NATURAL MODES OF WEAKLY GUIDING OPTICAL FIBER

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Abstract – **The eigenvalue problem for generalized natural modes of an inhomogeneous optical fiber is formulated as a problem for the Helmholtz equation with Reichardt condition at infinity in the cross-sectional plane. The generalized eigenvalues of this problem are the complex propagation constants on a logarithmic Reimann surface. The original problem is reduced to a spectral problem with compact integral operator. Theorem on spectrum localization is proved, and then it is proved that the set of all eigenvalues of the original problem can only be a set of isolated points on the Reimann surface, and it also proved that each eigenvalue depends continuously on the frequency and can appear and disappear only at the boundary of the Reimann surface. The existence of the surface modes is proved. The Galerkin method for numerical calculation of the surface modes is proposed. Some results of the numerical experiments are presented.**

I. INTRODUCTION

Optical fibers are dielectric waveguides (DWs), i.e., regular dielectric rods, having various cross sectional shapes, and where generally the refractive index of the dielectric may vary in the waveguide's cross section [1]. The study of the source-free electromagnetic fields, called natural modes, that can propagate on DWs necessitates that longitudinally the rod extend to infinity. Since often DWs are not shielded, the medium surrounding the waveguide transversely forms an unbounded domain, typically taken to be free space. This fact plays an extremely important role in the mathematical analysis of natural waveguide modes, and brings into consideration a variety of possible formulations. They differ in the form of the condition imposed at infinity in the cross-sectional plane, and hence in the functional class of the natural-mode field. This also restricts the localization of the eigenvalues in the complex plane of the eigenparameter [2]. All of the known natural-mode solutions (i.e., guided modes, leaky modes, and complex modes) satisfy the Reichardt condition at infinity. The wavenumbers β may be generally considered on the appropriate logarithmic Reimann surface. For real wavenumbrs on the principal ("proper") sheet of this Reimann surface, one can reduce the Reichardt condition to either the Sommerfeld radiation condition or to the condition of exponential decay. The Reichardt condition may be considered as a generalization of the Sommerfeld radiation condition and can be applied for complex wavenumbers. This condition may also be considered as the continuation of the Sommerfeld radiation condition from a part of the real axis of the complex parameter β to the appropriate logarithmic Reimann surface.

II. STATEMENT OF THE PROBLEM

We consider the natural modes of an inhomogeneous optical fiber. Let the three-dimensional space be occupied by an isotropic source-free medium, and let the refractive index be prescribed as a positive real-valued function $n = n(x_1, x_2)$ independent of the longitudinal coordinate x_3 and equal to a constant n_∞ outside a cylinder. The axis of the cylinder is parallel to the x_3-axis, and its cross-section is a bounded domain Ω with a Lipschitz boundary Γ on the plane $R^2 = \{(x_1, x_2): -\infty < x_1, x_2 < \infty\}$. Denote by Ω_∞ the unbounded domain $\Omega_\infty = R^2 \setminus \overline{\Omega}$, and denote by n_+ the maximum of the function n in the domain Ω, where $n_+ > n_\infty$. Let the function n belongs to the space of real-valued continuous and continuously differentiable in Ω functions. By U denote the space of twice continuously differentiable in Ω and Ω_∞, continuous and continuously differentiable in $\overline{\Omega}$ and $\overline{\Omega}_\infty$ real-valued functions. The modal problem can be formulated as an eigenvalue problem for the Helmholtz equation

$$\left[\Delta + \left(k^2 n^2 - \beta^2\right)\right] u = 0, \quad x \in R^2 \setminus \Gamma. \tag{1}$$

Here $k^2 = \omega^2 \varepsilon_0 \mu_0$, ε_0, μ_0 are the free-space dielectric and magnetic constants, respectively. We consider the propagation constant β as a complex parameter and radian frequency ω as a positive parameter. We seek non-zero solutions u of set (1) in the space U. Functions u have to satisfy the conjugation conditions:

$$u^+ = u^-, \quad \frac{\partial u^+}{\partial \nu} = \frac{\partial u^-}{\partial \nu}, \quad x \in \Gamma. \tag{2}$$

Here ν is the normal vector. By Λ denote the Reimann surface of the function $\ln \chi(\beta)$, where $\chi(\beta) = \sqrt{k^2 n_\infty^2 - \beta^2}$, and by $\Lambda_0^{(1)}$ denote the principal ("proper") sheet of Λ, which is specified by the following conditions: $-\pi/2 < \arg \chi(\beta) < 3\pi/2$, $\operatorname{Im}\left(\chi(\beta)\right) \geq 0$, $\beta \in \Lambda_0^{(1)}$. We say that function u satisfies the Reichardt condition if the function u can be represented for all $|x| > R_0$ as

$$u = \sum_{l=-\infty}^{\infty} a_l H_l^{(1)}(r, \varphi) \exp(il\varphi), \tag{3}$$

where $H_l^{(1)}$ is the Hankel function of the first kind and index l, (r, φ) are the polar coordinates of the point x. The series in (1) should converge uniformly and absolutely.

Definition 1. *A nonzero function $u \in U$ is referred to as an eigenfunction (natural mode) of the problem (1)-(3) corresponding to some eigenvalues $\beta \in \Lambda$ and $\omega > 0$ if the relations of problem (1)-(3) are valid. The set of all eigenvalues of the problem (1)-(3) is called the spectrum of this problem.*

II. SPECTRUM PROPERTIES

If u is an eigenfunction of problem (1)-(3) corresponding to some eigenvalues $\beta \in \Lambda$ and $\omega > 0$, then

$$v(x) = \lambda^2 \int_\Omega \Phi(\beta; x, y) p(x) p(y) v(y) dy, \quad x \in \Omega, \tag{4}$$

where $v = up$, $p^2 = (n^2 - n_\infty^2)/(n_+^2 - n_\infty^2)$, $\lambda^2 = k^2/(n_+^2 - n_\infty^2)$, $\Phi = i/4 H_0^{(1)}\left(\chi(\beta)|x-y|\right)$. The original problem (1)-(3) is spectrally equivalent [3] to the problem (4). Let frequency ω has a fixed positive value. Rewrite problem (4) in the form of spectral problem for operator-valued function

$$A(\beta)v = 0, \quad A(\beta) = I - \lambda^2 B(\beta): L_2(\Omega) \to L_2(\Omega), \tag{5}$$

where B is the operator, defined by the right side of equation (4), I is the identical operator.

Definition 2. *A nonzero vector $v \in L_2(\Omega)$ is called an eigenvector of operator-valued function $A(\beta)$ corresponding to an eigenvalue $\beta \in \Lambda$ if the relation (5) is valid. The set of all $\beta \in \Lambda$ for which the operator $A(\beta)$ does not have the bounded inverse operator in $L_2(\Omega)$ is called the spectrum of operator-valued function $A(\beta)$. Denote by $\sigma(A) \subset \Lambda$ the spectrum of operator-valued function $A(\beta)$.*

Theorem 1. *For all $\omega > 0$ and $\beta \in \Lambda$ the operator $B(\beta)$ is compact. If ω has a fixed positive value, then the spectrum of the operator-valued function $A(\beta)$ can be only a set of isolated points on Λ, moreover on the principal sheet $\Lambda_0^{(1)}$ it can belong only the set $G = \left\{\beta \in \Lambda_0^{(1)} : kn_\infty < |\beta| < kn_+, \operatorname{Im}\beta = 0\right\}$. Each eigenvalue β of the operator-valued function $A(\beta)$ depends continuously on $\omega > 0$ and can appear and disappear only at the boundary of Λ, i.e., at $\beta = \pm kn_\infty$ and at infinity on Λ.*

This theorem was proved in [3]. The well known surface modes satisfy to propagation constants $\beta \in G$. In this case $\chi(\beta) = i\sigma(\beta)$, where $\sigma(\beta) = \sqrt{\beta^2 - k^2 n_\infty^2} > 0$. Let transverse wavenumber σ has a fixed positive value. Rewrite problem (4) in the form of usual liner spectral problem with integral compact operator

$$v = \lambda^2 B(\sigma)v, \quad B: L_2(\Omega) \to L_2(\Omega), \tag{6}$$

Definition 3. *A nonzero function $v \in L_2(\Omega)$ is called an eigenfunction of operator B corresponding to a characteristic value λ^2 if the relation (6) is valid.*

Theorem 2. *For all positive σ the following statements are valid. There exist the denumerable set of positive characteristic values λ_l^2, $l = 1, 2, \ldots$, with only cumulative point at infinity. The set of all eigenfunctions v_l, $l = 1, 2, \ldots$, can be choose as the orthonormal set. The smallest characteristic value λ_1^2 is positive and simple, corresponding eigenfunction v_1 is positive. Each eigenvalue λ_l^2, $l = 1, 2, \ldots$, depends continuously on $\sigma > 0$, and $\lambda_1^2 \to 0$, if $\sigma \to 0$.*

This theorem is proved by the methods of the spectral theory of compact integral operators. The well known fundamental mode satisfies to smallest characteristic value λ_1^2. If some values of the parameters λ^2 and σ are known, then β and ω can be calculated by evidence formulas.
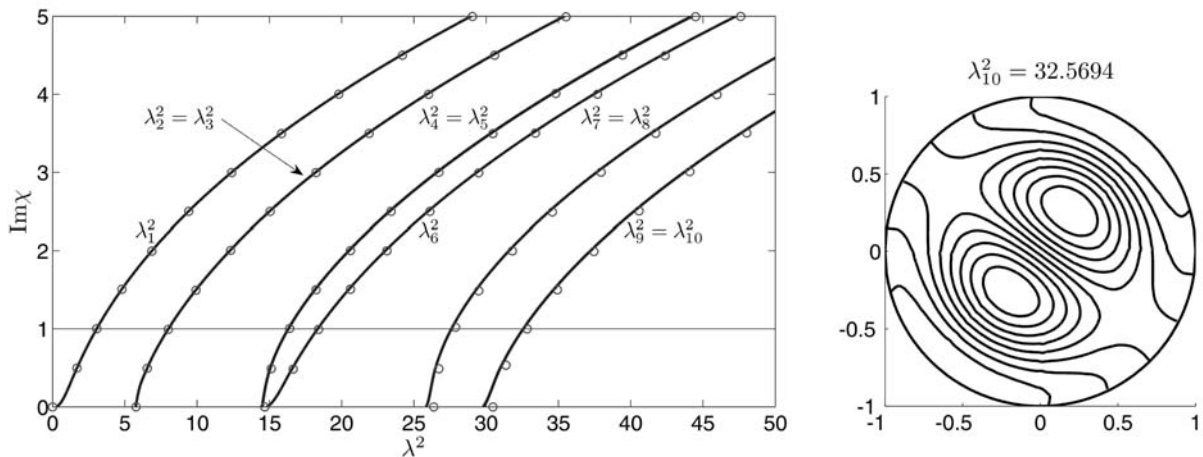

Fig. 1.   The first ten dispersion curves (on the left) for circular step-index fiber calculated by Galerkin method (plotted by sold lines) with comparison to exact solutions (marked by circles). The tenth eigenfunction v (on the right), calculated for $\sigma = \operatorname{Im} \chi = 1$.

II. GALERKIN METHOD

Consider the Galerkin method for numerical approximation of integral equation (6). We cover Ω with small triangles Δ_i and denote by Ω_n the sub-domain $\Omega_n = \bigcup_{i=1}^{n} \Delta_i \subseteq \Omega$. We seek the approximate solution v_n of equation (6) in the form of linear combination $v_n(x) = \sum_{i=1}^{n} a_i f_i(x)$, $x \in \Omega_n$, where f_i are basis functions, $f_i(x) = 1$, if $x \in \Delta_i$, $f_i(x) = 0$, if $x \notin \Delta_i$. We seek the non-zero approximate solution v_n in the space $H_n = \operatorname{span}\{f_1, \ldots, f_n\}$. The unknown coefficients a_i can be determined from the set of linear algebraic equations:

$$\sum_{i=1}^{n} a_i \left((I - \lambda^2 B) f_i, f_j \right) = 0, \quad j = 1, \ldots, n, \tag{7}$$

where $(\cdot, \cdot)$ denotes inner product in $L_2(\Omega)$. The singular Galerkin elements $\left((I - \lambda^2 B) f_i, f_i \right)$ are calculated analytically. Therefore, using Galerkin method for solving linear spectral problem for integral equation (6), we obtain finite–dimensional linear spectral problem (7), that we can rewrite in the operator form:

$$v_n = \lambda^2 B_n(\sigma) v_n, \quad B_n : H_n \to H_n, \tag{8}$$

where the operator $B_n(\sigma)$ is determined by (7). The numerical results are presented on the figures 1-3 with comparison to known exact solutions and some results of other authors.
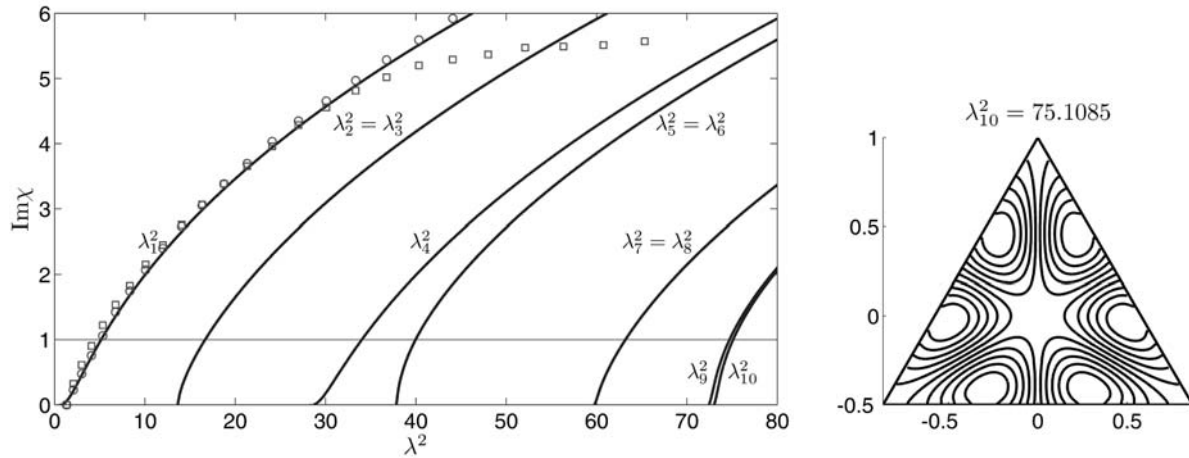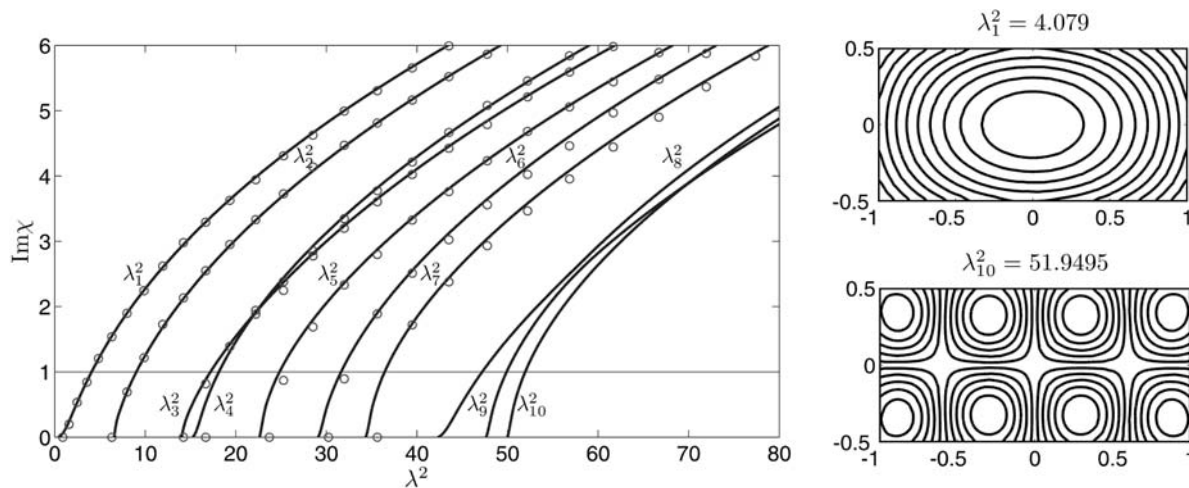

Fig. 2.   The first ten dispersion curves (on the left) for triangular step-index fiber calculated by Galerkin method (plotted by sold lines) with comparison to results of other authors (marked by circles and squares). The tenth eigenfunction v (on the right), calculated for $\sigma = \operatorname{Im}\chi = 1$.

Fig. 3.   The first ten dispersion curves (on the left) for rectangular step-index fiber calculated by Galerkin method (plotted by sold lines) with comparison to results of other authors (marked by circles). The first and the tenth eigenfunctions v (on the right), calculated for $\sigma = \operatorname{Im}\chi = 1$.

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