t \in \Gamma$, that is, F_H is null-integrator. But its distributional $\overline{\partial}-$derivative

$$\frac{\partial F_H}{\partial \overline{\zeta}} : C_0^\infty(\mathbb{C}) \ni \omega \mapsto -\iint\limits_{\mathbb{C}} F_H \frac{\partial \omega}{\partial \overline{z}} dz d\overline{z},$$

is not zero, because otherwise, F_H is identical null by virtue of the Weyl lemma (see [26]). Then there exists $\omega_0 \in C_0^\infty(\mathbb{C})$ such that

$$\left\langle \frac{\partial F_H}{\partial \overline{\zeta}}, \omega_0 \right\rangle \neq 0.$$

Thus, if F is a holomorphic integrator of a function f on a curve Γ, then $F_c := F + cF_H$ also is its holomorphic integrator for any constant c, and all Stokes integrals (3) generated by these integrators are different.

On the other hand, Dolzhenko [25] proved the following result:

- if E is a compact subset of a domain $D \subset \mathbb{C}$, and a function $\Phi(z)$ is holomorphic in $D \setminus E$ and satisfies the Hölder condition with exponent $\nu > \mathrm{dm}_{\mathrm{H}}(E) - 1$ in the whole domain D, then F is holomorphic in D.

This result implies uniqueness of holomorphic Stokes integrators satisfying near Γ the Hölder condition with sufficiently large exponents. Indeed, if F and G are holomorphic Stokes integrators for function f on path Γ and both these functions satisfy the Hölder condition with exponent $\nu > \mathrm{dm}_{\mathrm{H}}(\Gamma) - 1$ in half-neighborhoods of points of Γ', then their difference is holomorphic on Γ'. As the difference vanishes at infinity point, then it is identical zero. In the case of non-closed arc we obtain the same conclusion by virtue of integrability of $F - G$ near points $t_{1,2}$.

Let us prove analogous result on uniqueness of Stokes integrations with non-holomorphic integrators.

Lemma 2

If two integrators for function f on path Γ satisfy the Hölder condition with exponent $\nu > \mathrm{dm}_{\mathrm{H}}(\Gamma) - 1$ near Γ, then these integrators generate the same Stokes integration $\int_\Gamma^{(S)} f \cdot dz$.

Proof

Let Γ be an open arc. We fix $\alpha \geq \mathrm{dm}_{\mathrm{H}}\Gamma$ such that $\mathfrak{h}_\alpha(\Gamma) = 0$; then for any $\varepsilon > 0$, we can cover Γ by a family of disks $B_j = B(z_j, r_j), j = 1, 2, \dots$, such that $r_j < \varepsilon$ and $\sum_{j>0} r_j^\alpha < \varepsilon$. Then the length of boundary Λ of the union $\mathbf{B} = \cup_{j>0} B_j$ is less than $\varepsilon^{1-\alpha}$.

Now, let $F_{1,2}$ be two integrators for f on Γ and $I_{1,2}$ are corresponding integrations, that is,

$$\langle I_k, \omega \rangle = -\iint_{\mathbb{C}} \frac{\partial F_k \omega}{\partial \overline{z}} dz d\overline{z}, \quad k = 1, 2.$$

Then for any $\omega \in C_0^\infty(\mathbb{C})$, we have

$$|\langle I_1 - I_2, \omega \rangle| \leq \left| \iint_{\mathbf{B}} \frac{\partial (F_1 - F_2)\omega}{\partial \overline{z}} dz d\overline{z} \right| + \left| \int_\Lambda (F_1 - F_2)\omega dz \right|,$$

and if $\nu > \alpha - 1$, then both terms of the right side vanish for $\varepsilon \to 0$. For a closed Γ, the proof is analogous.

Let $\mathfrak{d}(t), t \in \Gamma$ be a real function, $0 \leq \mathfrak{d}(t) < 2$. We refer a curve Γ to class $U_\mathfrak{d}$ if for any $t \in \Gamma$, there exists $r = r(t) > 0$ such that $\mathrm{dm}_{\mathrm{H}}(B(t, r) \cap \Gamma) \leq \mathfrak{d}(t)$, and a function F to class $\mathfrak{H}_\mathfrak{v}$ if $F|_{B^\pm(t,r)} \in H_\mathfrak{v}(B^\pm(t, r)), t \in \Gamma'$. Clearly, the previous lemma implies □

Theorem 5

If two integrators of function f over path $\Gamma \in U_\mathfrak{d}$ belong to class $\mathfrak{H}_\mathfrak{v}$ and $\mathfrak{v}(t) > \mathfrak{d}(t) - 1, t \in \Gamma$, then these integrators generate the same Stokes integration $\int_\Gamma^{(S)} f \cdot dz$.

In the proof of Theorem 1, we have constructed an integrator $F \in \mathfrak{H}_\mathfrak{v}$ for $f \in H_\mathfrak{v}(\Gamma)$.

3.3. Local Hölder orders

As known (see, for instance, [27]), order of a distribution φ is integer number k such that $|\langle \varphi, \omega \rangle| \leq A\|\omega\|_{C^k}$ with independent on ω value $A > 0$. Clearly, the Stokes integration is a distribution of first order (unlike the customary integration along rectifiable curve, which is distribution of null order). However,t we can introduce a concept of order for any family of spaces X_λ such that $C^\infty \subset X_{\lambda_1} \subset X_{\lambda_2}$ for $\lambda_1 > \lambda_2$: the order of φ with respect to family X_λ is $\inf\{\lambda : |\langle \varphi, \omega \rangle| \leq A\|\omega\|_{X_\lambda}\}$. If this inequality is valid for ω with support in a sufficiently small neighborhood of a point t, then we say that it is the local order at this point.

Let $f \in H_{\mathfrak{v}}(\Gamma)$, where Γ is a closed curve. By virtue of Theorem 1 under restriction (4), there exists integrator $F \in \mathfrak{H}_{\mathfrak{v}}$. We consider a function $\omega \in C_0^{\infty}(\mathbb{C})$ with supports in disk $B = B(t, r)$ such that $F|_{B^{\pm}}$ satisfies the Hölder condition with exponent $\mathfrak{v}(t)$. We restrict ω on Γ and denote $\tilde{\omega}$ the Whitney continuation of the restriction. One can verify easily that

$$\int_{\Gamma}^{(S)} f(z)\omega(z)dz = -\iint_{\mathbb{C}} \frac{\partial F\omega}{\partial \bar{z}} dz d\bar{z} = -\iint_{\mathbb{C}} \frac{\partial F\tilde{\omega}}{\partial \bar{z}} dz d\bar{z}.$$

By virtue of the property (c) of the Whitney extension, we obtain

$$\left| \int_{\Gamma}^{(S)} f(z)\omega(z)dz \right| \leq A\|\omega\|_{H_{\mathfrak{v}(t)}}$$

for $\mathfrak{v}(t) > 1 - \mathfrak{m}^*(\Gamma; t)$. Consequently, the local Hölder order of the Stokes integration at a point $t \in \Gamma$ is less or equal $1 - \mathfrak{m}^*(\Gamma; t)$. The global Hölder order (i.e., the order with respect to scale $H_{\nu}(\Gamma)$) is less or equal $1 - \mathfrak{m}^*(\Gamma)$. Thus, if Γ has fractional dimension, then the corresponding Stokes integration has fractional order.

3.4. Sequential integrations

As mentioned earlier, the idea to define curvilinear integral over non-rectifiable path as limit of integrals along converging to its rectifiable curves is considered in numerous papers. Here, we establish certain results on its existence and connections with the Stokes integrability.

We restrict ourself by closed curves. Let Γ be a closed non-rectifiable Jordan curve bounding finite domain D. We say that a sequence $\Gamma = \{\Gamma_1, \Gamma_2, \dots\}$ of rectifiable closed curves increases and converges to Γ, if curve Γ_j bounds finite domain D_j, $D_1 \subset D_2 \subset \dots \subset D$ and $\cup_{j>0} D_j = D$. If $D_1 \supset D_2 \supset \dots \supset D$ and $\cap_{j>0} D_j = D$, then we say that Γ decreases and converges to Γ. Both classes of sequences we call monotone convergent.

A curvilinear integral $\int_{\Gamma} f\, dz$ over a rectifiable curve Γ determines distribution

$$I(f, \Gamma) : C^{\infty}(\mathbb{C}) \ni \omega \to \int_{\Gamma} f(t)\omega(t)dt.$$

Definition 5

Let Γ be a closed non-rectifiable curve, Γ is a monotone convergent sequence of rectifiable curves, a function f is defined on Γ, and F is its continuation onto the whole complex plane. If $\lim_{j\to\infty} I(F, \Gamma_j)$ exists, then we call it sequential integral and denote $\int_{\Gamma}^{(se)} f(t) \cdot dt$.

Theorem 6

If Γ is a closed Jordan curve of null plane measure, then the following propositions are valid.

1. A function $f \in H_{\mathfrak{v}}(\Gamma)$ has continuation F such that $\lim_{j\to\infty} I(F, \Gamma_j)$ exists for any increasing convergent sequence of rectifiable curves Γ if
$$\mathfrak{v}(t) > 1 - \mathfrak{m}^{+}(\Gamma; t), \quad t \in \Gamma. \tag{6}$$
2. A function $f \in H_{\mathfrak{v}}(\Gamma)$ has continuation F such that $\lim_{j\to\infty} I(F, \Gamma_j)$ exists for any decreasing convergent sequence of rectifiable curves Γ if
$$\mathfrak{v}(t) > 1 - \mathfrak{m}^{-}(\Gamma; t), \quad t \in \Gamma. \tag{7}$$
3. If a function f has a continuation $F \in H_{\nu}(\mathbb{C})$ such that $\lim_{j\to\infty} I(F, \Gamma_j)$ exists for certain monotone convergent sequence of rectifiable curves Γ, then f is Stokes-integrable.

   In all these cases, $\int_{\Gamma}^{(se)} f(t) \cdot dt = \int_{\Gamma}^{(S)} f(t) \cdot dt$.

Proof

Let $f \in H_{\mathfrak{v}}(\Gamma)$. As previously, we fix a finite covering of Γ by small disks $B_j = B(t_j, r_j)$ and put $f_j = f\psi_j$, where $\psi_j, j = 1, 2, \dots$ is corresponding decomposition of unit and $F = \sum_{j>0} f_j^w$. Clearly, $F_{\bar{z}}$ is integrable in the bounded by Γ domain D under condition (6) and in its complement under condition (7). In the first case, $\lim_{j\to\infty} I(F, \Gamma_j)$ exists for any increasing convergent sequence of rectifiable curves, and in the second case – for any decreasing one. Indeed, in both cases

$$\left| \int_{\Gamma_m} F(t)\omega(t)dt - \int_{\Gamma_n} F(t)\omega(t)dt \right| = \left| \iint_{D_{m,n}} \frac{\partial (F\omega)}{\partial \bar{\zeta}} d\zeta d\bar{\zeta} \right|,$$

where $D_{m,n}$ is a ring-like domain between curves Γ_m and Γ_n. The first and second propositions are proved.

Then we prove the third proposition. For definiteness, we consider an increasing convergent sequence of rectifiable curves Γ. We denote D_j a ring-like domain between curves Γ_j and Γ_{j+1}, and F_j the Whitney continuation of restriction of F onto $\partial D_j = \Gamma_j \cup \Gamma_{j+1}$, $j = 1, 2, \dots$. In addition, let F_0 be a Whitney continuation of $F|_{\Gamma_1}$ and D_0 is finite domain bounded by Γ_1. The function F satisfies the Hölder condition with certain exponent $\nu > 0$ near Γ, but we can consider without loss of generality that this condition fulfils in the whole domain D bounded by Γ. Then the first derivatives of F_j are integrable in D_j, $j = 0, 1, 2, \dots$. We introduce function F^* equaling F_j in D_j, $j = 0, 1, \dots$, and 0 in the complement of D. Then

$$\iint_{\mathbb{C}} F^*_{\bar{z}}\, dz\, d\bar{z} = \sum_{j=0}^{\infty} \iint_{D_j} (F_j)_{\bar{z}}\, dz\, d\bar{z} = -\lim_{j\to\infty} I(F, \Gamma_j).$$

By assumption, the last limit exists, that is, $F^*_{\bar{z}}$ is integrable. Hence, F^* is a Stokes integrator. Theorem is proved. □

4. Applications

4.1. The Cauchy type integral

Let $z \in \mathbb{C} \setminus \Gamma$. We fix a function $\omega_z(\zeta) \in C_0^\infty(\mathbb{C})$ such that $\omega_z(\zeta) = (\zeta - z)^{-1}$ for $\zeta \in \Gamma$, apply distribution $\int_\Gamma^{(S)} f(t) \cdot dt$ to $(2\pi i)^{-1}\omega_z$, and obtain Cauchy type integral over non-rectifiable curve Γ. Thus, the Cauchy integral in the present paper is

$$\Phi(z) = \frac{1}{2\pi i} \int_\Gamma^{(S)} \frac{f(\zeta)\, d\zeta}{\zeta - z} := \frac{1}{2\pi i} \int_\Gamma^{(S)} f(t)\omega_z(t)\, dt. \tag{8}$$

We will study its boundary properties and apply them for solving of certain boundary value problems on non-rectifiable curves.

We know that the Cauchy integral (8) exists if $f \in H_{\mathfrak{v}}(\Gamma)$ and $\mathfrak{v}(t) > 1 - \mathfrak{m}^*(\Gamma; t)$ for $t \in \Gamma'$; if Γ is non-closed arc, then there arise additional restrictions on curls at end points or factor q (see Section 3.1).

Theorem 7
Let Γ satisfy assumptions of one of Theorems 1 and 2. If

$$\mathfrak{v}(t) > 1 - \frac{1}{2}\mathfrak{m}^*(\Gamma; t) \tag{9}$$

at a point $t \in \Gamma'$, then there exist limit values $\Phi^\pm(t)$ from the left and from the right, and

$$\Phi^+(t) - \Phi^-(t) = f(t). \tag{10}$$

Under assumptions of Theorem 4, the right side of the last equality has to be replaced by fq.

Proof
Let Γ be a closed curve and F is an integrator built in the proof of Theorem 1. Then we rewrite representation (8) in the form

$$\Phi(z) = F(z) - \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\partial F}{\partial \bar{\zeta}} \frac{d\zeta d\bar{\zeta}}{\zeta - z}.$$

The integral operator

$$T : f \mapsto \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{f(\zeta) d\zeta d\bar{\zeta}}{\zeta - z}$$

is well known (see, for instance, [28]). If $f \in L^p$ has compact support and $p > 2$, then Tf is continuous in $\mathbb{C}$ function satisfying the Hölder condition with exponent $1 - 2p^{-1}$ and $(Tf)_{\bar{z}} = f$. Clearly, $F_{\bar{z}}$ is integrable near point t with any exponent $p < \mathfrak{m}^*(\Gamma; t)(1 - \mathfrak{v}(t))^{-1}$. Under restriction (9), the right side of the last inequality exceeds 2. Thus, Φ is holomorphic in $\mathbb{C} \setminus \Gamma$ and has the boundary values $\Phi^\pm(t)$ satisfying condition (10). In the case of non-closed arc the proof is analogous. □

The known properties of operator T imply that $\Phi^\pm$ belong to $\mathfrak{H}_{\mathfrak{w}}$ if

$$\mathfrak{w}(t) < 1 - \frac{2(1 - \mathfrak{v}(t))}{\mathfrak{m}^*(\Gamma; t)}. \tag{11}$$

4.2. Boundary value problems

We apply the previous result for solving of certain boundary values problems on non-rectifiable curves for analytic problems and begin our consideration from the so- called jump problem. It is well known in the case of piecewise smooth curves (see, for instance, monographs [1–4]) and consists in evaluation of holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function $\Phi(z)$ such that its limit values $\Phi^\pm(t)$ from the left and from the right exist at any point $t \in \Gamma'$ and satisfy on Γ' the boundary condition (10); here, f is a given function. If curve Γ is not closed, then we have to add certain restrictions on growth of the desired function at end points. The Cauchy type integral solves the problem for piecewise smooth curves [1–4]. For non-rectifiable curves, it was solved first in [29] without using curvilinear integration.

it was shown that it is solvable under assumption $f \in H_\nu(\Gamma)$, $\nu > \frac{1}{2}\mathrm{dm}(\Gamma)$. Then, this problem was solved for non-rectifiable curves in terms of generalized curvilinear integrals (see survey [15]). Theorem 7 implies that the jump problem is solvable under restriction (9), and consequently, for $f \in H_\nu(\Gamma)$, $\nu > 1 - \frac{1}{2}\mathfrak{m}^*(\Gamma)$. The last result is obtained in another manner by Katz in an unpublished paper entitled *Weighted Marcinkiewicz exponents with applications*.

Clearly, these solvability conditions sharpen the aforecited results of the paper [29]. The relation (11) enables us to describe uniqueness of solutions of the jump problem in terms of the aforementioned Dolzhenko theorem.

Analogously, we can apply Theorem 7 for solving of the jump problem on arcs and other boundary value problems. For instance, let Γ be a non-rectifiable arc with end points $t_{1,2}$. We consider a problem on evaluation of holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function $\Phi(z)$ such that its limit values $\Phi^{\pm}(t)$ from the left and from the right exist at any point $t \in \Gamma'$ and satisfy the boundary value condition $\Phi^+(t) + \Phi^-(t) = f(t), t \in \Gamma'$ and end condition $\Phi(z) = O(|z - t_j|^\beta), \beta < 1, z \to t_j, j = 1, 2$ (the sum problem). Our considerations immediately yields

Theorem 8

Let Γ be a directed Jordan arc of null plane, $f \in H_\nu(\Gamma)$, $\nu > 1 - \frac{1}{2}\mathfrak{m}(\Gamma; t_{1,2})$ and $\nu > 1 - \frac{1}{2}\mathfrak{m}^*(\Gamma; t)$ for $t \in \Gamma'$. Then function

$$\Phi(z) = \frac{1}{2\pi i q(z)} \int_\Gamma^{(S)} \frac{f(\zeta)q(\zeta)d\zeta}{\zeta - z}$$

is a solution of the sum problem.

Note that we do not restrict here the end curls of Γ.

Clearly, the Stokes integrations allow us to solve the Riemann boundary value problem and other boundary value problems of the Riemann–Hilbert type on non-rectifiable curves.

5. Examples

Here, we cite certain examples illustrating the relations between Marcikievicz exponents and Minkowskii dimension.

Example 1

Let us divide segment $I = \{x + iy : 0 \le x \le 1, y = 0\}$ into parts $I_n := \{2^{-n} \le x \le 2^{-n+1}, y = 0\}, n = 1, 2, \ldots$, fix values $\alpha \ge 1$ and $\beta \ge 1$, and divide each of segments I_n on $2^{[n\beta]}$ equal parts; here, $[\cdot]$ stands for the entire part. We denote the points of division x_{nj} in decreasing order. Let $p_{nj} := \{x + iy : x_{nj} - C_n \le x \le x_{nj}, 0 \le y \le 2^{-n}\}$. Here $C_n = \frac{1}{2}a_n^\alpha$, where a_n is the distance between neighboring points of division of segment I_n, that is, $a_n = 2^{-n-[n\beta]}$. Then rectangles p_{nj} are mutually disjoint. We put $D_1 := S \bigcup \left(\bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{[n\beta]}} p_{nj} \right)$, where $S = \{x + iy : 0 \le x \le 1, -1 \le y \le 0\}$ and denote boundary of domain D_1 by $\Gamma_1(\alpha, \beta)$. Clearly, this curve consists of an infinite number of vertical and horizontal segments condensing to the origin. The summary length of the vertical segments is infinite.

This curve was constructed first in [30]. As shown there, $\mathrm{dm}(\Gamma_1(\alpha, \beta)) = \dfrac{2\beta}{\beta + 1}$, that is, it does not depend on α.

If point $t \in \Gamma_1(\alpha, \beta)$ does not coincide with the origin, then its sufficiently small neighborhood contains a rectifiable arc. Hence, $\mathfrak{m}^+(\Gamma_1(\alpha, \beta); t)$ and $\mathfrak{m}^-(\Gamma_1(\alpha, \beta)); t)$ are equal to 1. For $t = 0$, we have $\mathfrak{m}^+(\Gamma_1(\alpha, \beta); 0) = 1 - \dfrac{\beta - 1}{(\beta + 1)\alpha}$ and $\mathfrak{m}^-(\Gamma_1(\alpha, \beta); 0) = \dfrac{2}{\beta + 1}$. By virtue of Lemma 1, we have $\mathfrak{m}^+(\Gamma_1(\alpha, \beta)) = 1 - \frac{\beta - 1}{\alpha(\beta + 1)} > 2 - \mathrm{dm}(\Gamma_1(\alpha, \beta))$, $\mathfrak{m}^-(\Gamma_1(\alpha, \beta)) = \frac{2}{\beta + 1} = 2 - \mathrm{dm}(\Gamma_1(\alpha, \beta))$. Finally, $\mathfrak{m}^*(\Gamma_1(\alpha, \beta); t) = 1$ for $t \neq 0$, and

$$\mathfrak{m}^*(\Gamma_1(\alpha, \beta); 0) = 1 - \frac{\beta - 1}{\alpha(\beta + 1)} > 2 - \mathrm{dm}(\Gamma_1(\alpha, \beta)).$$

Example 2

The curve $\Gamma_1(\alpha, \beta)$ consists of two vertical segments $[0, -i]$ and $[-i + 1, 1]$, horizontal segment $[-i, -i + 1]$, and arc $\Gamma_2(\alpha, \beta)$ bounding domain D_1 from the top. We put $\Gamma_3(\alpha, \beta) := I \cup \Gamma_2(\alpha, \beta)$, where I is an arbitrarily fixed rectilinear segment with end point 0 and without another common point with $\Gamma_2(\alpha, \beta)$. Clearly, $\mathrm{dm}(\Gamma_2(\alpha, \beta)) = \mathrm{dm}(\Gamma_3(\alpha, \beta)) = \dfrac{2\beta}{\beta + 1}$. Sufficiently small neighborhood of any inner point $t \neq 0$ of each of these arcs is a rectifiable arc. Hence, $\mathfrak{m}^+(\Gamma_2(\alpha, \beta); t) = \mathfrak{m}^-(\Gamma_2(\alpha, \beta)); t) = \mathfrak{m}^+(\Gamma_3(\alpha, \beta); t) = \mathfrak{m}^-(\Gamma_3(\alpha, \beta)); t) = 1$. The left and right Marcinkiewicz exponents are undefined at end points of an arc; thus, $\mathfrak{m}^*(\Gamma_2(\alpha, \beta); 0)$ is undefined and

$$\mathfrak{m}(\Gamma_2(\alpha, \beta); 0) = 2 - \mathrm{dm}(\Gamma_2(\alpha, \beta)) = \frac{2}{\beta + 1}.$$

For arc $\Gamma_3(\alpha, \beta)$, the origin is an inner point and

$$\mathfrak{m}^*(\Gamma_3(\alpha, \beta); 0) = 1 - \frac{\beta - 1}{\alpha(\beta + 1)} > 2 - \mathrm{dm}(\Gamma_3(\alpha, \beta)).$$

Thus, there exists plane curves Γ such that $\mathfrak{m}^*(\Gamma) > 2 - \mathrm{dm}(\Gamma)$.

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