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Analysis of the Eigenmode Spectra of Dielectric Waveguides

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Abstract—Localization and discreteness of the eigenmode spectra are examined for cylindrical dielectric waveguides with a small discontinuity in the refractive index. A projective method for calculating the propagation constants, based on the use of simple-layer potentials, is suggested and analyzed.

This paper is devoted to an analysis of the nonlinear spectral problem for the Helmholtz equation on a plane whose solutions are the surface and outgoing eigenmodes of dielectric waveguides of arbitrary cross section with a small discontinuity in the refractive index. The problem is reduced to a spectral problem for a Fredholm holomorphic operator-valued function. The reduction is based on a representation of the solutions of the original problem as simple layer potentials. The equivalence of these problems is analyzed. The localization domains of the spectrum are found. By using the results of [1], it is proved that the spectrum can only consist of isolated points. To solve the problem numerically, Galerkin's system with a trigonometric basis is constructed. The zeros of the determinant of the system's matrix are taken as approximate eigenvalues. To justify the method, the results of [2] are employed. An analogous approach has been previously applied to problems of the eigenmodes of slot and microstrip lines [3], eigenmodes of open resonators [4, 5], and surface eigenmodes of dielectric waveguides [6].

1. Let the three-dimensional space $\mathbb{R}^3 = \{(x, y, z): -\infty < x, y, z < \infty\}$ be occupied by an isotropic medium without sources, and let the refractive index be prescribed as a real, piecewise constant function n independent of the coordinate z, equal to n_1 inside a cylinder, and $n_2 < n_1$ outside the cylinder. The axis of the cylinder is parallel to the z-axis, and its cross section is a domain S_1 bounded by a twice continuously differentiable contour Γ. Let us find the eigenmodes of the form $e^{i(\beta z - \omega t)}$ propagating along the z-axis, where the propagation constant β is an unknown complex parameter, and $\omega > 0$ is a given frequency of electromagnetic oscillations. Assuming that closeness between the refractive indexes of the waveguide and the environment are close, we reduce the problem (see [7, Section 2]) to determination of the values of β for which there exist nontrivial solutions to the Helmholtz equation

$$\Delta u + \chi_j^2 u = 0, \quad (x, y) \in S_j, \quad j = 1, 2, \quad S_2 = \mathbb{R}^2 \backslash \bar{S}_1, \tag{1}$$

satisfying the conjugation conditions

$$u^+ = u^-, \quad \partial u^+/\partial \nu = \partial u^-/\partial \nu, \quad (x, y) \in \Gamma, \tag{2}$$

and an appropriate condition at infinity. Here, $\chi_j = \sqrt{k_0^2 n_j^2 - \beta^2}$, $k_0^2 = \omega^2 \varepsilon_0 \mu_0$, ε_0 is the electric constant, and μ_0 is the magnetic constant.

Following [8] (see also [9]), we assume that u satisfies the partiality condition at infinity, i.e., at sufficiently large r, it can be represented as

$$u(r, \varphi) = \sum_{n=-\infty}^{\infty} \alpha_n H_n^{(1)}(\chi_2 r) e^{in\varphi}, \quad x = r\cos\varphi, \quad y = r\sin\varphi, \tag{3}$$

where $H_n^{(1)}$ is the nth order Hankel function of the first kind.

We seek nontrivial solutions to problem (1)–(3) in the classes of continuous and continuously differentiable functions on $\bar{S}_1$ and $\bar{S}_2$ and twice continuously differentiable functions on S_1 and S_2.

We assume that the points of the spectrum of β belong to the set Λ defined as the intersection of the Riemann surfaces of the functions $\ln\chi_j(\beta)$, $j = 1, 2$. By Λ_0, we denote the intersection of the principal (physical) sheets of these surfaces. We also define $\Lambda_j = \{\beta \in \Lambda_0: \operatorname{Im}\chi_j < 0\}$, $j = 1, 2$ and $G = \{\beta \in \Lambda_0: \operatorname{Im}\chi_2 > 0, \operatorname{Im}\beta = 0, k_0n_2 < |\beta| < k_0n_1\}$.

The following theorem can be proved by analogy with [10, p. 133]:

Theorem 1. *Within Λ_0, the points of the spectrum of problem* (1)–(3) *can lie only in $G \cup \Lambda_2$.*

Note that real values of $\beta \in G$ correspond to surface waves (u decays exponentially as $r \longrightarrow \infty$), and complex $\beta \in \Lambda_2$ correspond to outgoing waves (u grows exponentially as $r \longrightarrow \infty$). This theorem generalizes the results of [11] on the localization of the eigenmode spectra of waveguides with circular cross sections, which were obtained by analyzing the characteristic equation with the use of separation of variables.

2. Let us now reduce the original problem to a spectral problem for an operator-valued function. Consider the functions

$$\Phi_j(\beta; M, M_0) = \frac{i}{4\pi}H_0^{(1)}(\chi_j r_{MM_0}), \quad \beta \in \Lambda, \quad j = 1, 2, \tag{4}$$

where $r_{MM_0} = \sqrt{(x-x_0)^2 + (y-y_0)^2}$, $M = (x, y)$, and $M_0 = (x_0, y_0)$. By applying Graf's sum theorem (see, for example, [12, p. 201]), it can be readily demonstrated that $\Phi_2(\beta; M, M_0)$ satisfies condition (3) at any $\beta \in \Lambda$ and $M_0 \in \mathbb{R}^2$.

We seek solutions to problem (1)–(3) as simple-layer potentials,

$$u(M) = \int_\Gamma \Phi_j(\beta; M, M_0)\varphi_j(M_0)dl_{M_0}, \quad M \in S_j, \tag{5}$$

with densities φ_j belonging to the space $C^{0,\alpha}$ of Hölder continuous functions. For any $\beta \in \Lambda$ and $\varphi_j \in C^{0,\alpha}$, the function u defined by (5) satisfies the desired smoothness conditions, equation (1), and condition (3). Using boundary conditions (2) and the limit properties of simple-layer potentials, we obtain a nonlinear spectral problem for the system of integral equations

$$\int_\Gamma [\Phi_1(\beta; M, M_0)\varphi_1(M_0) - \Phi_2(\beta; M, M_0)\varphi_2(M_0)]dl_{M_0} = 0, \quad M \in \Gamma, \tag{6}$$

$$\begin{aligned} &\frac{1}{2}[\varphi_1(M) + \varphi_2(M)] + \int_\Gamma \Big(\frac{\partial\Phi_1}{\partial\nu_M}(\beta; M, M_0)\varphi_1(M_0) \\ &- \frac{\partial\Phi_2}{\partial\nu_M}(\beta; M, M_0)\varphi_2(M_0)\Big)dl_{M_0} = 0, \quad M \in \Gamma. \end{aligned} \tag{7}$$

Let the contour Γ be defined parametrically: $r = r(t)$, $t \in [0, 2\pi]$. Changing to the integration variable t and isolating the logarithmic singularities of the kernels $\Phi_j(M, M_0)$, we transform system (6), (7) into

$$Sx^{(1)} + R^{(1,1)}(\beta)x^{(1)} + R^{(1,2)}(\beta)x^{(2)} = 0, \quad t \in [0, 2\pi], \tag{8}$$

$$x^{(2)} + R^{(2,1)}(\beta)x^{(1)} + R^{(2,2)}(\beta)x^{(2)} = 0, \quad t \in [0, 2\pi]. \tag{9}$$

Here,

$$Sx^{(1)} = -\frac{1}{2\pi}\int_0^{2\pi} \ln\left|\sin\frac{t-t_0}{2}\right|x^{(1)}(t_0)dt_0, \quad t \in [0, 2\pi],$$

$$R^{(i,j)}(\beta)x^{(j)} = \frac{1}{2\pi}\int_0^{2\pi} h^{(i,j)}(\beta; t, t_0)x^{(j)}(t_0)dt_0, \quad t \in [0, 2\pi],$$

$$x^{(1)}(t_0) = [\varphi_1(M_0) - \varphi_2(M_0)]|r'(t_0)|, \quad x^{(2)}(t_0) = \varphi_1(M_0) + \varphi_2(M_0),$$

$$h^{(1,1)}(\beta; t, t_0) = 2\pi[G^{(1,1)}(\beta; t, t_0) + G^{(1,2)}(\beta; t, t_0)],$$

$$h^{(1,2)}(\beta; t, t_0) = 2\pi[G^{(1,1)}(\beta; t, t_0) - G^{(1,2)}(\beta; t, t_0)]|r'(t_0)|,$$

$$h^{(2,1)}(\beta; t, t_0) = 4\pi[G^{(2,1)}(\beta; t, t_0) + G^{(2,2)}(\beta; t, t_0)],$$

$$h^{(2,2)}(\beta; t, t_0) = 4\pi[G^{(2,1)}(\beta; t, t_0) - G^{(2,2)}(\beta; t, t_0)]|r'(t_0)|,$$

$$G^{(1,j)}(\beta; t, t_0) = \Phi_j(\beta; M, M_0) + \frac{1}{2\pi}\ln\left|\sin\frac{t-t_0}{2}\right|,$$

$$G^{(2,j)}(\beta; t, t_0) = \frac{\partial}{\partial \nu_M}\Phi_j(\beta; M, M_0), \quad M = M(t), \quad M_0 = M_0(t_0).$$

It is well known (see, for example, [13, p. 10]) that the continuous linear operator S: $C^{0,\alpha} \longrightarrow C^{1,\alpha}$ ($C^{1,\alpha}$ is the space of Hölder continuously differentiable functions) is continuously invertible, and the inverse operator S^{-1}: $C^{1,\alpha} \longrightarrow C^{0,\alpha}$ is given by the formula

$$S^{-1}(y; t) = \frac{c_0(y)}{\ln 2} + 2\sum_{k=-\infty}^{\infty}|k|c_k(y)e^{ikt}, \quad y \in C^{1,\alpha}, \tag{10}$$

where

$$c_k(y) = \frac{1}{2\pi}\int_0^{2\pi} y(t_0)e^{-ikt_0}dt_0$$

are the Fourier coefficients of the function y. It is clear that the operators $R^{(2,1)}(\beta)$, $R^{(2,2)}(\beta)$: $C^{0,\alpha} \longrightarrow C^{0,\alpha}$, $R^{(1,1)}(\beta)$, and $R^{(1,2)}(\beta)$: $C^{0,\alpha} \longrightarrow C^{1,\alpha}$ are completely continuous at any $\beta \in \Lambda$. Therefore, system (8), (9) is equivalent to the operator equation

$$A(\beta)y \equiv [I + R(\beta)]y = 0, \tag{11}$$

where $y = (y^{(1)}, y^{(2)})$, $y^{(1)} = Sx^{(1)} \in C^{1,\alpha}$, $y^{(2)} = x^{(2)} \in C^{0,\alpha}$; the completely continuous operator R acting in the Banach space $H = C^{1,\alpha} \times C^{0,\alpha}$ is defined by the equation

$$Ry = (R^{(1,1)}S^{-1}y^{(1)} + R^{(1,2)}y^{(2)},\ R^{(2,1)}S^{-1}y^{(1)} + R^{(2,2)}y^{(2)}); \tag{12}$$

and I is the identity operator.

By the complete continuity of $R(\beta)$, the operator $A(\beta)$ is a Fredholm operator at any $\beta \in \Lambda$. Using the well-known properties of the Hankel functions (e.g., see [12]), it can be readily verified that $h^{(i,j)}(\beta; t, t_0)$ are analytic functions on Λ at any point $(t, t_0) \in [0, 2\pi][0, 2\pi]$. This implies (see [3, p. 71]) that the operator-valued function $A(\beta)$ is holomorphic on Λ. Therefore, problem (11) is a spectral problem for a Fredholm holomorphic operator-valued function.

3. Let us now analyze the equivalence of problems (1)–(3) and (11). To do this, consider the following four problems: find the values of $\beta \in \Lambda$ for which there exist nontrivial, continuous in S_i, and twice continuously differentiable on $\bar{S}_i$ solutions to the Helmholtz equation

$$\Delta u + \chi_j^2 u = 0, \quad M \in S_i, \quad i, j = 1, 2,$$

that satisfy the homogeneous Dirichlet boundary conditions on Γ and, in the case of $M \in S_2$, the partiality condition. We denote the interior and exterior problems by $D_1^{(j)}$ and $D_2^{(j)}$, respectively. The sets of values of $\beta \in \Lambda$ that admit nontrivial solutions to $D_i^{(j)}$ are denoted by $\sigma(D_i^{(j)})$, $i, j = 1, 2$. It is well known that the sets $\sigma(D_1^{(j)})$ can only consist of isolated points lying on the imaginary axis or the interval $(-k_0 n_j, k_0 n_j)$ of the real axis. A similar result for $\sigma(D_2^{(j)})$ is stated by the following lemma.

Lemma 1. *The sets* $\sigma(D_2^{(j)})$ $(j = 1, 2)$ *can only consist of isolated points. Moreover, within* Λ_0, *the points of the spectrum* $\sigma(D_2^{(j)})$ *can lie only on* Λ_j.

This lemma is validated by invoking results presented in [10, 14].

Lemma 2. *For* $\beta \in \Lambda\backslash(\bigcup_{i=1,2}\sigma(D_i^{(j)}))$, *any solution to problem* (1)–(3) *on* S_j *can be represented by* (5) *as a simple-layer potential.*

Proof. It is clear that if there exists a nontrivial solution u to problem (1)–(3) for some $\beta \in \Lambda$, then u is not strictly zero on the contour Γ, and the trace of u on Γ belongs to $C^{1,\alpha}$. Define $f = u$ for $M \in \Gamma$. The function u satisfies the inhomogeneous problem $D_i^{(j)}$ inside the domain S_j. Seeking a solution to problem (1)–(3) on S_j as a simple-layer potential of the form of (5) with density $\varphi_j \in C^{0,\alpha}$, we derive the equation

$$A_j(\beta)x \equiv Sx + R_j(\beta)x = f, \quad t \in [0, 2\pi], \quad j = 1, 2, \tag{13}$$

where

$$R_j(\beta)x = \frac{1}{2\pi}\int_0^{2\pi} h_j(\beta;\, t, t_0)x(t_0)dt_0, \quad t \in [0, 2\pi],$$

$$h_j(\beta;\, t, t_0) = \Phi_j(\beta;\, M, M_0) + \frac{1}{2\pi}\ln\left|\sin\frac{t-t_0}{2}\right|,$$

$$x(t) = \varphi(M)|r'(t)| \in C^{0,\alpha}, \quad f(t) \in C^{1,\alpha}.$$

The corresponding homogeneous equation has only the trivial solution for $\beta \in \Lambda\backslash(\bigcup_{i=1,2}\sigma(D_i^{(j)}))$. Consequently, since $A(\beta)$: $C^{0,\alpha} \longrightarrow C^{1,\alpha}$ is a Fredholm operator, the inhomogeneous equation has a solution for any right hand side. Therefore, in the domain S_j, a nontrivial solution to problem (1)–(3) can be expressed as simple layer potential (5) with density $\varphi_j \in C^{0,\alpha}$. The lemma is proved.

The following lemma can be proved by analogy with Theorem 2 in [15].

Lemma 3. *If the potential u defined by* (5) *with kernel* Φ_j *vanishes in* S_j *for some* $\beta \in \Lambda\backslash\sigma(D_{3-j}^{(j)})$ $(j = 1, 2)$, *then its density* φ_j *is strictly zero on* Γ.

Lemmas 2 and 3 entail the following theorem.

Theorem 2. *Let problem* (11) *have a nontrivial solution for some* $\beta \in \Lambda\backslash(\sigma(D_2^{(1)}) \cup \sigma(D_1^{(2)}))$. *Then, there also exists a nontrivial solution to problem* (1)–(3) *for this* β. *Let problem* (11) *have a nontrivial solution for some* $\beta \in \Lambda\backslash(\bigcup_{i,j=1,2}\sigma(D_i^{(j)}))$. *Then, there also exists a nontrivial solution to problem* (11) *for this* β.

Thus, it is demonstrated that problems (1)–(3) and (11) are equivalent on Λ, except for the discrete (by Lemma 1) set of points $\bigcup_{i,j=1,2}\sigma(D_i^{(j)})$. Note that problems (1)–(3) and (11) are completely equivalent on G (see [6]).

Theorem 3. *The spectrum of problem* (1)–(3) *can consist of isolated points only.*

The validity of this theorem follows from Theorems 1 and 2 (see [1]).

4. Let us now describe a method for the numerical solution of problem (11). We seek an approximate solution $y_n = (y_n^{(1)}, y_n^{(2)})$ to equation (11) in the form

$$y_n^{(j)}(t) = \sum_{k=-n}^{n}\alpha_k^{(j)}e^{ikt}, \quad n \in N, \quad j = 1, 2.$$

The coefficients $\alpha_k^{(j)}$ are determined by the Galerkin method:

$$\int_0^{2\pi}(Ay_n)^{(k)}(t)e^{-ijt}dt = 0, \quad j = -n, \dots, n, \quad k = 1, 2. \tag{14}$$

By (10), we have

$$S^{-1}(y_n^{(1)};\, t) = \frac{\alpha_0^{(1)}}{\ln 2} + 2\sum_{k=-n}^{n}|k|\alpha_k^{(1)}e^{ikt},$$

Therefore, equations (14) are equivalent to the following system of linear algebraic equations:

$$\alpha_j^{(1)} + \sum_{k=-n}^{n} h_{jk}^{(1,1)}(\beta)d_j\alpha_k^{(1)} + \sum_{k=-n}^{n} h_{jk}^{(1,2)}(\beta)\alpha_k^{(2)} = 0, \quad j = -n, \dots, n, \tag{15}$$

$$\alpha_j^{(2)} + \sum_{k=-n}^{n} h_{jk}^{(2,1)}(\beta)d_j\alpha_k^{(1)} + \sum_{k=-n}^{n} h_{jk}^{(2,2)}(\beta)\alpha_k^{(2)} = 0, \quad j = -n, \dots, n. \tag{16}$$

Here, $d_j = \{\ln^{-1}2 \text{ for } j = 0, 2|j| \text{ for } j \neq 0\}$,

$$h_{jk}^{(l,m)}(\beta) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} h^{(l,m)}(\beta; t, t_0)e^{-ijt}e^{ikt_0}dtdt_0.$$

Let H_n^T be the set of all trigonometric polynomials of order not higher than n. Denote by H_n the subspace of H spanned by the elements $(y_n^{(1)}, y_n^{(2)})$, where $y_n^{(1)}, y_n^{(2)} \in H_n^T$. Define the projection operator p_n: $H \longrightarrow H_n$ as

$$p_n y = (\Phi_n y^{(1)}, \Phi_n y^{(2)}), \quad y = (y^{(1)}, y^{(2)}) \in H, \tag{17}$$

where Φ_n is the Fourier operator,

$$\Phi_n(\varphi; t) = \sum_{k=-n}^{n} c_k(\varphi)e^{ikt}.$$

The system of linear algebraic equations (15), (16) is equivalent to the linear operator equation

$$A_n(\beta)y_n \equiv p_n A(\beta)y_n \equiv [I + p_n R(\beta)]y_n \equiv [I + R_n(\beta)]y_n = 0. \tag{18}$$

Here, A_n: $H_n \longrightarrow H_n$, and I is the unit operator in the space H_n.

Denote by $\sigma(A_n)$ the set of singular points of the operator $A_n(\beta)$. We seek approximations β_n of the propagation constants β as the singular points of $A_n(\beta)$, i.e., as the zeros of the determinant of the matrix of system (15), (16). The convergence of the method described here is established by the following theorem, which follows from Theorem 1 in [2]:

Theorem 4. *The set $\sigma(A_n)$ consists of isolated points. Let the point β_0 belong to $\sigma(A)$, the set of singular points of the operator $A(\beta)$. Then, there exists such a sequence $\{\beta_n\}$, $\beta_n \in \sigma(A_n)$ that $\beta_n \longrightarrow \beta_0$, $n \longrightarrow \infty$. If there exists a sequence $\{\beta_n\}$, where $\beta_n \in \sigma(A_n)$, $\beta_n \longrightarrow \beta_0 \in \Lambda$ and $n \longrightarrow \infty$, then $\beta_0 \in \sigma(A)$.*

Numerical experiments have demonstrated that this method is highly accurate and requires relatively small computing resources. For example, to calculate the propagation constants of the basic surface waves for dielectric waveguides with various cross-sectional contours up to the fourth significant digit, it was sufficient to take n not greater than three [16, 6].

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