Transactions of A. Razmadze
Mathematical Institute
Vol. **173** (2019), 1–??

ASYMPTOTIC AND QUALITATIVE ANALYSIS OF THE MULTIDIMENSIONAL GKP AND 3-DNLS EQUATION SOLUTIONS FOR THEIR CLASSIFICATION

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Abstract. In this paper, basing on our results obtained by us earlier, we consider an approach to study of structure of possible multidimensional solutions of the Belashov-Karpman (BK) system which includes as partial cases the generalized Kadomtsev–Petviashvili (GKP) equation and the 3D derivative nonlinear Schrödinger (3-DNLS) equation. For the GKP equation with due account of the arbitrary nonlinearity exponent we study the solutions asymptotes along the direction of the wave propagation. The problem of the asymptotic behavior of the solutions of the 3-DNLS equation along the direction of the wave propagation is more simple one because we can write at once its exact solutions in the explicit form in one-dimensional approximation on the basis of the results known earlier. We also present some considerations on constructing of the phase-plane portraits in the 8-dimensional phase space for the GKP equation on the basis of the results of qualitatively analysis of the generalized equations of the KdV-class.

1. Basic Equations

In this paper we study the types and structure of possible multidimensional solitary waves forming on the low-frequency branch of oscillations in fluids and plasma which are described by the Belashov-Karpman (BK) class of equations [4]

$$\partial_t u + A(t,u)u = f, \quad f = \kappa \int\limits_{-\infty}^{x} \Delta_\perp u dx, \quad \Delta_\perp = \partial_y^2 + \partial_z^2 \tag{1}$$

which with the operator

$$A(t,u) = \alpha u \partial_x - \partial_x^2(\mu - \beta\partial_x - \delta\partial_x^2 - \gamma\partial_x^3) \tag{2}$$

turns into the generalized Kadomtsev-Petviashvili (GKP) equation [10], and in the case, when operator

$$A(t,u) = 3s|p|^2 u^2 \partial_x - \partial_x^2(i\lambda + \nu) \tag{3}$$

eq. (1) turns into the 3-dimensional derivative nonlinear Schrödinger (3-DNLS) equation where $p = (1+ie)$, and e is "an eccentricity" of the polarization ellipse of the wave [4, 10]. The upper and lower signs of $\lambda = \pm 1$ correspond to the right and left circularly polarized wave, respectively; sign of nonlinearity is accounted by coefficient $s = \mathrm{sgn}(1-p) = \pm 1$ in nonlinear term.

The sets of equations (1), (2) and (1), (3) are not completely integrable ones, and a problem of existence of multidimensional stable soliton solutions and their structure requires especial investigation. In [6] we studied the problem of stability of possible multidimensional solutions for two particular cases of the BK system mentioned above. Here, we investigate the structure of possible solutions of the sets of equations (1), (2) and (1), (3) using the methods of both qualitative and asymptotic analysis basing on the results for the generalized equations of the KdV-class obtained in [9].

Consider at first the GKP equation and then discuss the analogous problem for the 3-DNLS equation. Let us write the GKP equation in form [10]:

$$\partial_\eta(\partial_t u + \alpha u \partial_\eta u - \mu\partial_\eta^2 u + \beta\partial_\eta^3 u + \delta\partial_\eta^4 u + \gamma\partial_\eta^5 u) = \kappa\Delta_\perp u \tag{4}$$
$$\Delta_\perp = \partial_{\zeta_1}^2 + \partial_{\zeta_2}^2$$

1991 *Mathematics Subject Classification.* 34C05, 35B40, 35K40.

Key words and phrases. Multidimensional nonlinear waves; multidimensional solitons; generalized KP and 3-DNLS equations; asymptotic analysis; qualitative analysis; multidimensional phase space; classification of solutions.

where ζ_1 and ζ_2 are the transverse coordinates. At $\mu = \delta = \gamma = 0$ equation (4) is the classic KP equation which is the completely integrable Hamiltonian system and has in case $\Delta_\perp = \partial^2_{\zeta_1}$ the solutions in form of the 1-dimensional (for $\beta\kappa < 0$) or 2-dimensional (for $\beta\kappa > 0$) solitons (see [10]). The structure and the dynamics of the solutions of model (4) nonintegrable analytically in case $\delta = 0$ has been investigated in detail in [2, 7] where it was shown that at $\mu = 0$ in dependence on the signs of coefficients β, γ and κ the 2-dimensional and 3-dimensional soliton type solutions with the monotonous or oscillatory asymptotics can take place which at the presence of the "viscous-type" dissipation in the medium ($\mu > 0$) lose their symmetry and damp with evolution (see [10] for details). In [9] with use of the methods of the both asymptotic and qualitative analysis the asymptotics of the one-dimensional analogue of equation (4) were studied in detail and the sufficiently complete classification of its solutions including the solutions of the both soliton and non-soliton type were constructed.

Now, our purpose is the generalization of the results obtained in [9] with due account of the results presented in [5, 7, 8] to the multidimensional cases.

For the avoidance of the unhandiness of obtaining expressions let us consider the equation (4) in the 2-dimensional form supposing that $\Delta_\perp = \delta^2_{\zeta_1}$. Generalization of using technique and the results obtaining at that to a case $\Delta_\perp = \delta^2_{\zeta_1} + \delta^2_{\zeta_2}$ is rather trivial [3]. Let us assume that $\zeta_1 \equiv \zeta$ and take for the distinctness that $\alpha = 6$ (that can be obtained easily using the scaling transformation $u \to (6/\alpha)u$ in the equation).

Let us introduce new variables, $\bar\eta = \eta + \zeta$, $\bar\zeta = \eta - \zeta$. As a result, changing the variables η and ζ in equation (4) at first by way of $\bar\eta$ and then by way of $\bar\zeta$, we obtain the pair of one-dimensional equations:

$$\begin{aligned}
&\partial_{\bar\eta}(\partial_t u + 6u\partial_{\bar\eta} u - \mu\partial^2_{\bar\eta} u + \beta\partial^3_{\bar\eta} u + \delta\partial^4_{\bar\eta} u + \gamma\partial^5_{\bar\eta} u) = \kappa\partial^2_{\bar\eta} u,\\
&\partial_{\bar\zeta}(\partial_t u + 6u\partial_{\bar\zeta} u - \mu\partial^2_{\bar\zeta} u + \beta\partial^3_{\bar\zeta} u + \delta\partial^4_{\bar\zeta} u + \gamma\partial^5_{\bar\zeta} u) = \kappa\partial^2_{\bar\zeta} u
\end{aligned} \tag{5}$$

writing in the co-ordinates with axes $\bar\eta$ and $-\bar\zeta$ rotated relative to axes η and ζ at the angle $+45°$. Representation (5) means in fact that the starting equation (4) admit two types of the 1-dimensional solutions, $u(\bar\eta, t)$ and $u(\bar\zeta, t)$, satisfying the first and second equations of set (5), respectively. But, at this, it is necessary to have in view that "1-dimensionality" of these solutions nevertheless implicity assumes the linear dependence of each either of the new variables $\bar\eta$ and $\bar\zeta$ on both coordinates, η and ζ.

Integrating equations (5) on $\bar\eta$, $-\bar\zeta$, respectively, we obtain two equiform the generalized KdV equations

$$\begin{aligned}
&\partial_t u + (-\kappa + 6u)\partial_{\bar\eta} u - \mu\partial^2_{\bar\eta} u + \beta\partial^3_{\bar\eta} u + \delta\partial^4_{\bar\eta} u + \gamma\partial^5_{\bar\eta} u = 0,\\
&\partial_t u + (-\kappa + 6u)\partial_{\bar\zeta} u - \mu\partial^2_{\bar\zeta} u + \beta\partial^3_{\bar\zeta} u + \delta\partial^4_{\bar\zeta} u + \gamma\partial^5_{\bar\zeta} u = 0
\end{aligned} \tag{6}$$

connected with each other by way of the change of the coordinates made above. Now, passing into the coordinates moving along the corresponding axis with velocity $-k$, i.e. making the change $\eta' = \bar\eta + \kappa t$, $\zeta' = \bar\zeta + \kappa t$ in equations (6) and leaving out the strokes, let us write equations (6) in the standard form:

$$\begin{aligned}
&\partial_t u + 6u\partial_\eta u - \mu\partial^2_\eta u + \beta\partial^3_\eta u + \delta\partial^4_\eta u + \gamma\partial^5_\eta u = 0,\\
&\partial_t u + 6u\partial_\zeta u - \mu\partial^2_\zeta u + \beta\partial^3_\zeta u + \delta\partial^4_\zeta u + \gamma\partial^5_\zeta u = 0.
\end{aligned} \tag{7}$$

So, we can now conduct the analysis for only one generalized equation of the set (6), and then, fulfilling the inverse change of the variables, extend the results to the 2-dimensional solutions $u(\eta, \zeta, t)$ of equation (4) with $\Delta_\perp = \partial^2_\zeta$.

As to the 3-DNLS equation written it, at first, in the differential form:

$$\partial_\eta[\partial_t h + s\partial_\eta(|h|^2 h) - i\lambda\partial^2_\eta h - \nu\partial^2_\eta h] = \sigma\Delta_\perp h, \quad \Delta_\perp = \partial^2_{\zeta_1} + \partial^2_{\zeta_2}, \tag{8}$$

then supposing for simplification of the statement that $\Delta_\perp = \partial^2_\zeta$ (it is clear that generalization to 3-dimensional case is trivial) and introducing, by analogy with the GKP equation, new variables

$\bar{\eta} = \eta + \zeta$, $\bar{\zeta} = \eta - \zeta$, we also obtain the pair of one-dimensional equations:

$$\partial_{\bar{\eta}}[\partial_t h + s\partial_{\bar{\eta}}(|h|^2 h) - i\lambda\partial_{\bar{\eta}}^2 h - \nu\partial_{\bar{\eta}}^2 h] = \sigma\partial_{\bar{\eta}}^2 h,$$
$$\partial_{\bar{\zeta}}[\partial_t h + s\partial_{\bar{\zeta}}(|h|^2 h) - i\lambda\partial_{\bar{\eta}}^2 h - \nu\partial_{\bar{\zeta}}^2 h] = \sigma\partial_{\bar{\zeta}}^2 h$$

writtten in the coordinates with axes $\bar{\eta}$ and $-\bar{\zeta}$ rotated relative to axes η and ζ at the angle $+45°$. Further transformations give us the set

$$\begin{aligned} &\partial_t h + s\partial_{\eta'}(|h|^2 h) - i\lambda\partial_{\eta'}^2 h - \nu\partial_{\eta'}^2 h = 0,\\ &\partial_t h + s\partial_{\zeta'}(|h|^2 h) - i\lambda\partial_{\zeta'}^2 h - \nu\partial_{\zeta'}^2 h = 0 \end{aligned} \tag{9}$$

written in the coordinates $\eta' = \bar{\eta} + \sigma t$, $\zeta' = \bar{\zeta} + \sigma t$ i.e. moving along the corresponding axis with velocity $-\sigma$.

So, as in case of the GKP equation, we can conduct the analysis for only one equation of the set (9) and then, fulfilling the inverse change of the variables, extend the results to the 2-dimensional solutions $h(\eta, \zeta, t)$ of equation (8) with $\Delta_\perp = \partial_\zeta^2$.

2. Generalization of Earlier Obtained Results to Multidimensional Cases

At first, let us consider the generalization of the results obtained in [10] to the equations of the GKP class (4). Following the results presented in ref. [5], consider more general case when the equations (7) were expended by introducing of the arbitrary positive nonlinearity exponent p and, for example, first equation of the set (7) takes the form

$$\partial_t u + 6u^p\partial_\eta u - \mu\partial_\eta^2 u + \beta\partial_\eta^3 u + \delta\partial_\eta^4 u + \gamma\partial_\eta^5 u = 0 \tag{10}$$

(see [10] for detail). Remind that in case $\mu = \delta = \gamma = 0$ it is the known KdV equation if $p = 1$, and the modified KdV equation (the MKdV equation) if $p = 2$. Note also that, analogously to the 1-dimensional case, the cases, when in equation (4) with the nonlinear term $6u^p\partial_\eta u$ the nonlinearity exponent $p = 1, 2$, are interesting from physical point of view, and the applications with $p > 2$ are unknown today. But, similarly to the generalized KdV equation considered in [9], in view of that the equations with arbitrary integer $p > 0$ display very largely similar mathematical characteristics, for the elucidation of dependence of the solution parameters on the value of the nonlinearity exponent we will consider the general case for $p > 0$.

With due account of the coefficients' signs, $\mu > 0$, $\delta > 0$ (in accordance with the physical sense of proper terms - see [4, 10] for detail), assuming without a loss of generality as in [5, 7, 8] that $\gamma > 0$, $\beta = \pm 1$ and making substitution $u = Vw$ (where V is a velocity of the wave propagation relatively coordinate axis η and ζ for the first and second equation of set (7), respectively), we can generalize the results obtained in [9] for different signs of V and β to the equations (5) and, accordingly, to equation (4) with $p \geq 1$ in the following way.

1. The value of the nonlinearity exponent p defines a character of dependence $V = f(u)$, namely: for $p > 1$ such dependence for equation (4), as in the 1-dimensional case (see ref. [9]), becomes nonlinear unlike of the known linear one for $p = 1$ (for example, in case of the KP equation). Moreover, for even p the solutions of equation (4) may have both positive and negative pulse direction ($u \gtrless 0$ for either sign of V).

2. In case of the **conservative equations** of class (4) (the cases when $\mu = \delta = 0$) the solutions asymptotics are defined by the following relations:

a) for the cases $V > 0$, $\beta = -1$ and $V < 0$, $\beta = -1$ (upper and lower signs, respectively):

$$w = A_1 \exp\left\{(2\gamma)^{-1/2}\left[C^2 + \sqrt{C^4 \pm 4\gamma}\right]^{1/2}\chi\right\}, \tag{11}$$

b) for case $V < 0$, $\beta = 1$:

$$\begin{aligned} w = A_2 &\exp\{(2C^{-1}\gamma^{-1/2})^{-1}(2C^{-2}\gamma^{1/2} - 1)^{1/2}\chi\}\\ &\times\cos\{(2C^{-1}\gamma^{-1/2})^{-1}(2C^{-2}\gamma^{1/2} + 1)\chi + \Theta\}, \end{aligned} \tag{12}$$

where A_1, A_2 and Θ are the arbitrary constants, $C = |V|^{-1/4}$, $\chi = (\eta \pm \zeta + (\kappa - V)t)$ (here the signs plus and minus relate to the first and second equations of the set (5), respectively). As one can see from expressions (11), (12)[1], in the solutions $u(\eta, \zeta, t)$ of equation (4) at $\mu = \delta = 0$ the solitons with both monotonous and oscillating asymptotics can take a place dependently on the signs of V and β. (Note that at $\beta = 0$ and any value of $\gamma > 0$ the solutions of the equations (5) with $\mu = \delta = 0$ have form $w = (A_1 + A_2 C^{-1}\chi)\exp(\gamma^{-1/4}C^{-1}\chi)$ and, consequently, also describe the soliton with monotonous asymptotics [7].) Fig. 1 shows the results of numerical simulation of equation (4) for $\mu = \delta = 0$ with the initial condition $u = u_0 \exp(-x^2/l_1^2 - y^2/l_2^2)$, that confirm the results of our asymptotic analysis.

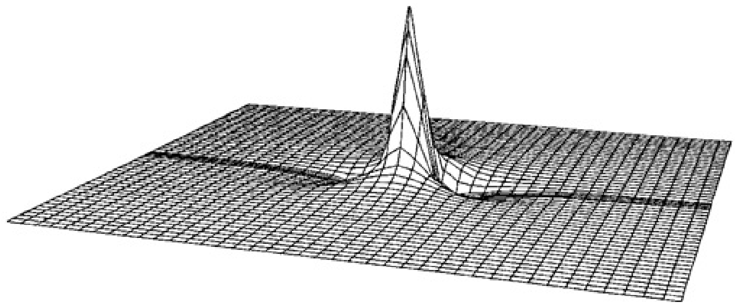

FIGURE 1. General view of a 2-dimensional soliton of eq. (4) with $\Delta_\perp = \partial_y^2$ for $\mu = \delta = 0$, $p = 1$, $\gamma = 1$, $\beta = -0.8$ at $t = 0.2$.

3. In case of the **dissipative equations** of class (4) with the instability (the cases when $\beta = \gamma = 0$) the solutions asymptotics are defined by the following relations:

a) for $\delta > (4/27)\mu^3 C^8$

$$\begin{aligned} w = A_1 \exp[(2\delta C)^{-1/3} Q_1^+ \chi] + \exp[-(16^{\delta C})^{-1/3} Q_1^+ \chi] \\ \times \Big\{ A_2 \cos[\sqrt{3}(16^{\delta C})^{-1/3} Q_1^- \chi + \Theta_1] + A_3 \sin[\sqrt{3}(16^{\delta C})^{-1/3} Q_1^- \chi + \Theta_2] \Big\}, \end{aligned} \tag{13}$$

b) for $\delta = (4/27)\mu^3 C^8$

$$w = A_1 \exp[(\delta C/4)^{-1/3}\chi] + A_2(1 + A_3\chi)\exp[-(2\delta C)^{-1/3}\chi], \tag{14}$$

c) for $\delta < (4/27)\mu^3 C^8$

$$\begin{aligned} w &= A_1 \exp[(\delta C/4)^{-1/3}\,\mathrm{Re}(Q^\pm)\chi] \\ &+ A_2 \exp\Big\{ -(2\delta C)^{-1/3}\chi\big[\mathrm{Re}(Q^\pm) - \sqrt{3}|\,\mathrm{Im}(Q^\pm)|\big]\Big\} \\ &+ A_3 \exp\Big\{ -(2\delta C)^{-1/3}\chi\big[\mathrm{Re}(Q^\pm) + \sqrt{3}|\,\mathrm{Im}(Q^\pm)|\big]\Big\} \end{aligned} \tag{15}$$

where A_1, A_2, A_3, Θ_1 and Θ_2 are the arbitrary constants, $Q_1^\pm = Q^+ \pm Q^-$,

$$Q^\pm = \Big[1 \pm \sqrt{1 - 4\mu^3 C^8/27\delta}\Big]^{1/3}$$

and $Q^\pm$ is real in the cases (a) and (b) and complex in case (c).

It is easy to see from formulae (13) – (15) that the solutions $u(\eta, \zeta, t)$ of equation (4) have the oscillating asymptotics in case (a) and the exponential ones in the cases (b) and (c). Fig. 2 shows the numerical solutions of equation (4) corresponding to the cases (c) and (a), respectively, obtained for the initial condition $u = u_0 \exp(-x^2/l_1^2 - y^2/l_2^2)$.

4. As to the proper transformation of the phase portraits and "binding" them for the 2-dimensional equation, as here, the fact that in case $\mu = \delta = 0$ the phase space is 8-dimensional, and in case $\beta = \gamma = 0$ it is 6-dimensional, we are obliged to the results obtained in ref. [5] binding the characteristics of each singular point of each equation of set (5) accordingly in the 8-dimensional and 6-dimensional

[1] The cases of another correlation of signs of V and β are not realized (see ref. [5]).

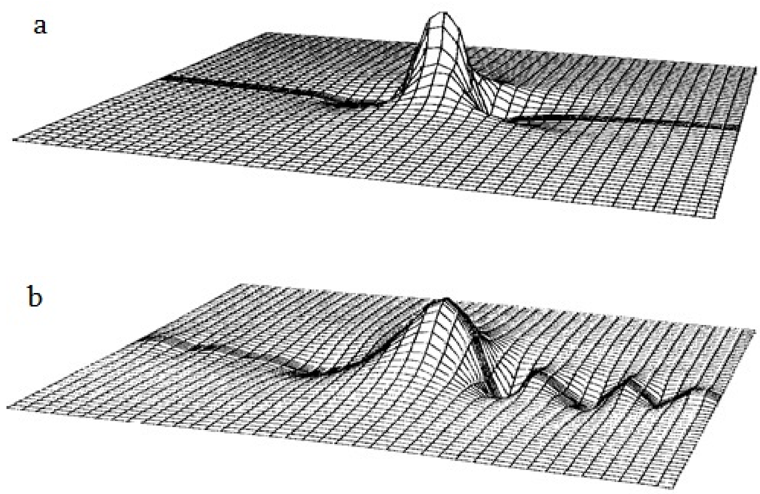

FIGURE 2. General view of a 2-dimensional soliton described by eq. (4) with $\Delta_\perp = \delta_y^2$ for $\beta = \gamma = 0$, $V > 0$ at $t = 0.3$: (a) $\mu = 1$, $\delta = 1 \times 10^{-6}$ [$\delta \leq (4/27)\mu^3 C^8$-case (c)]; (b) $\mu = 1$, $\delta = 1$ [$\delta > (4/27)\mu^3 C^8$-case (a)].

spaces. Moreover, the type of singularities in either of 4-dimensional or 3-dimensional subspaces (see ref. [5]) under the inverse transform of the coordinates, $\eta = (\bar\eta + \bar\zeta)/2$, $\zeta = (\bar\eta - \bar\zeta)/2$, will not be changed, and only those parameters of the phase portraits change which correspond for the solutions of the same class to changing of such parameters as the amplitude, the fronts steepness, frequency of the oscillations etc.

Now, let us make some our observations concerning the 3-DNLS equation (8) with $\Delta_\perp = \partial_\zeta^2$. Because, as it was shown in [10], equation (8) may be represented in form of set (9) , and, as it is known from [4, 11], exact solution of the 1-dimensional DNLS equation may be represented in form

$$h(x,t) = (A/2)^{1/2}[\exp(-Ax) + i\exp(Ax)]\exp(-iA^2t)\cosh^{-2}(2Ax), \tag{16}$$

where A is the amplitude of the wave (see [4, 10] for detail), we can fulfilling the inverse change of the variables, $\eta = (\bar\eta+\bar\zeta)/2$, $\zeta = (\bar\eta-\bar\zeta)/2$ and extending solution (16) to the 2-dimensional case (equation (8) with $\Delta_\perp = \partial_\zeta^2$) write at once for $\nu = 0$

$$h(\eta,\zeta,t) = (A/2)^{1/2}[\exp(-A\chi) + i\exp(A\chi)]\exp(-iA^2t)\cosh^{-2}(2A\chi), \tag{17}$$

where, as for the GKP equation, $\chi = (\eta \pm \zeta + (\sigma - V)t)$, and V is a velocity of the wave propagation relatively coordinate axis η and ζ for the first and second equations of set (9), respectively. Fig. 3 shows a character of solution for the first equation of set (9) with $\nu = 0$.

The dependence of the form of solution on dissipation and its dynamical characteristics for $\nu > 0$ have considered in [4, 10] in detail.

We think that there is no need to discuss here the problem of the qualitative analysis of solutions of the 3-DNLS equation, because unlike the GKP equation (4) and the corresponding set (7) the exact solution of either equation of set (9) is known, and there is no need to construct any special classification of its solutions in phase space.

3. Concluding Remarks

In conclusion, note that in this chapter for the GKP equation we have considered the special cases when $\mu = \delta = 0$ and $\beta = \gamma = 0$ in equation (4), and for other values of the coefficients more complicated wave structures resulting from the presence of all considered effects in the whole may be observed. So, the results obtained numerically in [1] (see also [4, 10]) show that for β, μ, $\delta \neq 0$

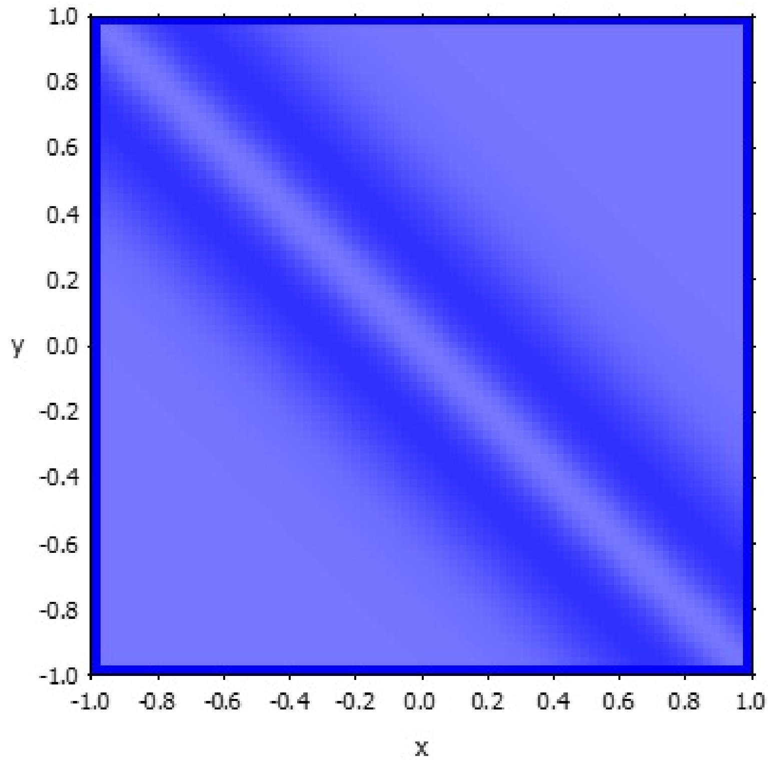

FIGURE 3. General view of solution $|h^2|$ of the first equation of set (9) for $A = 1$, $t = 0$.

at the time evolution in the presence of the Gaussian random fluctuations of the wave field for the harmonic initial conditions and the initial conditions in form of the solitary pulse the stable wave structures of the soliton type can be formed too. Furthermore, the stable soliton structures may be formed also at $\gamma \neq 0$. However, the analytical study of such cases is highly complicate, though the approach considered above can be also used. Note also that the results obtained in [5] and presented here for the GKP equation may be highly useful when studying the solutions and interpreting the multidimensional phase portraits of more complicated multidimensional model equations (see, for example, [8]).

Acknowledgements. The work is performed according to the Russian Government Program of Competitive Growth of Kazan Federal University. This work was also supported by the Shota Rustaveli National Science Foundation (SRNF), grant no. FR17 252.

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(Received ?.??.20??)

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