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Commutation of Projections and Trace Characterization on von Neumann Algebras. II

A. M. Bikchentaev*

Kazan (Volga Region) Federal University

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Abstract—We obtain new necessary and sufficient commutation conditions for projections in terms of operator inequalities. These inequalities are applied for trace characterization on von Neumann algebras for the class of all positive normal functionals.

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Keywords: *projection, von Neumann algebra, Hilbert space, trace, commutation of operators, normal functional, linear bounded operator, operator inequality.*

1. INTRODUCTION

The creation of the theory of noncommutative integration was stimulated by problems dealing with the mathematical justification of quantum mechanics. Fundamental results in this direction are contained in a series of works by J. von Neumann on operator algebras (in the 30s and 40s of the 20th century), some of which was coauthored by F. J. Murray. Full of constructive ideas, these works laid the foundation for the general integration theory in operator algebras. The general theory of integration with respect to unitarily invariant measures in semifinite von Neumann algebras was finalized by I. E. Segal in 1953. Segal's theory includes integration with respect to a normal trace. He also embedded classical integration theory for measure spaces into this scheme.

In connection with new results in the theory of von Neumann algebras and its widening sphere of applications, the extension of Segal's integration theory to normal weights in arbitrary von Neumann algebras became an important problem. The solution of this problem (in the 70s and 80s of the 20th century) was based on the fundamental results of the general theory of von Neumann algebras (Tomita–Takesaki modular theory (1970), Haagerup's characterization of normal weights (1975)); see, for example, [1].

Studies in trace characterizations for the class of normal weights or functionals on von Neumann algebras began in the 70s of the 20th century.

The present paper is a extension of [2], [3]; we use the notation and terminology from these works. In [2], a commutation criterion for a pair of projections was obtained in terms of their upper (lower) bound in the lattice of all projections of the algebra and it was shown that each skew-Hermitian element of the properly infinite von Neumann algebra $\mathscr{M}$ can be expressed as a finite sum of the commutators of projections from $\mathscr{M}$. The impossibility of such representations for a finite von Neumann algebra is related to the existence of a nontrivial finite trace on this algebra. In the finite-dimensional case, a multitude of operators with zero canonical trace tr has been described in terms of finite sums of the commutators of projections.

A trace characterization on von Neumann algebras in terms of the commutation of the products of projections under the weight sign was obtained in [3].

In the present paper, we establish new criteria for the commutation of projections in terms of operator inequalities. We obtain trace characterization for the class of all positive normal functionals on a von Neumann algebra.

Other trace characterization may be found in [3]–[9]; see also the references therein.

*E-mail: Airat.Bikchentaev@ksu.ru