

Forced Oscillations of the Elastic Strip with a Longitudinal Crack

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Abstract

The problem of forced oscillations of an elastic strip containing a longitudinal crack of a finite length is considered. The diffraction problem is reduced to the system of paired integral equations. The system of integral equations is reduced to the system of linear algebraic equations by using the Galerkin method. Singularities of integrand functions, through which coefficients of the system matrix are calculated, are determined.

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1 Introduction

Problems of diffraction by cracks in layers arise in various applications. For example, such problems arise in non-destructive testing of pipes. Similar problems often are reduced to the solving of the hyper singular integral equations. The detailed review of these methods is carried out in [1]. In our work, the method which reduces the initial problem to a system of the pair integral equations defined on an infinite interval is offered.

The obtained system of the pair integral equations is reduced by the Galerkin method to system of linear algebraic equations. The Hermitian functions are chosen as basic functions.

Boundary conditions in our work represent the set of displacements on border. It corresponds the enclosed oscillations to a layer wall. Note that often boundary conditions can be conditions of the third type. Such conditions arise in problems of diffraction of an elastic wave by the layers of various structure both for a normal incidence [2], and for the incidence under any angle [3], [4].

Coefficients of a matrix of system are expressed through integrals of functions which contain singularities. Singularities of each integrand function are marked out.

2 Statement of the problem

The Cartesian coordinate is chosen in such way that walls of the strip coincide with straight lines $y = 0$ and $y = H_2$. The straight line $y = H_1$ conditionally splits the strip of density ρ , longitudinal velocity v_P and transverse velocity v_S into two parts. The crack's location is on this straight line at $x \in (0, L)$. It is assumed that the crack edges oscillate freely. A material of the strip is assumed to be homogeneous and isotropic. A source of the oscillations is assumed to be located on the upper boundary of the layer at $y = 0$: $u_x(x, 0) = u_x^0(x)$ and $u_y(x, 0) = u_y^0(x)$. The lower boundary is assumed to be a free boundary: $\sigma_y(x, H_2) = 0$, $\tau(x, H_2) = 0$. It is required to find a scattered field.

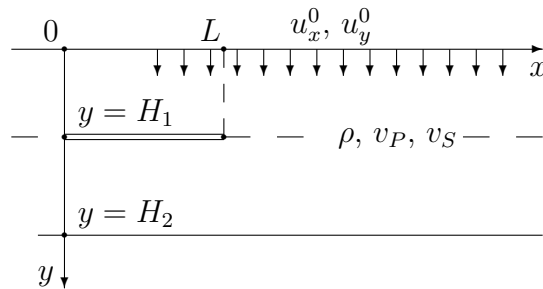


Figure 1. The elastic strip with a longitudinal crack.

Mathematical formulation of the problem of diffraction of the elastic wave by a longitudinal crack located inside the strip is the following: it is required to find a solution to the system of equations from the dynamic theory of elasticity satisfying the boundary conditions and the condition at the crack. The system of equations from the dynamic theory of elasticity has the form

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} + \rho \omega^2 u_x = 0, \quad \frac{\partial \tau}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \rho \omega^2 u_y = 0,$$

$$\sigma_x = (\lambda + 2\mu) \frac{\partial u_x}{\partial x} + \lambda \frac{\partial u_y}{\partial y}, \quad \sigma_y = \lambda \frac{\partial u_x}{\partial x} + (\lambda + 2\mu) \frac{\partial u_y}{\partial y}, \quad (1)$$

$$\tau = \mu \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right).$$

The boundary conditions at the strip edges have the form

$$\begin{aligned} u_x(x, 0) = u_x^0(x), \quad u_y(x, 0) = u_y^0(x), \\ \sigma_y(x, H_2) = 0, \quad \tau(x, H_2) = 0, \end{aligned} \quad (2)$$

the boundary conditions at the crack have the form

$$\sigma_y(x, H_1) = 0, \quad \tau(x, H_1) = 0, \quad x \in (0, L), \quad (3)$$

and the the boundary conditions on the outside of the crack at $x \notin (0, L)$ in the following form:

$$\begin{aligned} u_x(x, H_1 - 0) = u_x(x, H_1 + 0), \\ u_y(x, H_1 - 0) = u_y(x, H_1 + 0), \\ \sigma_y(x, H_1 - 0) = \sigma_y(x, H_1 + 0), \\ \tau(x, H_1 - 0) = \tau(x, H_1 + 0). \end{aligned} \quad (4)$$

3 Reducing the system to the system of paired integral equations

After the Fourier transformation (or with the account for dependence of the functions to find on the x coordinate), the general solution to system (1) takes the form [5]:

$$u_x(\xi, y) = -e^{iy\gamma_1} \xi A_1(\xi) + e^{-iy\gamma_1} \xi B_1(\xi) + e^{iy\gamma_2} \gamma_2 C_1(\xi) + e^{-iy\gamma_2} \gamma_2 D_1(\xi),$$

$$u_y(\xi, y) = e^{iy\gamma_1} \gamma_1 A_1(\xi) + e^{-iy\gamma_1} \gamma_1 B_1(\xi) + e^{iy\gamma_2} \xi C_1(\xi) - e^{-iy\gamma_2} \xi D_1(\xi),$$

$$\begin{aligned} \sigma_y(\xi, y) = i \left(e^{iy\gamma_1} (\rho\omega^2 - 2\mu\xi^2) A_1(\xi) - e^{-iy\gamma_1} (\rho\omega^2 - 2\mu\xi^2) B_1(\xi) + \right. \\ \left. + 2e^{iy\gamma_2} \mu\xi\gamma_2 C_1(\xi) + 2e^{-iy\gamma_2} \mu\xi\gamma_2 D_1(\xi) \right), \end{aligned}$$

$$\begin{aligned} \tau(\xi, y) = -i \left(2e^{iy\gamma_1} \mu\xi\gamma_1 A_1(\xi) + 2e^{-iy\gamma_1} \mu\xi\gamma_1 B_1(\xi) - \right. \\ \left. - e^{iy\gamma_2} (\rho\omega^2 - 2\mu\xi^2) C_1(\xi) + e^{-iy\gamma_2} (\rho\omega^2 - 2\mu\xi^2) D_1(\xi) \right), \end{aligned}$$

where $\gamma_j = \gamma_j(\xi) = \sqrt{k_j^2 - \xi^2}$, $j = 1, 2$ (k_j is a wave number) and $A(\xi)$, $B(\xi)$, $C(\xi)$, $D(\xi)$ are arbitrary functions.

Using the boundary conditions and the condition of continuity of stresses σ_y and τ at $y = H_1$, the arbitrary functions $B(\xi)$, $C(\xi)$, $D(\xi)$ can be expressed through $A(\xi)$ in the upper and lower parts of the strip. The obtained expressions can be substituted into the conditions at the crack as well as into the condition of continuity of displacements u_x and u_y at $y = H_1$. The inverse Fourier transformation gives a system of paired integral equations.

Theorem 1. The problem of diffraction by a longitudinal crack of width L (1)–(4) is reduced to the system of paired integral equations with amplitudes $A_1(\xi)$ and $A_2(\xi)$ being unknown functions:

$$\begin{aligned} \int_{-\infty}^{+\infty} A_1(\xi) K_{1j}(\xi) e^{-i\xi x} d\xi + \int_{-\infty}^{+\infty} A_2(\xi) K_{2j}(\xi) e^{-i\xi x} d\xi = \\ = \int_{-\infty}^{+\infty} f_j(\xi) e^{-i\xi x} d\xi, x \notin (0, L), \quad j = 1, 2, \quad (5) \end{aligned}$$

$$\begin{aligned} \int_{-\infty}^{+\infty} A_1(\xi) K_{1j}(\xi) e^{-i\xi x} d\xi + \int_{-\infty}^{+\infty} A_2(\xi) K_{2j}(\xi) e^{-i\xi x} d\xi = \\ = \int_{-\infty}^{+\infty} f_j(\xi) e^{-i\xi x} d\xi, x \in (0, L), \quad j = 3, 4, \quad (6) \end{aligned}$$

where $K_{ij}(\xi)$ and $f_j(\xi)$ are some known functions.

4 Reducing the system of paired integral equations to the system of linear equations

For solving the system of integral equations (5), (6), the Galerkin method can be used with the scalar product defined by an integral along the entire axis.

The functions $h_n(x) = H_n(x)e^{-x^2/2}$ are basis functions, where $H_n(x)$ are the Hermitian polynomials. The chosen system of functions is orthogonal

$$\int_{-\infty}^{+\infty} h_m(x) h_n(x) dx = \int_{-\infty}^{+\infty} H_m(x) H_n(x) e^{-x^2} dx = \sqrt{\pi} 2^n n! \delta_{mn},$$

where δ_{mn} is the Kronecker delta.

The required functions of paired integral equations will be sought in the following form:

$$A_1(\xi) \approx \sum_{n=0}^N B_n h_n(\xi), \quad A_2(\xi) \approx \sum_{n=0}^N C_n h_n(\xi). \quad (7)$$

The representations (7) for the unknown functions can be substituted into the first equation of the system of paired integral equations (5),(6) and multiplied scalarly by $h_m(x)$:

$$\begin{aligned} & \sum_{n=0}^N B_n \int_{-\infty}^{+\infty} h_n(\xi) \left[\int_{-\infty}^0 K_{11}(\xi) e^{-i\xi x} h_m(x) dx + \int_L^{+\infty} K_{11}(\xi) e^{-i\xi x} h_m(x) dx + \right. \\ & \qquad \qquad \qquad \left. + \int_0^L K_{13}(\xi) e^{-i\xi x} h_m(x) dx \right] d\xi + \\ & + \sum_{n=0}^N C_n \int_{-\infty}^{+\infty} h_n(\xi) \left[\int_{-\infty}^0 K_{21}(\xi) e^{-i\xi x} h_m(x) dx + \int_L^{+\infty} K_{21}(\xi) e^{-i\xi x} h_m(x) dx + \right. \\ & \qquad \qquad \qquad \left. + \int_0^L K_{23}(\xi) e^{-i\xi x} h_m(x) dx \right] d\xi = \\ & = \int_{-\infty}^0 \int_{-\infty}^{+\infty} f_1(\xi) e^{-i\xi x} d\xi h_m(x) dx + \int_L^{+\infty} \int_{-\infty}^{+\infty} f_1(\xi) e^{-i\xi x} d\xi h_m(x) dx + \\ & \qquad \qquad \qquad + \int_0^L \int_{-\infty}^{+\infty} f_3(\xi) e^{-i\xi x} d\xi h_m(x) dx, \quad m = 0..N. \end{aligned}$$

The obtained linear equations for $m = 0..N$ are transformed to the following form:

$$\begin{aligned} & \sum_{n=0}^N B_n \left[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} K_{11}(\xi) e^{-i\xi x} h_m(x) dx h_n(\xi) d\xi + \right. \\ & \qquad \qquad \qquad \left. + \int_0^L \int_{-\infty}^{+\infty} (K_{13}(\xi) - K_{11}(\xi)) e^{-i\xi x} h_m(x) h_n(\xi) d\xi dx \right] + \end{aligned}$$

$$\begin{aligned}
& + \sum_{n=0}^N C_n \left[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} K_{21}(\xi) e^{-i\xi x} h_m(x) dx h_n(\xi) d\xi + \right. \\
& \quad \left. + \int_0^L \int_{-\infty}^{+\infty} (K_{23}(\xi) - K_{21}(\xi)) e^{-i\xi x} h_m(x) h_n(\xi) d\xi dx \right] = \\
& = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_1(\xi) e^{-i\xi x} d\xi h_m(x) dx + \int_0^L \int_{-\infty}^{+\infty} (f_3(\xi) - f_1(\xi)) e^{-i\xi x} d\xi h_m(x) dx.
\end{aligned}$$

Note that the Hermitian polynomials, obviously, are eigen functions of the Fourier transformation:

$$\int_{-\infty}^{+\infty} h_m(x) e^{-i\xi x} dx = \sqrt{2\pi} i^m h_m(\xi).$$

Thus, a system of linear algebraic equations is obtained.

Theorem 2. The problem of diffraction by a longitudinal crack is reduced to the system of linear equations for $m = 0..N$

$$\begin{aligned}
\sum_{n=0}^N B_n a_{mn}^{(1)} + \sum_{n=0}^N C_n a_{mn}^{(2)} &= c_{mn}^{(1)}, \\
\sum_{n=0}^N B_n b_{mn}^{(1)} + \sum_{n=0}^N C_n b_{mn}^{(2)} &= c_{mn}^{(2)},
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
a_{mn}^{(j)} &= \sqrt{2\pi} i^m \int_{-\infty}^{+\infty} K_{j1}(\xi) h_m(\xi) h_n(\xi) d\xi + \\
& \quad + \int_0^L \int_{-\infty}^{+\infty} (K_{j3}(\xi) - K_{j1}(\xi)) e^{-i\xi x} h_m(x) h_n(\xi) d\xi dx,
\end{aligned}$$

$$\begin{aligned}
b_{mn}^{(j)} &= \sqrt{2\pi} i^m \int_{-\infty}^{+\infty} K_{j2}(\xi) h_m(\xi) h_n(\xi) d\xi + \\
& \quad + \int_0^L \int_{-\infty}^{+\infty} (K_{j4}(\xi) - K_{j2}(\xi)) e^{-i\xi x} h_m(x) h_n(\xi) d\xi dx,
\end{aligned}$$

$$c_{mn}^{(j)} = \sqrt{2\pi}i^m \int_{-\infty}^{+\infty} f_j(\xi)h_m(\xi)d\xi + \int_0^L \int_{-\infty}^{+\infty} (f_{j+2}(\xi) - f_j(\xi)) e^{-i\xi x} d\xi h_m(x) dx.$$

5 Peculiarities of integrand functions

Coefficients of a matrix of system (8) are calculated as integrals from functions which contain singularities. All singularities are singularities of the first order and are contained in functions $K_{ij}(\xi)$.

$$\begin{aligned} K_{11}(\xi) &= \frac{Q_{11}(\xi)}{\xi Z(\xi)} + \frac{P_{11}(\xi)}{Z(\xi)} + \frac{R_{11}(\xi)}{\xi} + S_{11}(\xi), \\ K_{21}(\xi) &= \frac{Q_{21}(\xi)}{\xi Z(\xi)} + \frac{P_{21}(\xi)}{Z(\xi)} + \frac{R_{21}(\xi)}{\xi}, \\ K_{12}(\xi) &= \frac{P_{21}(\xi)}{Z(\xi)} + S_{12}(\xi), \quad K_{22}(\xi) = \frac{P_{22}(\xi)}{Z(\xi)} + S_{22}(\xi), \\ K_{13}(\xi) &= \frac{Q_{13}(\xi)}{\xi Z(\xi)} + \frac{R_{13}(\xi)}{\xi} - S_{13}(\xi), \quad K_{23}(\xi) = \frac{Q_{23}(\xi)}{\xi Z(\xi)}, \\ K_{14}(\xi) &= \frac{P_{14}(\xi)}{Z(\xi)} + S_{14}(\xi), \quad K_{24}(\xi) = \frac{P_{24}(\xi)}{Z(\xi)}, \end{aligned}$$

where functions $Q_{ij}(\xi)$, $P_{ij}(\xi)$, $R_{ij}(\xi)$ and $S_{ij}(\xi)$ are some continuous functions.

Function $Z(\xi)$ has two zeros: $\xi = \pm\xi^*$. Thus, all $K_{ij}(\xi)$ have a singularity at $\xi = \pm\xi^*$. Functions $K_{11}(\xi)$, $K_{13}(\xi)$, $K_{21}(\xi)$ and $K_{23}(\xi)$ have singularities at $\xi = 0$ also.

Singularities at zero, in difference from case $\xi = \pm\xi^*$ can be found analytically. For example, the function $K_{11}(\xi)$ has the singularity of the first order:

$$\begin{aligned} \lim_{\xi \rightarrow 0} \xi K_{11}(\xi) &= (k_2^2 \cos k_1 H_1 \cos k_2 H_2) \times \\ &\times \left(e^{ik_1(H_2-H_1)} \cos k_2 H_1 - \cos k_2 H_2 - i \sqrt{\frac{\lambda + 2\mu}{4\mu}} e^{-ik_1 H_1} \sin k_2(H_2 - H_1) \right)^{-1} \end{aligned}$$

It should be noted that all integrals containing functions $K_{ij}(\xi)$ exist. It is necessary to calculate the integrals accurately considering their singularities.

6 Conclusions

The problem of diffraction of an elastic wave by the longitudinal crack in a layer is reduced to system of the linear algebraic equations. It is shown that coefficients of a matrix of system are calculated as integrals of functions with one or three features. These features are marked out.

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