MULTIDIMENSIONAL NONLINEAR ION-ACOUSTIC WAVES IN A PLASMA IN VIEW OF RELATIVISTIC EFFECTS

V. Yu. Belashov*

Kazan Federal University, Kazan, 420008 Russia

**e-mail: vybelashov@yahoo.com*

Received September 2, 2016

On the basis of the model of the Kadomtsev-Petviashvili (KP) equation, the structure and dynamics of ion-acoustic waves in an unmagnetized plasma including a case of weakly relativistic collisional plasma, when the high energy flows of particles, as observed in the magnetospheric plasma, should be taken into account, are studied analytically and numerically. It is shown that if the velocity of plasma particles approaches the speed of light, the relativistic effects start to strongly influence the wave characteristics, such as its phase velocity, amplitude and the characteristic wavelength, at propagation of the two-dimensional solitary ion-acoustic wave. The results obtained can have application in study of the nonlinear wave processes in the magnetosphere and also in the laser and astrophysical plasma.

1. INTRODUCTION. STATEMENT OF A PROBLEM

Study of nonlinear wave processes in real media with dispersion, despite of essential progress taking place in the last years in this research field (see, for example, [Belashov and Vladimirov, 2005; McKerr et al., 2014, 2016] and numerous references in these works) still remains actual. In particular, it concerns dynamics of fluctuations in cases, when in the medium (magnetosphere, compact astrophysical systems, for example, white dwarfs, laser plasma [Haas, 2014; Shukla et al., 1984]) there are the high energy flows of particles take place essentially changing such parameters of propagating wave structures as their phase velocity, amplitude and characteristic length. Rather big number of works is devoted to investigations of such relativistic effects, for example, [Canuto and Ventura, 1977; Giamarchi, 2003; McKerr et al., 2016; Passoni et al., 2010; Rahman and Ali, 2014; Shukla and Eliasson, 2008], however practically all of them consider only one-dimensional (1D) approach. In particular, in Refs. [McKerr et al., 2014, 2016] and in earlier Refs. [Washimi and Taniuti, 1966; Das and Paul, 1985] the relativistic effects for the ion-acoustic branch of oscillations have been investigated in an 1D plasma. The investigations [Nejon, 1987; Taniuti and Wei, 1968] are perhaps some exceptions, however in these papers were studied only some extreme cases.

The purpose of our work consists in investigating relativistic effects in dynamics of ion-acoustic multidimensional nonlinear wave structures in electron-ionic plasma, that is especially important in astrophysical applications and in the physics of magnetosphere. To solve this problem, in principle, we could start from general set of hydrodynamic equations for the relativistic case (see, for example, [Elsässer and Popel, 1997]), however since we is interested by the effects which are displayed at the relativistic velocities in comparison with the nonrelativistic case, it would be more logical to consider, at first, the nonrelativistic approach and further, introducing the relativistic factor (by analogy with Ref. [Nejon, 1987]), to consider its influence on the time-space characteristics of multidimensional nonlinear ion-acoustic wave. We shall undertake this approach further.

In the absence of the magnetic field and for the negligible ion temperature, the equations of motion and continuity for ions take form [Belashov, 1997]

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\,\mathbf{v} = -\frac{e}{M}\nabla\varphi, \quad \frac{\partial n_i}{\partial t} + \mathrm{div}\,(n_i\mathbf{v}) = 0, \tag{1}$$

where M is a mass of ion, φ is an electric potential. A comparison with the equations in generalized variables for ideal gas in neglect a dissipation,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\,\mathbf{v} + \frac{c^2}{\rho}\nabla\rho = 0, \quad \frac{\partial \rho}{\partial t} + \nabla\,(\rho\mathbf{v}) = 0,$$

where ρ and $c = c(\rho)$ have a sense of the generalized "density" and velocity of "sound", respectively, at density ρ in neglect a dispersion [Karpman, 1973], shows that in this case the role of ρ and c plays the ion density n_i and the ion-acoustic velocity $c_s = (T_e/M)^{1/2}$; the dispersion "length" is defined by $\delta^2 = D^2/2 = T_e/8\pi e^2 n_0$, where n_0 is the unperturbed electron density. The electrons in the ion-acoustic wave are Boltzmann distributed,

$$n_e = n_0 \exp(e\varphi/T_e), \tag{2}$$

And the densities of the ions and electrons are related to the electric potential φ via Poisson's equation

$$\Delta\varphi = 4\pi e\,(n_e - n_i). \tag{3}$$

The dispersion equation for the set (1)–(3) is written as [Danilov and Petviashvili, 1983]

$$\omega^2 = c_s^2 k^2 \big/ \left(1 + D^2 k^2\right). \tag{4}$$

Following further to the techniques developed in Ref. [Belashov and Vladimirov, 2005] and proceeding from the presented reasons, consider the basic equation describing the dynamics of the ion-acoustic waves in a unmagnetized collisional plasma, and discuss its possible solutions, and after that, introducing the relativistic factor, consider the effects related to particles moving with the velocities which are rather close to the speed of light.

2. BASIC EQUATIONS. NONRELATIVISTIC APPROXIMATION

Consider the wave packet propagating in the direction close to the x-axis. We assume that the wave numbers of its harmonics are small satisfying the inequalities,

$$kD \ll 1, \quad k_x^2 \gg k_\perp^2, v' \ll c_s, \tag{5}$$

where v' is the x-component of the ion velocity. It is well known that the weakly dispersive [see the first inequality of (5)] ion-acoustic wave steepens in the direction of its propagation, therefore, at some time moment the second inequality of (5) “switches on.” Conditions (5) enable us to reduce the dispersion relation (4) to the form

$$\omega \approx c_s k_x \left(1 + k_\perp^2 / 2k_x^2 - \delta^2 k_x^2\right),$$

therefore, limiting ourselves in the nonlinear expansion by the terms quadratic in the wave amplitude and considering the solution in the form of a propagating wave $u = u(t, x - c_s t, r_\perp)$, and applying the procedure described in Ref. [Belashov and Vladimirov, 2005], we obtain the nonlinear equation

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + c_s \frac{\partial v}{\partial x} - c_s \delta^2 \frac{\partial^3 v}{\partial x^3} + v v_x\right) = \pm \frac{c_s}{2} \Delta_\perp v, \tag{6}$$

that, after the homothetic transformation and in the reference frame moving along the x-axis with the velocity c_s, coincides with the Kadomtsev-Petviashvili (KP) equation in its standard form [Belashov, 1997]:

$$\partial_t u + \alpha u \partial_x u + \beta \partial_x^3 u = \kappa \int_{-\infty}^{x} \Delta_\perp u dx, \quad \Delta_\perp = \partial_y^2 + \partial_z^2, \tag{7}$$

where $\kappa = 1$ is related to the positive dispersion, and $\kappa = -1$ corresponds to the negative dispersion, respectively, and the factors at the equation terms for the case of ion sound are [Karpman, 1973]:

$$\alpha = \frac{3}{2} c_s / n_i, \quad \kappa = c_s / 2, \quad \beta = c_s D^2 / 2,$$
$$c_s = \sqrt{T_e / M}, \quad D^2 = T / 4\pi n_0 e^2.$$

Generally speaking, for the ion-acoustic wave the sign on the right-hand side of (6) is positive so that the dispersion is negative, $\kappa = -1$ in (7). However, for other modes there are cases when the dispersion is positive, i.e. there is the ‘minus’ sign on the right-hand side of (6). The term $c_s\partial_x V$ describes the wave propagation along the *x*-axis with the “sound” velocity, and other terms responsible for dispersion, nonlinearity, and diffraction describe slow changes of the acoustic field on the background of the wave motion with the velocity c_s. Such acoustic waves is mainly characteristic for isotropic media (e.g., an unmagnetized plasma), but sometimes it can be observed in anisotropic media as well. Thus, if the characteristic frequency of the ion-acoustic wave packet significantly exceeds the ion-cyclotron frequency in a magnetized plasma, ω_{Hi}, the plasma anisotropy can be neglected and therefore (6) can be reduced to the KP equation (7) [Danilov and Petviashvili, 1983]. In the opposite case, when $\omega << \omega_{Hi}$ the anisotropy cannot be neglected. In this case the additional term $\omega_{Hi}[\mathbf{i}, \mathbf{v}]$ ($\mathbf{i}$ is the dimensionless vector of the *x*-axis) appears in the right-hand side of the equation of motion (1), and sign of the second term in the dispersion equation changes to minus. In this case we also have the equation of the KP class but with right-hand side of form $\mu\Delta_\perp\partial_x u$ [Zakharov and Kuznetsov, 1974]. Upper sign in this equality, as in the equation (6), corresponds to the case of negative dispersion, and lower sign corresponds to positive one.

In Ref. [Belashov, 1997] the isotropic case of (7) for the ion-acoustic waves in un unmagnetized plasma was considered. Further, in Ref. [Belashov and Vladimirov, 2005] the results were generalized for more wide spectrum of nonlinear systems. Using the approaches proposed in these Refs., the numerical simulation based on the specially developed high-accuracy methods [Belashov, 1991; Belashov and Vladimirov, 2005], was spent for the initial condition in form of the solitary pulse of form $u(0, x, y) = u_0 \exp[-(x/l_x + y/l_y)^2 / L^2]$ with the periodic boundary conditions. Figure shows an example of the numerical results obtained for the two-dimensional (2D) ($\Delta_\perp = \partial_y^2$) equation (7).

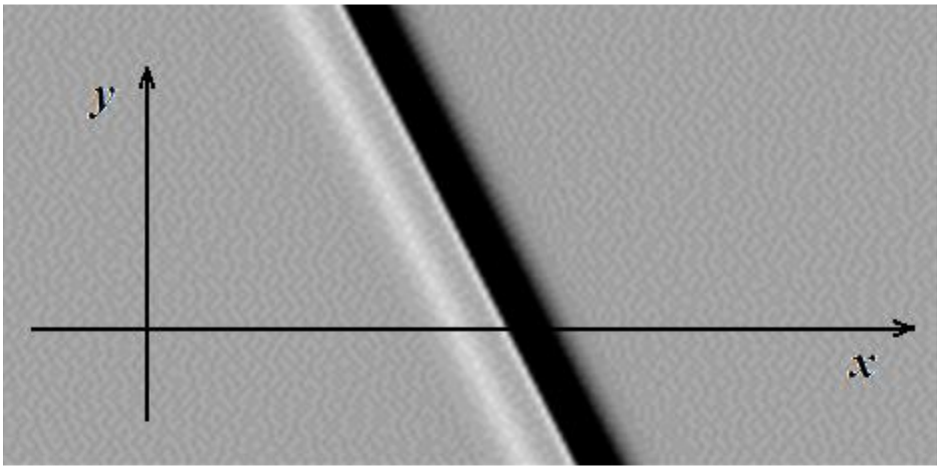

Figure. General view of 2D solution of the KP equation (7) at $\alpha = 6,\ \beta = 1$.

We can see that as a result of the evolution of the 2D acoustic perturbation $u(0, x, y)$ in an isotropic plasma, the 1D soliton of KP equation is formed. The form of the soliton corresponds to that obtained analytically for the negative dispersion in Ref. [Kadomtsev and Petviashvili, 1970] by employing the Krylov-Bogolyubov method and in Ref. [Zakharov et al., 1980] by using the inverse scattering transform (IST) method. It was shown in our numerical simulation that for sufficiently large *t*, the soliton velocity and the first three integrals of the 2D KP equation are conserved within the limits of the accuracy of the numerical simulation, namely:

$$\partial_t I_1 = \partial_t \iint u\,dxdy = 0 + O(\tau^2, h_{x.y}^4),$$

$$\partial_t I_2 = \partial_t \iint u^2 dxdy = 0 + O(\tau^2, h_{x.y}^4),$$

$$\partial_t I_3 = \partial_t \iint \left[\frac{1}{2}(\partial_x u)^2 - \tfrac{1}{2}\kappa \left(\int_{-\infty}^{x} \partial_y u\,dx \right)^2 - u^3 \right] \times$$
$$\times\ dxdy = 0 + O(\tau^2, h_{x,y}^4),$$

where τ and $h_{x,y}$ are the steps on the time and space grids, respectively (the last two integrals has a sense of momentum and energy of medium described by the KP equation). It confirms our earlier obtained results [Belashov, 1997] and the fact of existence of 2D ion-acoustic soliton in such physical system. Consider now the problem of influence of relativistic effects on the evolution of ion-acoustic wave.

3. WEAKLY RELATIVISTIC EFFECTS

As we already demonstrated, the ion-acoustic waves in a plasma can be described by the KP equation (7). However, if the velocity of plasma particles approaches the speed of light, the relativistic effects start to strongly influence the wave characteristics (such as its phase velocity, amplitude and the characteristic wavelength) in the propagation of the two-dimensional solitary ion-acoustic wave.

For the 2D ion-acoustic solitary waves in a weakly relativistic collisional plasma, the KP equation in form (7) taking into account the relativistic factor *u/c* can be obtained [Nejon, 1987] using the reduced perturbation method [Taniuti and Wei, 1968]. We can rewrite it in the following form:

$$\frac{\partial \Phi_1}{\partial \tau} + \alpha(\vartheta_1)\Phi_1 \frac{\partial \Phi_1}{\partial \xi} + \frac{1}{2}\beta(\vartheta_1)\frac{\partial^3 \Phi_1}{\partial \xi^3} = = -\frac{1}{2}\int_{-\infty}^{\xi} \frac{\partial^2 \Phi_1}{\partial \eta^2} d\xi, \quad (8)$$

where $\Phi_1 = \vartheta_1^{1/2} u_1$ is a small perturbation of the electrostatic potential $\Phi = \varepsilon\Phi_1 + \varepsilon^2\Phi_2 + \ldots$, ε is the small expansion parameter; u_1 is the perturbation of the plasma particle velocity ($u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots$),

$$\alpha(\vartheta_1) = \beta(\vartheta_1)\left(1 - \frac{\vartheta_2}{\vartheta_1^{3/2}}\right), \quad \beta(\vartheta_1) = \vartheta_1^{-1/2},$$
$$\vartheta_1 = 1 + \frac{3}{2}\left(\frac{u_0}{c}\right)^2, \quad \vartheta_2 = \frac{3}{2}\frac{u_0}{c^2}. \tag{9}$$

Equation (8) is written in the reference frame moving along the x-axis: $\xi = \varepsilon^{1/2}(x - \lambda t)$, $\eta = \varepsilon y$, $\tau = \varepsilon^{3/2} t$, where λ is the phase velocity. Note that the coefficient at nonlinear term $\alpha > 0$, since $\vartheta_1 >> \vartheta_2$. In this case we can obtain the stationary solution as the propagating in the system solitary wave. Introducing the new variable $\zeta = k_x\xi + k_y\eta - \omega\tau$ and substituting it into (8), write the solution in the form of a 2D wave

$$\Phi_1 = \Phi_0 \mathrm{sech}^2\left\{\frac{1}{W}\left(\xi + \frac{k_y}{k_x}\eta - \frac{\omega}{k_x}\tau\right)\right\}, \tag{10}$$

where the amplitude Φ_0 and the characteristic size W are defined by the expressions

$$\Phi_0 = \frac{3\delta}{\alpha(\vartheta_1)}, \quad W = \left[\frac{2\beta(\vartheta_1)}{\delta}\right]^{1/2}, \tag{11}$$

and $\delta = \omega/k_x - \frac{1}{2}(k_y/k_x)^2$, and the boundary conditions are $\Phi_1 \to 0$, $\partial_\zeta^n \Phi_1 \to 0$ for n = 1, 2 and $|\zeta| \to \infty$. The dispersion law for these waves is given by

$$\omega = k_x\left[2\beta(\vartheta_1)k_x^2 + k_y^2/2k_x^2\right].$$

We see from (9) that the factors at the nonlinear term as well at the dispersion term are defined by the relativistic factor ϑ_1, equations (11) shows the dependence of the amplitude and the characteristic length of the 2D ion-acoustic soliton of the KP equation on the weakly relativistic effects. Comparison of the results following from (9)–(11) with those for the three extreme cases considered in Refs. [Кадомцев и Петвиашвили, 1970; Nejon, 1987; Washimi and Taniuti, 1966; Das and Paul, 1985] [16,105,106,207] is given in the Table. Here

$$s = (\omega/k) \cong v_0 + \vartheta_1^{-1/2}\left(1 - \frac{1}{2}k^2\right),$$

v_0 is the velocity of the ion flow (if v_0 ~ 0 and the relativistic effects are ignored, we have

$s \cong 1-\frac{1}{2}k^2$). One can see that the obtained results include all three extreme cases too, but they are essentially more general, because describe the influence of the relativistic effects on such parameters as the amplitude, characteristic size and the phase velocity of the 2D solitary wave, which, in its turn, are defined by the dependencies of the factors at the nonlinear term as well at the dispersion term of KP equation: $\alpha(\vartheta_1)$ and $\beta(\vartheta_1)$, respectively, on the particles' velocity *u*. One can also see that the dependencies of the amplitude and characteristic size of the wave on relativistic factor essentially differ for the 2D and 1D cases (compare the second and the last column in the Table): in the expressions for Φ_0 and *W* we have the parameters δ and *s*, respectively.

Table

Comparison of the results obtained with the results for three extreme cases

Parameter	Obtained results	Results of refs. [Kadomtsev and Petviashvili, 1970; Nejon, 1987; Washimi and Taniuti, 1966; Das and Paul, 1985]		
		$u_0/c = 0, \eta \neq 0$	$u_0/c = 0, \eta = 0$	$u_0/c \neq 0, \ \eta = 0$
λ	$u_0 + \vartheta_1^{-1/2}$	1	1	$u_0 + \vartheta_1^{-1/2}$
α	$\vartheta_1^{-1/2}(1-\vartheta_2/\vartheta_1^{3/2})$	1	1	$\vartheta_1^{-1/2}(1-\vartheta_2/\vartheta_1^{3/2})$
β	$\vartheta_1^{-1/2}$	1	1	$\vartheta_1^{-1/2}$
Φ_0	$3\delta\vartheta_1^{1/2}(1-\vartheta_2/\vartheta_1^{3/2})$	3δ	3s	$3s\, 3s\vartheta_1^{1/2}(1-\vartheta_2/\vartheta_1^{3/2})$
W	$\vartheta_1^{-1/4}(2/\delta)^{1/2}$	$(2/\delta)^{1/2}$	$(2/s)^{1/2}$	$\vartheta_1^{-1/4}(2/s)^{1/2}$

4. CONCLUSION

So, in this paper on the basis of model of the KP equation the structure and dynamics of the ion-acoustic waves in an unmagnetized plasma, including a case of collisional weakly relativistic plasma, when the high energy flows of particles should be taken into account, were studied analytically and numerically. In particular, when kinetic energy of ions $M u_0^2/2$ at $u_0/c \cong 0.1$ reaches values $\cong 4.7$ MeV, the 2D weakly relativistic ion-acoustic solitary waves will describe a motion of the high energy protons with velocity which is significant in comparison with speed of light that is observed in the magnetospheric plasma (trapping region, outer radiation belt, plasma sheet) [Vette, 1970]. We showed, that if velocity of plasma particles approaches the tenth shares of speed of light (for example, in the region of the maximum of the outer radiation belt on L-shell L = 3.1 [Krimigis and Van Allen, 1967]), the relativistic effects start to strongly influence the wave characteristics, such as its phase velocity, amplitude and the characteristic wavelength, at propagation of the 2D solitary ion-acoustic wave. Let's also note that, besides physics of nonlinear processes in the magnetosphere, the study of the relativistic nonlinear waves has also the applications in such physical systems as laser plasma [Shukla et al., 1984] and astrophysics [Canuto and Ventura, 1977; Arons, 1979; Haas, 2014; Rahman and Ali, 2014].

The work is performed according to the Russian Government Program of Competitive Growth of Kazan Federal University.

REFERENCES

- *Белашов В.Ю.* Уравнение КП и его обобщения. Теория, приложения. Магадан: СВКНИИ ДВО РАН. 162 с. 1997.
- *Данилов Ю.А., Петвиашвили В.И.* Солитоны в плазме // Итоги науки и техники. Физика плазмы. М.: ВИНИТИ, Т. 4. С. 5–47. 1983.
- *Захаров В.Е., Кузнецов Е.А.* О трехмерных солитонах // ЖЭТФ. Т. 66. Вып. 2. С. 594–597. 1974.
- *Захаров В.Е., Манаков С.В., Новиков С.П., Питаевский Л.П.* Теория солитонов: Метод обратной задачи. М.: Наука, 319 с. 1980.
- *Кадомцев Б.Б., Петвиашвили В.И.* Об устойчивости уединенных волн в слабо диспергирующих средах // ДАН СССР. Т. 192. № 4. С. 753–756. 1970.
- *Карпман В.И.* Нелинейные волны в диспергирующих средах. М.: Наука, 175 с. 1973.
- *Arons J.* Some problems of pulsar physics// Space Sci. Rev. V. 24. P. 417–510. 1979.
- *Belashov V.Yu.* The methods for numerical integration of nonlinear evolutional KP-class equations // XX Int. Conf. on Phenomena in Ionized Gases, Pisa, Italy, 1991. Contributed papers. V. 6. P. 1241–1242. 1991.
- *Belashov V.Yu.* Numerical study of dynamics of 3D ion-acoustic and FMS nonlinear waves in plasma using spectral approach // Proc. of 5th Int. School/Symp. for Space Simulation (ISSS-5), Kyoto, Japan, 1997. RASC, Kyoto Univ., Kyoto, Japan. P. 118–122. 1997.
- *Belashov V.Yu., Vladimirov S.V.* Solitary Waves in Dispersive Complex Media. Theory, Simulation, Applications. Springer-Verlag, Berlin–Heidelberg–New York–Tokyo. 305 p. 2005.
- *Canuto V., Ventura J.* Quantizing magnetic fields in astrophysics // Fundamentals of Cosmic Physics. V. 2. Gordon and Breach Science Publishers, UK. P. 203–353. 1977.
- *Das G.C., Paul S.N.* Ion-acoustic solitary waves in relativistic plasmas // Phys. Fluids. V. 28. P. 823–837. 1985.

– *Elsässer K., Popel S.* Plasma equations in general relativity // Phys. Plasmas. V. 4. P. 2348–2356. 1997.
– *Giamarchi T.* Quantum Physics in One Dimension. Oxford University Press, New York. 448 p. 2003.
– *Haas F.* Relativistic hydrodynamic equations for fully degenerate plasma / 17th Int. Congress on Plasma Physics – ICPP 2014, Lisbon, Portugal, 2014. Book of Abstracts. BAP.15. 2014.
– *Krimigis S.M., Van Allen J.A.* Geomagnetically trapped alpha particles // J. Geophys. Res. V. 72. P. 5779–5797. 1967.
– *McKerr M., Haas F., Kourakis I.* Relativistic theory for localized electrostatic excitations in degenerate electron-ion plasmas // Phys. Rev. E. V. 90. Iss. 3. 033112. 2014.
– *McKerr M., Haas F., Kourakis I.* Ion-acoustic envelope modes in a degenerate relativistic electron-ion plasma // Phys. Plasmas. V. 23. 052120. 2016.
– *Nejon Y.* A two-dimensional ion acoustic solitary wave in a weakly relativistic plasma // J. Plasma Phys. V. 38. Part 3. P. 439–444. 1987.
– *Passoni M., Bertagna L., Zani A.* Target normal sheath acceleration: theory, comparison with experiments and future perspectives // New J. Phys. V. 12. 0450122. 2010.
– *Rahman A., Ali S.* Solitary and rogue waves in Fermi-Dirac plasmas: relativistic degeneracy effects// Astrophys. Space Sci. V. 351. Iss. 1. P. 165–172. 2014.
– *Shukla P.K., Yu M.Y., Tsintsadze N.L.* Intense solitary laser pulse propagation in a plasma // Phys. Fluids. V. 27. P. 327–334. 1984.
– *Shukla P.K., Eliasson B.* Nonlinear theory for a quantum diode in a dense Fermi magnetoplasma // Phys. Rev. Lett. V. 100. 036801. 2008.
– *Taniuti T., Wei C.C.* Reductive perturbation method in nonlinear wave propagation, I // J. Phys. Soc. Japan. V. 24. P. 941–946. 1968.
– *Vette J.I.* Particles and Fields in the Magnetosphere / Ed. McCormac B.M., Reidel. 305 p. 1970.
– *Washimi H., Taniuti T.* Propagation of ion-acoustic solitary waves of small amplitude // Phys. Rev. Lett. V. 17. № 17. P. 966–971. 1966.