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SPECTRAL ORDER FOR POSITIVE CONTRACTIONS IN OPERATOR ALGEBRAS

Let $\mathcal{A}$ be a C^*-algebra. By $E(\mathcal{A})$ we will denote the set of all positive elements of $\mathcal{A}$ with norm less than one. The standard operator order, $\leqslant$, on $E(\mathcal{A})$ is given by the positive cone $\{a^*a \mid a \in \mathcal{A}\}$. This order structure is a lattice if and only if $\mathcal{A}$ is abelian. In contrast to this, if $\mathcal{M}$ is a von Neumann algebra, then its projection structure $(P(\mathcal{M}), \leqslant)$ forms even a complete lattice. (Projection is a self-adjoint idempotent.) There is a natural extension of the standard order from projections to positive contractions that organizes $E(\mathcal{M})$ into complete lattice as well. This, so called, spectral order, $\preceq$, is defined as follows. We denote by $(e_\lambda(x))_\lambda$ the spectral resolution of $x \in E(\mathcal{M})$. For $a, b \in E(\mathcal{M})$, $a \preceq b$ if, for each $\lambda \geqslant 0$, we have

$$e_\lambda(b) \leqslant e_\lambda(a)\,.$$

Spectral order plays an important role in foundation of quantum theory, where the structure $E(\mathcal{M})$ embodies unsharp propositions about the quantum system. Besides, the spectral order is widely used in matrix analysis. There is therefore a natural interest in describing maps preserving the spectral order. Let us call a bijection $\varphi : E(\mathcal{M}) \to E(\mathcal{M})$ a spectral automorphism if it preserves spectral order in both directions. Spectral automorphisms were first described for the set of all positive contractions acting on a Hilbert space H by Molnar and Šemrl [3]. For unbounded operators the preservers of the spectral order were characterized in [4]. We have described spectral automorphims for general von

Neumann algebras under mild condition of preserving multiples of projections. We say that a bijection φ acting on $E(\mathcal{M})$ preserves multiples of projections if for each projection p there is a projection q such that φ maps the set $\{\lambda p \,|\, \lambda \in [0,1]\}$ onto the set $\{\lambda q \,|\, \lambda \in [0,1]\}$.

Having a lattice isomorphism $\tau : P(\mathcal{M}) \to \mathcal{P}(\mathcal{M})$, we define the spectral automorphism φ_τ by condition

$$e_\lambda(\varphi_\tau(a)) = \tau(e_\lambda(a)), \qquad \lambda \geqslant 0,\, a \in E(\mathcal{M}).$$

Theorem 1. *Let φ be a spectral automorphism of $E(\mathcal{M})$ preserving multiples of projections. Then there is a unique lattice isomorphism $\tau : P(\mathcal{M}) \to \mathcal{P}(\mathcal{M})$ and a unique strictly increasing bijection $f : [0,1] \to [0,1]$ such that*

$$\varphi(a) = \varphi_\tau(f(a)), \qquad a \in E(\mathcal{M}).$$

It was shown in [5] that any nonfactorial von Neumamm algebra admits a spectral automorphism that does not preserve multiples of projections. An open question remains whether assumption on preserving multiples of projections in the previous theorem can be relaxed for factors.

There is a natural geometric orthogonality relation defined on $E(\mathcal{M})$. We say that two positive contractions a and b are orthogonal if $ab = 0$. Spectral automorphisms preserving orthogonality of elements in both directions were first studied in [1]. Such spectral automorphisms are called spectral orthoautomorphisms. They are analogous to orthoisomorphisms of projection lattices (bijections preserving orthogonality of projections in both directions). First author showed that any spectral orthoautomorphism preserving scalar operators (multiples of the unit) can be described in the same way as in Theorem 1 with τ being specified to be an orthoisomorphism. Orthoisomorphisms for projection lattices of von Neumann algebras are described by famous Dye's theorem as being Jordan $*$-isomorphisms. Jordan $*$-isomorphism is a linear bijection preserving squares of elements and the star operation. This theorem was extended to a more general context of AW^*-algebras by the first author in [2]. AW^*-algebras are algebraic counterpart of von Neumann algebras and were introduced by Kaplansky. Let us remark that spectral order can be defined for AW^*-algebras in the same way as for von Neumann algebras thanks to the fact that any self-adjoint element in an AW^*-algebra admits a spectral resolution. Generalizing our methods for von Neumann algebras and using [2], we obtain the following result.

Theorem 2. *Let $\mathcal{A}$ be an AW^*-algebra without Type I_2 direct summand. Let*

$\varphi : E(\mathcal{A}) \to \mathcal{E}(\mathcal{A})$ be a spectral orthoautomorphism mapping the set $\{\lambda 1 \,|\, \lambda \in [0,1]\}$ onto itself. Then there is a unique Jordan $*$-isomorphism J acting on $\mathcal{A}$ and a unique strictly increasing bijection $f : [0,1] \to [0,1]$ such that

$$\varphi(a) = J(f(a)) \qquad \text{for all } a \in \mathcal{A}\,.$$

R E F E R E N C E S

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