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Uniformization of simply connected compact Riemann surfaces by rational functions[1]

S. R. Nasyrov (Kazan)
snasyrov@kpfu.ru

We investigate the problem of describing a one-parametric family of rational functions $R(z,t)$ depending on a real parameter $t \in [0;1]$ and, for a fixed t, mapping the Riemann sphere $\widehat{\mathbb{C}}$ onto a Riemann surface $S(t)$. We assume that all critical points $a_k = a_k(t)$ of $R(z,t)$ a simple and the corresponding critical values $A_k(t) = R(a_k(t),t)$ depend smoothly on the parameter t. Besides, $R(z,t)$ has poles of order n_j at some points $b_j = b_j(t)$. Under assumption that $A_k(t)$ are known, we find a system of ODE for the critical points $a_k(t)$ and poles $b_j(t)$.

It should be noted that in [1] we solved a similar problem for the case when each surface $S(t)$ has only one point of maximal multiplicity over ∞, therefore, the mapping functions are polynomials. In [2, 3] we investigated the case of complex tori (Riemann surfaces of genus 1 over the Riemann sphere).

Thus, we consider a family of rational functions

$$R(z,t) = C_0 \int_{z_0}^{z} \frac{\prod_{k=1}^{m}(\zeta - a_k)\, d\zeta}{\prod_{j=1}^{N}(\zeta - b_j)^{n_j+1}} + C_1 \tag{1}$$

with simple critical points a_k and poles b_j of order n_j. Without loss of generality we can assume that $C_0 = 1$, because we can achieve this by an additional linear map $z \mapsto \alpha z$. The value of the constant $C_1 = R(z_0,t)$ can be easily found: e.g., if we put $z_0 = a_1$, then $C_1 = A_1$. We will also assume that the condition

$$\sum_{k=1}^{m} a_k - \sum_{j=1}^{N} (n_j+1) b_j = 0 \tag{2}$$

fulfills for all functions of the family. Actually, if $m = n := \sum_{j=1}^{N}(n_j+1)$, then it is valid because the residue of the derivative $R'(z,t)$ at the infinity equals zero. If $m \neq n$, we see that under the shift of the complex z-plane by a number α_0, the value of $\sum_{k=1}^{m} a_k - \sum_{j=1}^{N} (n_j+1)b_j$ changes by

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$(m-n)\alpha_0$; therefore, we can get validity of (2) by choosing an appropriate value of α_0.

Theorem 1. *Under the above assumptions, the family* (1) *satisfies the equation*

$$\frac{\partial R(z,t)}{\partial t} = \frac{\partial R(z,t)}{\partial z}\, h(z,t)$$

where

$$h(z,t) = \sum_{k=1}^{m} \frac{\dot{A}_k}{D_k(z-a_k)}\,,$$

$$D_k = \frac{\prod_{l\neq 1}^{m}(a_k-a_l)}{\prod_{j=1}^{N}(a_k-b_j)^{n_j+1}}\,,$$

$A_k = A_k(t)$ *being the trajectories of critical values of* $R(z,t)$.

Theorem 2. *The critical points* $a_k = a_k(t)$ *and the poles* $b_j = b_j(t)$ *of the rational functions* (1) *satisfy the system of ODE*

$$\dot{a}_k = \frac{\dot{A}_k}{D_k}\left[\sum_{l\neq k}\frac{1}{a_l-a_k} - \sum_{j=1}^{N}\frac{n_j+1}{b_j-a_k}\right] + \sum_{l\neq k}\frac{\dot{A}_l}{D_l(a_l-a_k)}\,, \tag{3}$$

$$\dot{b}_j = \sum_{k=1}^{m}\frac{\dot{A}_k}{D_k(a_k-b_j)}\,. \tag{4}$$

Theorem 2 give an approximate method to find the critical points and the poles of a rational function uniformizing a compact Riemann surface S_1 of genus 0 with simple branch-points over the plane $\mathbb{C}$. In the space of ramified coverings of the Riemann sphere we connect S_1 with a similar surface S_0 by a smooth curve $S(t)$, $0\le t\le 1$, so that $S(0)=S_0$, $S(1)=S_1$. If we know for $S(0)=S_0$ the critical points a_{j0} and the poles b_{j0} of the corresponding uniformazing rational function, then we solve the Cauchy problem for the system of ODE (3), (4) with initial data $a_j(0)=a_{j0}$, $b_j(0)=b_{j0}$, and find positions of the critical points $a_j(1)$ and the poles $b_j(1)$ for a rational function uniformizing the surface S_1. We note that while solving the Cauchy problem we obtain uniformization of the whole family $S(t)$, $0\le t\le 1$.

We can consider (3), (4) as variational formulas which connect variations of the critical values of rational functions with variations of their critical points and poles. The formulas can be used for solving extremal problems for rational functions.

Bibliography

1. Nasyrov S.R. *Determination of the polynomial uniformizing a given compact Riemann surface* // Mathematical Notes. 2012, V. 91, No 3. P. 558–567.

2. Nasyrov S.R. *One-parametric families of elliptic functions uniformizing complex tori* // Proc. of VII Petrozavodsk Int. Conf. 'Complex analysis and applications'. Petrozavodsk, Petr. Univ., 2014. P. 78–79.

3. Nasyrov S.R. *One-parametric families of multivalent functions and Riemann surfaces.* Proc. of Int. Conf. 'Modern Methods of Function Theory and Related Topics', Voronezh, Voronezh Univ., 2015. P. 83–85 (in Russian).